Automorphic Representations and L-functions

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References and historical notes will be added later, maybe.

Many of the statements made here without proof are very difficult to prove! Just because no mention of proof is made it should not be presumed that it’s ‘just an exercise’!
1. Decomposition by central characters

Let $Z$ be the center of a reductive linear group $G$ defined over a number field $k$. Let $G_k$ denote the $k$-valued points of $G$ and let $G_A$ denote the adele points of $G$. Likewise, let $Z_k$ be the $k$-rational points of the center $Z$, and let $Z_A$ denote the adele points of $Z$.

Since $G_k, G_A, Z_k$ and $Z_A$ are all unimodular, there exists a right $G_A$-invariant measure on $G_k \setminus G_A$ and on $Z_A G_k \setminus G_A$, unique up to constant multiples.

A central character is simply a continuous group homomorphism

$$\omega : Z_A \to \mathbb{C}^\times$$

Often we will want to assume that such $\omega$ is trivial on $Z_k$, so gives rise to

$$\omega : Z_A \to Z_k \setminus Z_A \to \mathbb{C}^\times$$

And often we will suppose that $\omega$ is unitary, meaning that for all $z \in Z_A$ we have $|\omega(z)| = 1$.

Let $\omega$ be a central character. Let $f$ be a complex-valued function on $G_A$ so that

$$f(zg) = \omega(z) f(g)$$

for all $z \in Z_A$ and for all $g$. In this case, say that $f$ has central character $\omega$.

The space $L^2(G_k \setminus G_A)$ may be decomposed as a direct integral of Hilbert spaces, according to central characters, in the sense that $L^2(G_k \setminus G_A)$ is a direct integral of the Hilbert spaces

$$L^2(Z_A G_k \setminus G_A, \omega) = \{ |f| \in L^2(Z_A G_k \setminus G_A) and f has central character \omega \}$$

2. Square-integrable cuspforms

Let $G$ be a reductive linear group defined over a number field $k$. Let $f$ be a complex-valued function on $G_A$.

The first hypothesis that we impose upon $f$ is that $f$ is square-integrable with central character $\omega$, in the sense that

$$f \in L^2(Z_A G_k \setminus G_A, \omega)$$

for some character $\omega$ on the adele points $Z_A$ of the center $Z$ of $G_A$.

Next, for every $k$-parabolic subgroup $P$ of $G$ with unipotent radical $N$ we suppose that for almost all $g \in G_A$ (in a measure-theoretic sense)

$$\int_{N_k \setminus N_A} f(ng) \, dn = 0$$

where the $dn$ refers to a right $N_A$-invariant measure on the quotient $N_k \setminus N_A$. The space of $f \in L^2(Z_A G_k \setminus G_A, \omega)$ satisfying the latter condition for all $k$-parabolics $P$ is denoted by

$$L^2_0(Z_A G_k \setminus G_A, \omega)$$

Functions in these spaces are called square-integrable cuspforms with central character $\omega$.

3. Smoothness of cuspforms
A vector \( v \) in a complex-linear representation \( \pi \) of a Lie group \( H \) is smooth if the \( V \)-valued function

\[
g \mapsto \pi(g)v
\]
on \( H \) is infinitely differentiable.

A vector \( v \) in a complex-linear representation \( \pi \) of a totally disconnected group \( H \) is smooth if the \( V \)-valued function

\[
g \mapsto \pi(g)v
\]
is uniformly locally constant, meaning that there is an open subgroup \( N \) so that

\[
f(g) = f(g\theta)
\]
for all \( g \in G \) and for all \( \theta \in N \).

The adeles \( \mathbb{A} \) are a product

\[
\mathbb{A} = \mathbb{A}_{\text{inf}} \times \mathbb{A}_{\text{fin}}
\]
where \( \mathbb{A}_{\text{inf}} \) is the product of the archimedean (i.e., infinite) prime completions, and \( \mathbb{A}_{\text{fin}} \) is the finite prime part of the adeles. Often \( \mathbb{A}_{\text{fin}} \) is called the finite adeles and \( \mathbb{A}_{\text{inf}} \) is called the infinite adeles.

An adele group \( G_{\mathbb{A}} \) is a product of a Lie group and a totally disconnected group, namely,

\[
G_{\mathbb{A}} = G_{\text{inf}} \times G_{\text{fin}}
\]
where \( G_{\text{inf}} \) is the product of the archimedean-prime completions, and where \( G_{\text{fin}} \) is the finite-adele points of \( G \). We need to use coordinates

\[
g_{\text{inf}} \in G_{\text{inf}} \quad g_{\text{fin}} \in G_{\text{fin}}
\]
on these two factors of \( G_{\mathbb{A}} \). A function on \( G_{\mathbb{A}} \) is smooth if, as a function of the two variables \( g_{\text{inf}}, g_{\text{fin}} \), the function is smooth (in the two senses).

