We compute natural integrals giving intertwining operators among principal series of $G = SL(2, \mathbb{C})$.

1. Principal series representations
2. The main computation
   • Smooth vectors

1. **Principal series representations**

As usual, let

$$N = \{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \}, \quad M = \{ m_a = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \mathbb{C}^\times \}$$

and

$$P = NM = MN$$

For $s \in \mathbb{C}$ and integer $\kappa$, the $(s, \kappa)$th principal series representation $I_{s, \kappa}$ is the space of smooth functions $f$ on $G$ with prescribed left equivariance

$$I_{s, \kappa} = \{ f : f(pg) = \chi_{s, \kappa}(p)f(g) \text{ for all } p \in P, g \in G \}$$

(where $\chi_{s, \kappa}(a) = |a|^{4s}(a/|a|)^{\kappa}$)

with the normalization of the character to have the intertwining operator $T_{s, \kappa}$ below map from $I_{s, \kappa}$ to $I_{1-s, -\kappa}$ rather than have $s$ transform in some other fashion. The group $G$ acts on $I_{s, \kappa}$ by the right regular representation, that is, by right translation of functions:

$$(g \cdot f)(x) = f(xg) \quad \text{(for } g, x \in G)$$

The *standard intertwining operator* $T = T_{s, \kappa} : I_{s, \kappa} \to I_{1-s, -\kappa}$ is defined, for Re$(s)$ sufficiently large, by the integral

$$(T_{s, \kappa}f)(g) = \int_N f(w_0 n \cdot g) \, dn$$

where the long Weyl element $w_o$ is

$$w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The integration is on the left, so does not disturb the right action of $G$. To verify that (assuming convergence) the image really does lie inside $I_{1-s, -\kappa}$, observe that $T_{s, \kappa}f$ is left $N$-invariant by construction, and that for $m \in M$

$$(T_{s, \kappa}f)(mg) = \int_N f(w_0 n \cdot mg) \, dn = \int_N f(w_0 m m^{-1}nm \cdot g) \, dn = \chi_1(m) \cdot \int_N f(w_0 mn \cdot g) \, dn$$

by replacing $n$ by $mnm^{-1}$, taking into account the change of measure $d(mnm^{-1}) = \chi_1(m) \cdot dn$ coming from

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2z \\ 0 & 1 \end{pmatrix}$$

Then this is
That is, for
Conversely, for
The class of
The Cartan element

\[ h \] is a non-negative integer

\( o \) is a non-negative integer

\[ n \]

The complexified Lie algebra \( \mathfrak{su}(2) \otimes _{R} \mathbb{C} \approx \mathfrak{sl}_2(\mathbb{C}) \) has standard \( \mathbb{C} \)-basis

\[ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

The Cartan element \( h \) decomposes finite-dimensional complex representations \( \sigma \) of \( SU(2) \) into eigenspaces, and there is a unique (up to scalars) non-zero \( h \)-eigenvector \( v_o \) annihilated by \( x \), a \textit{highest-weight vector of} \( \sigma \).

The \( h \)-eigenvalue of \( v_o \) is a non-negative integer \( \ell \), the \textit{highest weight} of \( \sigma \), and determines the isomorphism class of \( \sigma \). Application of \( y \) to an \( h \)-eigenvector with eigenvalue \( \lambda \) shifts the eigenvalue to \( \lambda - 2 \), or else annihilates the vector. The collection of all \( h \)-eigenvalues in the irreducible \( \sigma_\ell \) with highest weight \( \ell \) has eigenvalues exactly

\[ -\ell, -\ell + 2, -\ell + 4, \ldots, \ell - 4, \ell - 2, \ell \] (with (non-zero) multiplicities all 1)

A convenient model for the irreducible \( \sigma_\ell \) with highest weight \( \ell \) is homogeneous polynomials of total degree \( \ell \) on \( \mathbb{C}^2 \) treated as row vectors, with the action

\[ (k \cdot f)(u,v) = \sigma_\ell(k)f(u,v) = f((u,v) \cdot k) \] (with \( k \in K \) and \( f \) on \( (u,v) \in \mathbb{C}^2 \))

The highest-weight vector is \( (u,v) \rightarrow u^\ell \). The biregular representation of \( K \times K \) on functions on \( K \) is

\[ (k \times k')f(x) = f(k^{-1}xk) \]

This decomposes the space of (for example) right \( K \)-finite functions as \( \bigoplus_\sigma \sigma \otimes \check{\sigma} \), where \( \check{\sigma} \) is the dual of \( \sigma \), and \( \sigma \) runs through the irreducibles of \( K \).

