Background

Satake (mid-1960’s) considered

\[ G \to \tilde{G} \]

where \( G \) and \( \tilde{G} \) are of hermitian type and the map is of hermitian type insofar as it respects this structure.

Then restriction of holomorphic automorphic forms from \( \tilde{G} \) to \( G \) yields holomorphic things.

Shimura (mid-1970’s) looked at examples

\[ SL(2, \mathbb{Q}) \to Sp(n, \mathbb{Q}) \]

\[ SL(2, \mathbb{Q}) \to SL(2, F) \quad (F \text{ totally real}) \]

wherein Fourier coefficients of restrictions are finite sums of Fourier coefficients on \( \tilde{G} \), so a Fourier-coefficient-wise notion of rationality is preserved by restriction.
Shimura combined this with his *canonical models* results to give initiate the modern arithmetic of (holomorphic) automorphic forms. In particular, he generalized a classical principle:

For holomorphic Hecke eigenfunction $f$ with totally real algebraic Fourier coefficients, and for $g$ another holomorphic automorphic form with algebraic Fourier coefficients, *not necessarily a Hecke eigenfunction*,

$$\frac{\langle g, f \rangle}{\langle f, f \rangle} \in \overline{\mathbb{Q}}$$

and for $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the Galois equivariance

$$\left(\frac{\langle g, f \rangle}{\langle f, f \rangle}\right)^\sigma = \frac{\langle g^\sigma, f^\sigma \rangle}{\langle f^\sigma, f^\sigma \rangle}$$

with Galois acting on Fourier coefficients.
In the simplest application, \( g = E \cdot h \) with \( E \) a holomorphic Eisenstein series and \( h \) a cuspform, and as in Rankin (who credits Ingham) for \( h \) a Hecke eigenfunction the integral \textit{unwinds} giving a \textit{special value} of an \( L \)-function

\[
\langle E \cdot h, f \rangle = L(h \otimes f, s_o)
\]

Combining the unwinding with the comparison of inner products gives

\[
\frac{L(h \otimes f, s_o)}{\langle f, f \rangle} \in \mathbb{Q}
\]

and Galois equivariance.

To get \textit{all} (or nearly all) predicted special values, Shimura took a \textit{lower-weight} holomorphic Eisenstein series \( E_{\text{low}} \) and differentiated it to raise its weight before integrating.

\[
\frac{L(h \otimes f, s_o - 2m)}{\langle f, f \rangle} = \frac{\langle D^m E_{\text{low}} \cdot h, f \rangle}{\langle f, f \rangle} \in \mathbb{Q}
\]
Casting about for more examples: *Multiplicative* imbeddings

\[
O(Q) \times Sp(V) \rightarrow Sp(Q \otimes V)
\]

are not usually of hermitian type, but *additive* maps such as

\[
Sp(V_1) \times Sp(V_2) \rightarrow Sp(V_1 \oplus V_2)
\]

\[
U(V_1) \times U(V_2) \rightarrow U(V_1 \oplus V_2)
\]

are. In coordinates,

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\times
\begin{pmatrix}
a' & b' \\
c' & d'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a & 0 & b & 0 \\
0 & a' & 0 & b' \\
c & 0 & d & 0 \\
0 & c' & 0 & d'
\end{pmatrix}
\]

We want simple automorphic forms (or representations) to restrict and decompose. Not thetas, although they do interesting things under multiplicative imbeddings. Siegel-type (degenerate) Eisenstein series, now widely appreciated, were less popular circa 1980.
With holomorphy a complete decomposition (not just $L^2$) is possible (1980). Decomposing a holomorphic Siegel Eisenstein series along

$$Sp(m, \mathbb{Z}) \times Sp(n, \mathbb{Z}) \rightarrow Sp(m + n, \mathbb{Z})$$

$$\sum_{0 \leq \ell \leq \min (m, n)} \sum_{f \text{ cfms } Sp(\ell)} L(f, s_o) \frac{E_f^{(m)} \otimes E_f^{(n)}}{\langle f, f \rangle}$$

where $E_f^{(n)}$ is a Klingen-type Eisenstein series made from cuspform $f$ on $Sp(\ell)$, and $L(f, s_o)$ is a special value of a standard $L$-function of $f$.

