Jacquet theory

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• Jacquet modules
• Jacquet’s lemmas

In effect, the present discussion is merely a slight abstraction of [Jacquet 1971]. This material has been treated in a number of places, but merits reiteration. In effect, this is an elaboration of the consequences of the fact that unipotent radicals of parabolic subgroups of p-adic reductive groups are ascending unions of compact open subgroups, and that the parabolic can act to contract any such subgroup to the identity.

We assume familiarity with simple general results concerning smooth representations of totally disconnected (locally compact, Hausdorff, separable) topological groups. As usual, the condition of smoothness of a representation \( \pi \) of a group \( G \) is that the fixing subgroup 

\[ G_x = \{ g \in G : g \cdot x = x \} \]

be open in \( G \) for every \( x \) in the representation space \( \pi \).

These results apply very broadly to reductive p-adic groups \( G \) and parabolic subgroups \( P \), but the general apparatus of reductive groups is a separate issue from our points here. Indeed, it is already instructive to consider \( G = GL(n, \mathbb{Q}_p) \) and maximal proper parabolics

\[ P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL(n_1, \mathbb{Q}_p), d \in GL(n_1, \mathbb{Q}_p) \right\} \]

or even simply the invertible 2-by-2 matrices \( GL(2, \mathbb{Q}_p) \) and the parabolic \( P \) of upper triangular matrices.

All vector spaces will be over a fixed field \( k \) of characteristic zero, which may be taken to be the complex numbers without much loss.

1. Jacquet modules

Let \( \pi \) be a smooth representation of a p-adic reductive group \( G \) on a \( k \)-vectorspace also denoted \( \pi \). We may suppress explicit reference to \( \pi \), and write 

\[ g \times v \rightarrow g \cdot v \]

for the action of \( g \in G \) on \( v \in \pi \). Let \( P \) be a parabolic subgroup with unipotent radical \( N \) and choice of Levi component \( M \). The Jacquet module \( \pi_N \) (or \( J_P \pi \)) of \( \pi \) is the \( N \)-co-isotype of \( \pi \) for the trivial representation of \( N \). That is, it is the largest quotient of \( \pi \) on which \( N \) acts trivially. Since \( P \) normalizes \( N, \pi_N \) is still a representation of \( P \), and the quotient map

\[ q : \pi \rightarrow \pi_N \]

is a \( P \)-intertwining.

**Proposition:** The Jacquet module \( \pi_N \) is the quotient of \( \pi \) by the \( P \)-subrepresentation \( \pi(N) \) generated by all expressions \( v - n \cdot v \) for \( v \in \pi \) and \( n \in N \).

**Proof:** Under any \( P \)-map \( r : \pi \rightarrow V \) where \( N \) acts trivially on \( V \), certainly

\[ r(v - nv) = rv - r(nv) = rv - n(rv) = rv - rv = 0 \]

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so these elements are in the kernel of the quotient map to the Jacquet module. On the other hand, it is easy to check that the linear span of these elements is stable under $P$, hence under $N$, so we may form the quotient $\pi/\pi(N)$ as a $P$-space. By construction, $N$ acts trivially.

Proposition: The Jacquet module $\pi_N$ is a smooth $P$-representation.

Proof: Given $v \in \pi_N$, let $u \in \pi$ be such that $q(u) = v$. Invoking the smoothness, let $G_u$ be the open subgroup of $G$ fixing $u$. Then $P_o = G_u \cap P$ is a compact open subgroup of $P$, and (since the quotient map is a $P$-morphism) $v$ is $P_o$-fixed.

Proposition: A vector $v \in \pi$ is in the kernel of the quotient $q : \pi \to \pi_N$ if and only if there is a compact open subgroup $N_o$ of $N$ such that

$$\int_{N_o} n \cdot v \, dn = 0$$

Proof: If there is such an $N_o$, let

$$N_1 = N_o \cap G_v$$

This is open, and because $G$ is totally disconnected it is closed, hence compact. Then

$$0 = \int_{N_o} n \cdot v \, dn = \int_{N_o/N_1} \int_{N_1} n n_1 \cdot v \, dn_1 \, dn = \text{meas}(N_1) \sum_{n \in N_o/N_1} n \cdot v$$

since $N_o/N_1$ is a finite set, say with $t$ elements. Then

$$v = v - 0 = v - \frac{1}{t} \sum_{n \in N_o/N_1} n \cdot v = \frac{1}{t} \sum_{n \in N_o/N_1} v - n \cdot v$$

expressing $v$ as a linear combination of the desired form.