A function \( f \) in \( I_{s,\kappa} \) is determined by its restriction to \( K \), and must lie in

\[ \text{Ind}_{P \cap K}^K \chi_{s,\kappa} \big|_{P \cap K} = \text{Ind}_{P \cap K}^K \chi_{\kappa} \]

Conversely, for \( s \in \mathbb{C} \), a smooth function \( f_o \) in \( \text{Ind}_{P \cap K}^K \chi_{\kappa} \) has a unique extension to \( f \in I_{s,\kappa} \), by

\[ f(pk) = \chi_s(p) \cdot f_o(k) \]

That is, for \( f \in I_{s,\kappa} \) the restriction \( f|_K \) is a \( \kappa \)-eigenvector for \( h \in \mathfrak{su}(2) \otimes _{R} \mathbb{C} \) under the left action

\[ (h \cdot f)(k) = \frac{\partial}{\partial t} \bigg|_{t=0} f(e^{th}k) \]

This gives the \textit{negative} of the eigenvalue under the left \textit{regular} action.
The irreducible $\hat{\sigma}_\ell$ has non-zero $-\kappa$ eigenspace $\hat{\sigma}_\ell[-\kappa]$ for $\ell \in 2\mathbb{Z}$ for $\ell \geq |\kappa|$ and of the same parity. The eigenspace is one-dimensional. Thus, under the right regular representation of $K$ on $\text{Ind}^K_{P \cap K} \chi_\kappa$, each irreducible appearing appears with multiplicity one:

$$\text{Ind}^K_{P \cap K} \chi_\kappa = \bigoplus_{|\kappa| \leq \ell \in \mathbb{Z}, \ell = \kappa \mod 2} \sigma_\ell \otimes \hat{\sigma}_\ell[-\kappa] \approx \bigoplus_{|\kappa| \leq \ell \in \mathbb{Z}, \ell = \kappa \mod 2} \sigma_\ell$$

(right regular of $K = SU(2)$)

Let

$$R_{s,\kappa} : I_{s,\kappa} \longrightarrow \text{Ind}^K_{P \cap K} \chi_\kappa \quad \text{(by } R_{s,\kappa} f = f|_K\text{)}$$

and

$$E_{s,\kappa} : \text{Ind}^K_{P \cap K} \chi_\kappa \longrightarrow I_{s,\kappa} \quad \text{(by } (E_{s,\kappa} f)o(k) = \chi_{s,\kappa}(p) \cdot f_o(k))$$

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2. The main computation

We compute the effect of the intertwining operator $T_{s,\kappa}$ on a function $f$ in $I_{s,\kappa}$ with a fixed $K$-type $\sigma = \sigma_\ell$. The map $T_{s,\kappa}$ is a $G$-homomorphism, so does not disturb right $K$-types. However, as observed above, the composite

$$\tau_{s,\kappa} = R_{1-s,-\kappa} \circ T_{s,\kappa} \circ E_{s,\kappa}$$

has the effect

$$\tau_{s,\kappa} : \sigma \otimes \hat{\sigma}[\kappa] \longrightarrow \sigma \otimes \hat{\sigma}[-\kappa]$$

For $\kappa = 0$, the two copies of $\sigma \otimes \hat{\sigma}[\pm 0]$ are identical, not merely isomorphic, so by Schur’s lemma $\tau_{s,0}$ is a scalar multiplication on $\sigma \otimes \hat{\sigma}[0]$. For $\kappa \neq 0$, the copies of $\sigma$ in the two induced representations require effort for comparison. We specify vectors $f_o \in \sigma \otimes \hat{\sigma}[-\kappa]$ as matrix coefficient functions, as follows.