(Circa 1981, Böcherer explicated the $L$-function here, and at about the same time Rallis and Piatetski-Shapiro systematically treated the projection of the restriction of not-necessarily holomorphic degenerate Eisenstein series to cuspforms for classical groups, obtaining meromorphic continuations of standard $L$-functions.)

(The full decomposition also suggested that Klingen-type holomorphic Eisenstein series had an arithmetical nature, which was proven by Harris, 1981, 1982.)
To get as many *special values* as possible one must differentiate the Eisenstein series *transversally* before restricting.

Many have played this differentiate-restrict-and-integrate game, and/or restrict-differentiate-integrate.

The archimedean factors of these integrals are nasty to evaluate.
Unitary groups

After the preliminary unwinding and factoring over primes, one is left in situations like the following. Let

\[ G = U(p, q) \quad K = U(p) \times U(q) \]

\[ p_+ = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \in \mathfrak{g} \mathbb{C} \quad p_- = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\} \in \mathfrak{g} \mathbb{C} \]

We must evaluate integrals

\[ T f(g) = \int_G f(gh) \overline{\eta(h)} \, dh \]

\( \eta \) is left-annihilated by \( p_+ \), right by \( p_- \), right \( K \)-type \( \tau \) (descended from the Eisenstein series)

(cuspform) \( f \) right-annihilated by \( p_- \) generating holomorphic discrete series \( \pi_\tau \) with extreme \( K \)-type \( \tau \)
If $\eta \in L^1(G)$ then $f \rightarrow Tf$ is an endomorphism of $\pi_\tau$ not depending upon the model.

Unfortunately, integrability fails in the critical strip, necessitating a more complicated argument there... But let’s suppose we have integrability.

The Harish-Chandra decomposition is

$$G \subset N_+ \cdot K_C \cdot N_- \subset G_C$$

with $N_\pm = \exp p_\pm$. Thus,

$$f(g) = f_{u,v}(g) = f_{u,v}(n_+ \theta n_-) = c_{u,v}(\theta)$$

a matrix coefficient function.

For extreme $K$-type $\tau$ of sufficiently high extreme weight the universal $(g, K)$-module generated by a vector $v_\tau$ of $K$-type $\tau$ and annihilated by $p_-$ is irreducible. Thus, take

$$\eta_{\mu,\nu}(n_+ \theta n_-) = c_{\mu,\nu}(\theta)$$
Theorem:

\[ T f(1) = \int_G f_{u,v}(h) \eta_{\mu,\nu}(h) \, dh \]

\[ = \pi^{pq} \cdot \langle u, \mu \rangle \cdot \langle v, \nu \rangle \cdot \text{(rational number)} \]

In particular, for example,

\[ \tau(k_1 \times k_2) = (\det k_1)^m (\det k_2)^{-n} \quad (m \geq p, \, n \geq q) \]

the rational number is

\[
\frac{\prod_{i=0}^{p+q-1} \Gamma(m + n - i)}{\prod_{i=0}^{p-1} \Gamma(m + n - p - i) \cdot \prod_{i=0}^{q-1} \Gamma(m + n - q - i)}
\]

The real point here is not explicit evaluation, but illustration of a qualitative argument for the rationality of integrals.
We have a Cartan decomposition

\[ G = C \cdot K \approx C \times K \]

where

\[ C = \{ g \in G = U(p, q) : g = g^* > 0 \} \]

Parametrize \( C \) by

\[ z \rightarrow g_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2}z^* & (1_q - z^*z)^{-1/2} \end{pmatrix} \]

where

\[ D_{p,q} = \{ z = p\text{-by-}q \text{ complex : } 1_p - zz^* > 0 \} \]

\( G = U(p, q) \) acts on \( G/K \approx D_{p,q} \) with invariant measure

\[ d^*z = \frac{dz}{\det(1_q - z^*z)^{p+q}} = \frac{dz}{\det(1_p - zz^*)^{p+q}} \]
To compute, use Cartan and Harish-Chandra, 
\( h = h_z k \) and \( h_z = n_z^+ \theta_z n_z^- \), where

\[
\begin{pmatrix}
(1_p - zz^*)^{-1/2} & z(1_q - z^* z)^{-1/2} \\
(1_q - z^* z)^{-1/2} z^* & (1_q - z^* z)^{-1/2}
\end{pmatrix}
\]

\[
\begin{bmatrix}
1 & z \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
(1 - zz^*)^{1/2} & 0 \\
0 & (1 - z^* z)^{-1/2}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\theta_z z^* & 1
\end{bmatrix}
\]