On the other hand, given a finite collection of expressions $v - n v$ with $n \in N$ and $v \in \pi$, there is a compact open subgroup $N_o$ of $N$ containing all the finitely-many $n$. Then

$$\int_{N_o} n'(v - n v) \, dn' = \int_{N_o} n' v \, dn' - \int_{N_o} n' n v \, dn' = \int_{N_o} n' v \, dn' - \int_{N_o} n' v \, dn' = 0$$

by replacing $n'$ by $n^{-1} n'$ in the second integral.

Note that for given $G$-morphism $f : A \to B$ the composite $q \circ f : A \to B_N$ is a map to a trivial $N$-space, so factors through $A_N$, giving a $P$-map $f_N : A_N \to B_N$ such that

$$f_N \circ q = q \circ f$$

We may suppress the subscript $N$.

Proposition: The functor $\pi \to \pi_N$ from $G$-representations to $P$-representations is exact, in the sense that short exact sequences

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
are sent to exact sequences

\[ 0 \to A_N \xrightarrow{f} B_N \xrightarrow{g} C_N \to 0 \]

**Proof:** The right half-exactness is a more general property of co-isotypes. That is, the surjectivity of \( g : B_N \to C_N \) is easy, since \( q \circ f : B \to C_N \) is a surjection. Likewise, since the composite \( g \circ f : A \to C \) is 0, certainly

\[ q \circ g \circ f : A \to C_N \]

is 0, so the composite \( A_N \to B_N \to C_N \) is 0.

The injectivity of \( A_N \to B_N \) and the fact that the image of \( A_N \) in \( B_N \) is the whole kernel of \( B_N \to C_N \) are less general, depending upon the special nature of the subgroup \( N \), as manifest in the previous proposition.

Let \( a \in A \) such that \( q(fa) = 0 \in B_N \). Then there is a compact open subgroup \( N_o \) of \( N \) such that

\[ \int_{N_o} n \cdot fa \, dn = 0 \]

Since \( f \) commutes with the action of \( N \), this gives

\[ f \left( \int_{N_o} n \cdot a \, dn \right) = 0 \]

By the injectivity of \( f \)

\[ \int_{N_o} n \cdot a \, dn = 0 \]

so \( qa = 0 \in A_N \). This proves exactness at the left joint.

Suppose \( g(qb) = 0 \). Then \( q(gb) = 0 \), so there is a compact open subgroup \( N_o \) in \( N \) such that

\[ \int_{N_o} n \cdot gb \, dn = 0 \]

and then \( ng = gn \) gives

\[ g \left( \int_{N_o} n \cdot b \, dn \right) = 0 \]

Thus, the integral is in the kernel of \( g \), so is in the image of \( f \). Let \( a \in A \) be such that

\[ fa = \int_{N_o} n \cdot b \, dn \]

Without loss of generality, \( \text{meas}(N_o) = 1 \). Then

\[ \int_{N_o} n' \cdot fa \, dn' = \int_{N_o} \int_{N_o} n' \cdot b \, dn \, dn' = \int_{N_o} \int_{N_o} n \cdot b \, dn \, dn' \]

by replacing \( n \) by \( n'^{-1}n \). Then this gives

\[ \int_{N_o} n \cdot (fa - b) \, dn = 0 \]

So \( q(fa - b) = 0 \) and \( f(qa) = qb \). This finishes the proof of exactness at the middle joint.

2. Jacquet’s lemmas
Let $K = N_1^{\text{opp}} M_\circ N_\circ$ be a compact open subgroup admitting an Iwahori factorization with respect to the parabolic $P$, where $M_\circ$ is a compact subgroup of a Levi component $M$ of $P$, $N_\circ$ is a compact open subgroup of the unipotent radical $N$ of $P$, and $N_1^{\text{opp}}$ is a compact open subgroup of an opposite unipotent radical $N^{\text{opp}}$ to $N$. Further, this presumes that $M_\circ$ normalizes $N_\circ$ and $N_1^{\text{opp}}$. Let $A$ be the maximal split torus in the center of the Levi component $M$. Let