Use the model of $\sigma_\ell$ by homogeneous holomorphic polynomials of total degree $\ell$ in two complex variables with hermitian inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_{C^2} \varphi_1(u, v) \cdot \overline{\varphi_2}(u, v) e^{-\pi|u|^2 + |v|^2} \ du \ dv$$

with the additive measure from $C \approx \mathbb{R}^2$. Take $f_o \in \sigma \otimes \hat{\sigma}$ to be a matrix coefficient function

$$f_o(k) = \langle k \cdot \varphi, \psi \rangle \quad (\varphi, \psi \in \sigma)$$

using the hermitian inner product to identify $\sigma$ with its dual. In that model, let

$$\varphi_{\ell,\kappa}(u, v) = u^{\ell+\kappa} \cdot v^{\ell-\kappa}$$

For $f_o(k) = \langle k \cdot \varphi, \psi \rangle$ to be a left $-\kappa$ eigenvector for $h$, $\psi$ must be in $\hat{\sigma}[-\kappa]$, so take

$$\psi(u, v) = \varphi_{\ell,-\kappa}(u, v) = u^{\ell+\kappa} \cdot v^{\ell-\kappa}$$

To make $(\tau_{s,\kappa} f_o)(1) \neq 0$, take

$$\varphi(u, v) = \varphi_{\ell,\kappa}(u, v) = u^{\ell+\kappa} \cdot v^{\ell-\kappa}$$

and

$$f_o(k) = \langle k \cdot \varphi, \psi \rangle = \langle k \cdot \varphi_{\ell,\kappa}, \varphi_{\ell,-\kappa} \rangle$$

For all $K$-types $\sigma_\ell$ appearing, for $v \in \sigma_\ell$, $\tau_{s,\kappa}$ maps $v \otimes \varphi_{\ell,-\kappa}$ to a scalar multiple of $v \otimes \varphi_{\ell,\kappa}$, with scalar depending only on $\sigma, s, \kappa$, and the scalar can be computed as

$$\langle \tau_{s,\kappa} f_o \rangle(1) / \langle \varphi_{\ell,\kappa}, \varphi_{\ell,\kappa} \rangle$$
Paul Garrett: Intertwinings among principal series of SL₂(𝐂) (July 17, 2014)

First,

\[
\langle \varphi_{\ell,\kappa}, \varphi_{\ell,\kappa} \rangle = \int_{\mathbb{C}^2} |u|^{\ell + \kappa} |v|^{\ell - \kappa} \, e^{-\pi(|u|^2 + |v|^2)} \, du \, dv = \int_{\mathbb{C}^2} |u|^{\ell + \kappa} |v|^{\ell - \kappa} \, e^{-\pi(|u|^2 + |v|^2)} \, du \, dv
\]

The latter will cancel, below, so we do not need further explication of this integral. To evaluate

\[
(T_{s,\kappa} \circ E_{s,\kappa} f_o)(1) = \int_N f_o(w_o n) \, dn = \int_{\mathbb{C}} f_o(w_o n_z) \, dz \quad \text{(with } n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix})
\]

we need the Iwasawa decomposition \( w_o n_z = pk:\)

\[
\begin{pmatrix} 0 & -1 \\ 1 & z \end{pmatrix} = w_o n_z = \begin{pmatrix} \frac{1}{\sqrt{1+|z|^2}} & \frac{-\pi}{\sqrt{1+|z|^2}} \\ 0 & \frac{1}{\sqrt{1+|z|^2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{1+|z|^2}} & \frac{-1}{\sqrt{1+|z|^2}} \\ \frac{-\pi}{\sqrt{1+|z|^2}} & \frac{z}{\sqrt{1+|z|^2}} \end{pmatrix}
\]