The special form of \( f_{u,v} \) gives

\[
f_{u,v}(h_z k) = f_{u,v}(n_z^+ \theta_z n_z^- k) = f_{u,v}(\theta_z k \cdot k^{-1} n_z^- k)
\]

and

\[
f_{u,v}(\theta_z k) = \langle \tau(\theta_z k) u, v \rangle
\]

and similarly for \( \eta_{\mu,\nu} \). Suppressing \( \tau \),

\[
T f(1) = \int_C \int_K \langle \theta_z k \cdot u, v \rangle \langle \theta_z k \cdot \mu, \nu \rangle \, dk \, d^* z
\]

\[
= \int_C \int_K \langle k \cdot u, \theta_z^* \cdot v \rangle \langle k \cdot \mu, \theta_z^* \cdot \nu \rangle \, dk \, d^* z
\]

Schur relations compute the integral over \( K \).
\[ T f(1) = \frac{\langle u, \mu \rangle}{\dim \tau} \cdot \int_C \langle \theta_z^* \cdot \nu, \theta_z^* \cdot v \rangle d^* z \]
\[ = \frac{\langle u, \mu \rangle}{\dim \tau} \cdot \langle \nu, \int_C \tau(\theta_z^2) d^* z \cdot v \rangle \]

since \( \tau(g^*) = \tau(g)^* \) for \( g \) in \( K_C \), and \( \theta_z^* = \theta_z \).

We compute the endomorphism

\[ S = S(\tau) = \int_C \tau(\theta_z^2) d^* z \]

where

\[ \theta_z^2 = \begin{pmatrix} 1_p - zz^* & 0 \\ 0 & (1_q - z^* z)^{-1} \end{pmatrix} \]

\( \tau \approx \tau_1 \otimes \tau_2 \) with irreducibles \( \tau_1 \) of \( U(p) \) and \( \tau_2 \) of \( U(q) \), so

\[ S = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^* z) d^* z \]

Mapping \( z \to \alpha z \beta^* \) with \( \alpha \in U(p) \), \( \beta \in U(q) \) in the integral shows that \( S \) commutes with \( \tau(k) \), so by Schur’s lemma \( S \) is scalar.
Let $z = \alpha r \beta$ with $\alpha \in U(p)$, $\beta \in U(q)$, and

$$r = p\text{-by-}q = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_q \end{pmatrix}$$

with $-1 < r_i < 1$. Let $\Delta(r) = \prod_{i<j} (r_i^2 - r_j^2)^2$.

Up to a constant $C$ (determined subsequently)

$$\int_{D_{p,q}} h(z) \frac{dz}{\det(1_q - z^* z)^{p+q}}$$

$$= C \cdot \int \int_{(-1,1)^q} h(\alpha r \beta) \ d\alpha \ d\beta \frac{\Delta(r) \ dr}{\det(1_q - r^* r)^{p+q}}$$

Thus, $S$ is

$$C \cdot \int_{U(p) \times U(q)} (\alpha \otimes \beta) \cdot I \cdot (\alpha \otimes \beta)^{-1} \ d\alpha \ d\beta$$

where the inner integral $I$ is

$$I = \int_{(-1,1)^q} (1-rr^*) \otimes (1-r^* r)^{-1} \frac{\Delta(r) \ dr}{\det(1 - r^* r)^{p+q}}$$
The inner integral $I$ in $S$ acts on weight spaces by scalars. The identity

$$(t^2 - u^2) = (t^2 - 1) - (u^2 - 1)$$

shows that each such scalar is a $\mathbb{Q}$-linear combination of products of integrals

$$\int_{-1}^{1} (1 - t^2)^n \frac{dt}{(1 - t^2)^{p+q}}$$

$$= 2^{2n+1-p-q} \frac{\Gamma(n-p-q+1) \Gamma(n-p-q+1)}{\Gamma(2n-2p-2q+2)}$$

$$= \text{rational}$$

so the inner integral $I$ acts by rational scalars on all weight spaces. In particular, $I$ so is a rational endomorphism of $\tau$.