$$A^- = \{ a \in A : a N_\circ a^{-1} \subset N_\circ \}$$

Let $q : \pi \rightarrow \pi_N$ be the quotient map to the Jacquet module of a smooth representation $\pi$ of $G$. For a compact (not necessarily open) subgroup $H$ of $G$, and a smooth representation $\pi$ of $G$, let

$$\text{pr}_H : \pi \rightarrow \pi^H$$

be the map to the $H$-fixed vectors given by

$$\text{pr}_H(v) = \frac{1}{\text{meas}(H)} \int_H h \cdot v \, dh$$

**Lemma: (Jacquet’s First Lemma)** Given $v \in \pi^{M_\circ N_1^{\text{opp}}}$,

$$\text{pr}_K v = \text{pr}_{N_\circ} v$$

and therefore

$$\int_{N_\circ} n \cdot (v - \text{pr}_K v) \, dn = \int_{N_\circ} n \cdot (v - \text{pr}_{N_\circ} v) \, dn = 0$$

Thus, under the quotient map $q : \pi \rightarrow \pi_N$ to the Jacquet module,

$$q(v - \text{pr}_K v) = q(v - \text{pr}_{N_\circ} v) = 0$$

**Proof:** The Iwahori factorization of $K$ and $M_\circ N_1^{\text{opp}}$-invariance of $v$ yields

$$\int_{N_\circ} n \cdot v \, dn = \int_{N_\circ} \int_{M_\circ} \int_{N_1^{\text{opp}}} n m n' \cdot v \, dm \, dn' = \int_K k \cdot v \, dk$$

That is, $\text{pr}_{N_\circ} v = \text{pr}_K v$. ///

**Lemma: (Jacquet’s Second Lemma)** Let $N_1$, $N_2$ be compact open subgroups of $N$, with $m \in M$ such that $m N_1 m^{-1} \subset N_2$. Then

$$\int_{N_1} n \cdot v \, dn = 0$$

implies

$$\int_{N_2} n \cdot m v \, dn = 0$$

**Proof:** This is by direct computation.

$$\int_{N_2} n \cdot m v \, dn = \int_{m^{-1} N_2 m} mn \cdot v \, dn = \int_{m^{-1} N_2 m / N_1} m n \cdot v \, dn = \int_{m^{-1} N_2 m / N_1} mn \cdot 0 \, dn = 0$$

as asserted. ///

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Proposition: Under the quotient map to the Jacquet module $\pi_N$, $\pi^K$ maps surjectively to $(\pi_N)^{M_o}$. In particular, $\pi_N$ is admissible if $\pi$ is. Further, if $\pi^K$ generates $\pi$ then $(\pi_N)^{M_o}$ generates $\pi_N$.

Proof: Let $V$ be a finite-dimensional complex subspace of $(\pi_N)^{M_o}$, and take a finite-dimensional complex subspace $U$ of $\pi$ mapping surjectively to $V$. There is a sufficiently small compact open subgroup $N_2^{opp}$ of $N^{opp}$ such that

$$U \subset \pi^{M_o}N_2^{opp}$$

Take $a$ in $A$ such that

$$aN_2^{opp}a^{-1} \subset N_1^{opp}$$

Then

$$a \cdot U \subset \pi^{M_o}N_1^{opp}$$

Then, on one hand,

$$q(a \cdot U) \subset q(\pi^{M_o}N_1^{opp}) = q(\pi^K)$$

by Jacquet’s first lemma. On the other hand, because $q$ is a $P$-morphism

$$q(a \cdot U) = \pi_N(a) \cdot q(U) = \pi_N(a) \cdot V$$

Thus,

$$\pi_N(a) \cdot V \subset q(\pi^K)$$

Thus, for all finite-dimensional subspaces $V$ of $\pi^{M_o}_N$

$$\dim V \leq \dim U^K \leq \dim \pi^K < \infty$$

by the assumed admissibility of $\pi$, giving the bound

$$\dim \pi^{M_o}_N \leq \dim \pi^K$$

As $a \in A$ centralizes $M$ and hence $M_o$, $\pi_N(a)$ stabilizes $\pi^{M_o}_N$ and gives an automorphism of it. Thus, taking $V$ to be the whole $\pi^{M_o}_N$ shows that $\pi^K$ maps surjectively to $\pi^{M_o}_N$.

3. Bibliography