Thus,

\[
(E_{s,\kappa} f_o)(w_o n_z) = (1 + |z|^2)^{-2s} \cdot f_o(k_z) \quad \text{(with } k_z = \begin{pmatrix} \frac{-\pi}{\sqrt{1+|z|^2}} & \frac{1}{\sqrt{1+|z|^2}} \\ \frac{z}{\sqrt{1+|z|^2}} & \frac{1}{\sqrt{1+|z|^2}} \end{pmatrix})
\]

Then

\[
(k_z \cdot \varphi_{\ell,\kappa})(u, v) = \left( \frac{u\pi}{\sqrt{1+|z|^2}} + \frac{v}{\sqrt{1+|z|^2}} \right)^{\ell + \kappa} \cdot \left( \frac{-u}{\sqrt{1+|z|^2}} + \frac{vz}{\sqrt{1+|z|^2}} \right)^{\ell - \kappa}
\]

and

\[
f_o(z) = \langle k_z \cdot \varphi_{\ell,\kappa}, \varphi_{\ell,-\kappa} \rangle = \int_{\mathbb{C}^2} (k_z \cdot \varphi_{\ell,\kappa})(u, v) \cdot \overline{\varphi_{\ell,-\kappa}(u, v)} \, e^{-\pi(|u|^2 + |v|^2)} \, du \, dv
\]

\[
= \int_{\mathbb{C}^2} \left( \frac{u\pi}{\sqrt{1+|z|^2}} + \frac{v}{\sqrt{1+|z|^2}} \right)^{\ell + \kappa} \cdot \left( \frac{-u}{\sqrt{1+|z|^2}} + \frac{vz}{\sqrt{1+|z|^2}} \right)^{\ell - \kappa} \cdot \frac{u^{\ell + \kappa} v^{\ell - \kappa}}{u^{\ell - \kappa} v^{\ell + \kappa}} \, e^{-\pi(|u|^2 + |v|^2)} \, du \, dv
\]

\[
= (1 + |z|^2)^{-\ell/2} \int_{\mathbb{C}^2} (u\pi + v)(u^{\ell + \kappa} v^{\ell - \kappa}) \cdot (u^{\ell - \kappa} v^{\ell + \kappa}) \, e^{-\pi(|u|^2 + |v|^2)} \, du \, dv
\]

\[
= (1 + |z|^2)^{-\ell/2} \sum_{j=0}^{\min\left(\frac{\ell + \kappa}{2}, \frac{\ell - \kappa}{2}\right)} \binom{\ell + \kappa}{j} \binom{\ell - \kappa}{j} (-1)^j |z|^{2j} \cdot \int_{\mathbb{C}^2} |u|^{\ell - \kappa} |v|^{\ell + \kappa} \, e^{-\pi(|u|^2 + |v|^2)} \, du \, dv
\]

The latter integral is \( \langle \varphi_{\ell,\kappa}, \varphi_{\ell,\kappa} \rangle, \) with roles of \( u, v \) reversed. Thus, letting

\[
\mu = \min\left(\frac{\ell + \kappa}{2}, \frac{\ell - \kappa}{2}\right) = \frac{\ell - |\kappa|}{2}
\]

the scalar by which the \( K, \sigma \)-isotype is multiplied under \( T_{s,\kappa} : I_{s,\kappa} \longrightarrow I_{1-s,-\kappa} \) is

\[
(r_{s,\kappa} f_o)(1) / \langle \varphi_{\ell,\kappa}, \varphi_{\ell,\kappa} \rangle = \int_{\mathbb{C}} (1 + |z|^2)^{-2s - \frac{\mu}{2}} \sum_{j=0}^{\mu} \binom{\ell + \kappa}{j} \binom{\ell - \kappa}{j} (-1)^j |z|^{2j} \, dz
\]