(Better give $\tau$ a rational structure...)
The outer integration is the projection

$$\text{End}_C(\tau) \to \text{End}_K(\tau)$$

where $\text{End}_C(\tau)$ has the $K$-structure

$$k \cdot \varphi = \tau(k) \circ \varphi \circ \tau(k)^{-1}$$

$\text{End}_C(\tau)$ has a rational structure compatible with

$$g_Q = \mathfrak{gl}(p, Q) \otimes \mathfrak{gl}(q, Q)$$
on the complexified Lie algebra

$$g_C = \mathfrak{gl}(p, C) \otimes \mathfrak{gl}(q, C)$$
of $K = U(p) \times U(q)$.

Poincaré-Birkhoff-Witt, the Harish-Chandra homomorphism, and Verma modules still work over $Q$. 

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Highest weights $\lambda - \rho$ for finite-dimensional irreducibles are \textit{integral} (and dominant), so are rational on a rational Cartan subalgebra. Give a finite-dimensional irreducible complex representation $\tau$ a rational structure

$$\tau = (M_\lambda/N_\lambda) \otimes_\mathbb{Q} \mathbb{C}$$

with \textit{rational} Verma module $M_\lambda$ and (unique) maximal proper submodule $N_\lambda$.

$Z(g_\mathbb{Q})$ distinguishes finite-dimensional irreducibles: given finite-dimensional irreducibles $V$ and $V'$ with highest weights $\lambda - \rho = \lambda' - \rho$, there is $z \in Z(g_\mathbb{Q})$ such that $z(\lambda) \neq z(\lambda')$.

Let $\Lambda$ be the finite collection of $\lambda$’s indexing irreducibles in $\text{End}_\mathbb{C}(\tau) = \text{End}_\mathbb{Q}(\tau_\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}$. Then

$$P = \prod_{\lambda \in \Lambda} z_\lambda \in Z(g_\mathbb{Q})$$

projects endomorphisms to the $K$-invariants. Thus, projection to $K$-endomorphisms preserves rationality.
To determine $C$ compute

$$S = S_\tau = \int_{D_{p,q}} \det(1_p - z z^*)^m \det(1_q - z^* z)^{-n} d^* z$$

For $0 < \ell \in \mathbb{Z}$, let

$$C_\ell = \{\ell\text{-by-}\ell\text{ complex } Y > 0\}$$

For real $s > \ell - 1$ define

$$\Gamma_\ell(s) = \int_{C_\ell} e^{-\text{tr} x} (\det x)^s \frac{dx}{(\det x)^\ell}$$

$$= \pi^{\ell(\ell-1)/2} \prod_{i=1}^\ell \Gamma(s - i + 1)$$

Imitating classical computations,

$$\Gamma_p(m + n - p) \Gamma_q(m + n - q) \cdot S =$$

$$\int_{C_{p+q}} e^{-\text{tr} Z} (\det Z)^{m+n} \frac{dZ}{(\det Z)^{p+q}} = \Gamma_{p+q}(m+n)$$
Thus, for this \( \tau \)

\[
S = \frac{\Gamma_{p+q}(m+n)}{\Gamma_p(m+n-p) \Gamma_q(m+n-q)} = \\
\frac{\prod_{i=0}^{p+q-1} \Gamma(m+n-i)}{\prod_{i=0}^{p-1} \Gamma(m+n-p-i) \cdot \prod_{i=0}^{q-1} \Gamma(m+n-q-i)} \times \\
\frac{\pi^{(p+q)(p+q-1)/2}}{\pi p(p-1)/2 \cdot \pi q(q-1)/2}
\]

The net exponent of \( \pi \) is

\[
(p+q)(p+q-1)/2 - p(p-1)/2 - q(q-1)/2 = pq
\]

as anticipated. Thus,

\[
C = \pi^{pq} \cdot \text{(rational)}
\]

and for arbitrary \( \tau \)

\[
S = \pi^{pq} \cdot \text{(rational scalar endomorphism of } \tau)\]