A vector \( \varphi \) in any complex-linear representation \( (\pi, V) \) of \( G_{\mathbb{A}} \) is smooth or a smooth vector if

\[
g_{\text{inf}} \times g_{\text{fin}} \mapsto \pi(g_{\text{inf}} \times g_{\text{fin}})\varphi
\]
as a function on \( G_{\mathbb{A}} \) is smooth in the coordinates \( g_{\text{inf}} \times g_{\text{fin}} \).

Due to the existence of approximate identities in the ‘local groups’ \( G_v \) for all completions \( k_v \), smooth vectors are dense in unitary representations of \( G_{\mathbb{A}} \), whether irreducible or not. Thus, the smooth vectors are dense in \( L^2(\mathbb{Z}_A G_k \backslash G_{\mathbb{A}}, \omega) \). Similarly, the smooth vectors are dense in the space of not-necessarily-cuspidal square-integrable function \( L^2(\mathbb{Z}_A G_k \backslash G_{\mathbb{A}}, \omega) \), but this is of little consequence for us.

Thus, without loss of generality, we may suppose that a square-integrable cuspform \( f \) is smooth.

### 4. Eigen-cusps and automorphic representations

For a smooth square-integrable cuspform \( \varphi \in L^2(\mathbb{Z}_A G_k \backslash G_{\mathbb{A}}, \omega) \) consider the collection of linear combinations of right translates

\[
g \mapsto \varphi(gg_o)
\]
of \( \varphi \) by elements \( g_o \in G_{\mathbb{A}} \). The completion in \( L^2(\mathbb{Z}_A G_k \backslash G_{\mathbb{A}}, \omega) \) of this space is the subrepresentation generated by \( \varphi \).

One fundamental application of reduction theory to this situation is the fact that, as a representation space for the adele group \( G_{\mathbb{A}} \), each space \( L^2(\mathbb{Z}_A G_k \backslash G_{\mathbb{A}}, \omega) \) decomposes discretely and with finite multiplicities. Thus, without loss of generality, we may suppose that a cuspform \( f \) generates an irreducible unitary
representation $\pi_f$ of $G_A$ (under right translation). That is, the representation space of $\pi_f$ is the completion in $L^2_b(Z_AG_k \backslash G_A, \omega)$ of the collection of all functions

$$g \mapsto \sum_i c_i f(g \cdot g_i)$$

where the sum is finite, where $c_i \in \mathbb{C}$, and the $g_i$ are fixed elements of $G_A$.

An irreducible unitary representation on a reductive adele group $G_A$ which occurs as a subrepresentation of some $L^2_b(Z_AG_k \backslash G_A, \omega)$ is said to be an automorphic cuspidal representation or cuspidal automorphic representation. As an extension of classical terminology, we might say that such (square-integrable) cuspform $f$ is an eigen-cuspform.

Further, as a corollary of the proof of discreteness and finite multiplicities in spaces of square-integrable cuspforms, we find that a smooth square-integrable cuspform $f$ generating an irreducible representation is of rapid decay at infinity in any Siegel set in $G_A$.

5. Dirichlet series versus zeta and L-functions

Here we set up and clarify standard terminology, and then describe some desiderata for zeta and L-functions. As an overview: the class of all Dirichlet series includes and is much larger than the class of all L-functions, which usually is interpreted to include the class of all zeta functions. And the class of Dirichlet L-functions is a very tiny subclass of the class of all L-functions.

One kind of fairly general definition of L-function, in terms of so-called ‘local data’, is given in a following section.

We note the potential for confusion between the phrases Dirichlet series and Dirichlet L-function. These two phrases are in no way synonymous.

Any series of the form

$$\frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \frac{a_5}{5^s} + \ldots$$

is a Dirichlet series. The $a_n$ are the coefficients. If for some exponent $k$ we have

$$a_n = O(n^k)$$

then by elementary estimates the series is absolutely convergent (and uniformly so on compacta) for $\Re(s) > k + 1$.

For that matter, the numbers $n$ and $n^s$ in the denominators can be replaced by

$$\lambda_1 < \lambda_2 < \lambda_3 \ldots \Rightarrow +\infty$$

and corresponding $s^\text{th}$ powers, giving what are sometimes called generalized Dirichlet series

$$\frac{a_1}{\lambda_1^s} + \frac{a_2}{\lambda_2^s} + \frac{a_3}{\lambda_3^s} + \frac{a_4}{\lambda_4^s} + \frac{a_5}{\lambda_5^s} + \ldots$$

But usually an object interpretable as a generalized Dirichlet series has a more useful aspect of another sort.