\[
= 2\pi \int_0^\infty (1 + r^2)^{-2s - \frac{\mu}{2}} \sum_{j=0}^{\mu} \binom{\ell + \kappa}{j} \binom{\ell - \kappa}{j} (-1)^j r^{2j} \, rdr = \pi \int_0^\infty (1 + t)^{-2s - \frac{\mu}{2}} \sum_{j=0}^{\mu} \binom{\ell + \kappa}{j} \binom{\ell - \kappa}{j} (-t)^j \, dt
\]
From this point, we follow a slightly simplified version of the device of [Duflo 1975] in the same computation, pp. 57-58. Use the identity
\[
\sum_{j=0}^{c} \binom{a}{j} (b)^{(c)} (-t)^j = \frac{1}{c!} \left( \frac{\partial}{\partial u} \right)^c \bigg|_{u=0} (1 + u)^a (u - t)^b \quad (c \leq \min(a, b))
\]

With \(a = \frac{t + \kappa}{2}, \ b = \frac{t - \kappa}{2},\) and \(c = \mu,\) the scalar is
\[
\frac{\pi}{\mu!} \int_0^\infty (1 + t)^{-2s - \frac{1}{2}} \left( \frac{\partial}{\partial \tau} \right) \mu \bigg|_{\tau = \frac{1}{1+t}} \left( (1 + t) (1 + (t)(\tau - 1) \right)^{\frac{\mu}{2}} dt
\]

We would like to have the differentiation be used in an integration by parts. To this end, separate variables in both \(1 + u\) and \(u - t,\) that is, for some function \(p(t)\) and functions \(q(\tau), r(\tau)\) of a new variable \(\tau,\) to have \(1 + u = p(t) \cdot q(\tau)\) and \(u - t = p(t) \cdot r(\tau).\) Subtraction gives
\[
1 + t = (1 + u) - (u - t) = p \cdot (q - r)
\]

which suggests \(p(t) = 1 + t\) and \(q(\tau) - r(\tau) = 1.\) Take \(q(\tau) = \tau\) and \(r(\tau) = \tau - 1,\) so \(u = (1 + t)\tau - 1,\) and
\[
1 + u = (1 + t)\tau \quad u - t = (1 + t)(\tau - 1) \quad \left( \frac{\partial}{\partial u} \right)^\mu = (1 + t)^{-\mu} \left( \frac{\partial}{\partial \tau} \right)^\mu
\]

The evaluation occurs at \(\tau = \frac{1}{1+t}.\) In these terms, the scalar is
\[
\frac{\pi}{\mu!} \int_0^\infty (1 + t)^{-2s - \frac{1}{2}} \cdot (1 + t)^{-\mu} \left( \frac{\partial}{\partial \tau} \right) \mu \bigg|_{\tau = \frac{1}{1+t}} \left( (1 + t)(\tau - 1) \right)^{\frac{\mu}{2}} dt
\]

Let \(v = \frac{1}{1+t},\) that is, \(t = \frac{1}{v} - 1,\) so \(dt = -\frac{dv}{v^2},\) and \(1 + t = \frac{1}{v},\) and the scalar is
\[
\frac{\pi}{\mu!} \int_0^\frac{1}{v} \left( \frac{1}{v} \right)^{-\mu + \frac{1}{2} - 2s} \left( \frac{\partial}{\partial \tau} \right) \mu \bigg|_{\tau = v} \left( \frac{\mu}{2} \right) \tau^{\frac{\mu}{2}} (1 - 1)^{\frac{\mu}{2}} \frac{dv}{v^2} = \frac{\pi}{\mu!} \int_0^1 v^{2s - \frac{1}{2} + \mu - 2} \left( \frac{\partial}{\partial v} \right) \mu \left( v^{\frac{\mu}{2}} (v - 1)^{\frac{\mu}{2}} \right) dv
\]

writing differentiation followed by evaluation in the more usual fashion. Integrate by parts \(\mu\) times, with boundary terms vanishing, obtaining
\[
\frac{\pi}{\mu!} (-1)^\mu \int_0^1 \left( \frac{\partial}{\partial v} \right) \mu \left( v^{2s - \frac{1}{2} + \mu - 2} \cdot v^{\frac{\mu}{2}} (v - 1)^{\frac{\mu}{2}} \right) dv
\]

\[
= \frac{\pi}{\mu!} (-1)^\mu \int_0^1 \left( (2s - \frac{\mu}{2} + 2) (2s - \frac{\mu}{2} + 3) \cdots (2s - \frac{\mu}{2} - 1) \right) v^{2s - \frac{1}{2} - 2} \cdot v^{\frac{\mu}{2}} (v - 1)^{\frac{\mu}{2}} dv
\]

\[
= \frac{\pi}{\mu!} (-1)^\mu \frac{\Gamma(2s - \frac{\mu}{2} + 2)}{\Gamma(2s - \frac{\mu}{2} - 1)} \int_0^1 v^{2s - \frac{1}{2} - 2} \cdot v^{\frac{\mu}{2}} (v - 1)^{\frac{\mu}{2}} dv
\]

Standard computational devices give
\[
\int_0^1 v^a (1 - v)^b dv = \frac{\Gamma(a + 1) \Gamma(b + 1)}{\Gamma(a + b + 2)}
\]
so the scalar is

\[
\pi \frac{(-1)^{\ell - |\kappa|}}{\Gamma(\frac{\ell - |\kappa|}{2} + 1)} \frac{\Gamma(2s - \frac{|\kappa|}{2} - 1)}{\Gamma(2s - \frac{\ell}{2} - 1)} \cdot (-1)^{\ell + |\kappa|} \frac{\Gamma(2s - \frac{\ell}{2} - 2 + \frac{\ell + |\kappa|}{2})}{\Gamma(2s - \frac{\kappa}{2} - 2 + \frac{\ell + |\kappa|}{2})} = \pi \frac{(-1)^{\ell - |\kappa|}}{\Gamma(\frac{\ell - |\kappa|}{2} + 1)} \frac{\Gamma(2s - \frac{|\kappa|}{2} - 1)}{\Gamma(2s - \frac{\ell}{2} - 1)} \frac{\Gamma(2s - \frac{\ell}{2} - 1)}{\Gamma(2s + \frac{\ell}{2})}
\]

For \( \kappa = 0 \) and \( \ell \in 2\mathbb{Z} \), this simplifies to

\[
\frac{\pi}{\Gamma(2s - 1)} \frac{\Gamma(2s - 1)}{\Gamma(2s - \frac{\ell}{2} - 1)} \frac{\Gamma(2s - \frac{\ell}{2} - 1)}{\Gamma(2s + \frac{\ell}{2})} \quad \text{(with } \kappa = 0 \text{ and } 0 \leq \ell \in 2\mathbb{Z})
\]

### 3. Smooth vectors

From the previous computation, and from

\[
\frac{\Gamma(z + a)}{\Gamma(z + b)} \sim z^{a-b} \quad \text{(fixed } a, b)\]

and

\[
\Gamma(1 - z) \cdot \Gamma(z) = \frac{\pi}{\sin \pi z}
\]

we see that, for fixed \( \kappa \) and \( s \in \mathbb{C} \), the scalar by which \( T_{s,\kappa} \) maps \( \pi \sigma_\ell \otimes \sigma_\ell [-\kappa] \) to \( \sigma_\ell \otimes \sigma_\ell [\kappa] \) is of polynomial growth in \( \ell \).

Thus, these intertwinings extend to the smooth vectors of the representation, since the \( L^2 \) norms of \( \sigma_\ell^{} \) Fourier components of smooth functions on \( SU(2) \) decrease rapidly with \( \ell \), and have sup-norms bounded by a constant multiple of \( \sqrt{\dim \sigma_\ell} \) times their \( L^2 \)-norms.

For \( SL(2, \mathbb{R}) \), see


I thank V. Drinfeld for the following bibliographic notes: \( SL_2(\mathbb{R}) \) is treated in 7.17, and the outcome is stated for \( SL_2(\mathbb{C}) \) in 7.23, in


The case of \( SL_2(\mathbb{C}) \) is Prop 3.7 pp 57-58 in Chap III of


An earlier treatment of this aspect of \( SL_2(\mathbb{C}) \) is