One basic point is that every L-function and zeta function is a Dirichlet series, but not every Dirichlet series qualifies as an L-function or zeta-function. And, for most purposes, the notion of ‘L-function’ includes all notions of ‘zeta function’ as special cases.

The potential source of confusion about the terminology is that there is a notion of Dirichlet L-function. Note that the phrase is ‘Dirichlet L-function’, not the more general ‘Dirichlet series’. These Dirichlet L-functions are the most elementary of all L-functions, and include the Riemann zeta-function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$
as the most special and elementary case. Before giving the general definition of Dirichlet L-function, we need a little preparation. Fix a positive integer $F$, and let

$$\chi : \mathbb{Z}/F^\times \to \mathbb{C}^\times$$

be a group homomorphism from the multiplicative group $\mathbb{Z}/F^\times$ of the quotient ring $\mathbb{Z}/F$ to non-zero complex numbers. Extend $\chi$ to a function (still denoted by $\chi$) on all of $\mathbb{Z}/F$ by defining it to be 0 off $\mathbb{Z}/F^\times$. By composing this extended $\chi$ with the quotient map

$$\mathbb{Z} \to \mathbb{Z}/F$$

we get a map (still denoted by $\chi$)

$$\chi : \mathbb{Z} \to \mathbb{Z}/F \to \mathbb{C}^\times$$

The latter map is what is called a Dirichlet character modulo $F$. The most general Dirichlet L-function is of the form

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$$

for a Dirichlet character $\chi$.

Assume an inequality $a_n = O(n^k)$ so that the function

$$D(s) = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \frac{a_5}{5^s} + \ldots$$

is holomorphic in the right half-plane $\Re(s) > k+1$. It is not hard to see that the coefficients $a_n$ are completely determined by the holomorphic function $D(s)$. This is useful in what follows.

For convenience, let’s suppose that $a_1 \neq 0$, so that we can divide through by it, and have

$$D(s) = 1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \ldots$$

With this normalization, we might demand an Euler product factorization

$$D(s) = 1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \ldots = \prod_p (1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \ldots)$$

where $p$ runs over primes, at least in the region of absolute convergence $\Re(s) > k + 1$. The factor

$$1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \frac{a_{p^3}}{p^{3s}} + \ldots$$

is the $p^{th}$ Euler factor.

It is not hard to see that such factorization is equivalent to a weak multiplicativity property of the coefficients $a_n$, namely

$$a_{mn} = a_m \cdot a_n \quad \text{for } m, n \text{ relatively prime}$$

In practice, in any scenario in which the issue is not trivial, such weak multiplicativity is not verified directly, but is a corollary of some exogenous considerations. And here we are making implicit use of the unique factorization in the rational integers $\mathbb{Z}$. In more general situations where Euler factors are indexed by (finite) primes in a number field, we use the unique factorization into prime ideals rather than prime numbers.

For example, Riemann’s zeta function has an Euler product

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots) = \prod_p \frac{1}{\frac{1}{1 - p^{-s}}}$$
Similarly, because of unique factorization and the multiplicativity of Dirichlet characters $\chi$, Dirichlet L-functions have Euler products

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p (1 + \frac{\chi(p)}{p^{s}} + \frac{\chi(p^2)}{p^{2s}} + \frac{\chi(p^3)}{p^{3s}} + \ldots) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

Assume that a Dirichlet series $D(s)$ has an Euler factorization over primes. Then we would hope further that for each prime $p$ the $p^{th}$ Euler factor is a rational function of $p^{-s}$, meaning that

$$1 + \frac{a_p}{p^s} + \frac{a_p^2}{p^{2s}} + \frac{a_p^3}{p^{3s}} + \ldots = \frac{1 + b_1 p^{-s} + \ldots + b_m p^{-ms}}{1 + c_1 p^{-s} + \ldots + c_n p^{-ms}}$$

for some complex numbers $b_1, \ldots, b_m, c_1, \ldots, c_n$ of course depending upon $p$. (For that matter, it may be that $m, n$ also depend upon $p$). For brevity, if this property holds, we would say that the Dirichlet series $D(s)$ has rational Euler factors.

For example, Riemann’s zeta function and Dirichlet L-functions certainly have rational Euler factors, and the Euler factors have the desirable property of depending upon $p$ in a very systematic way.

Without attempting to give a general description of what an ‘L-function’ or ‘zeta function’ might be, we can state as a general principle that Euler factorization and rationality of the Euler factors are prerequisites. Indeed, the modern definitions of most types of L-functions or zeta functions give them as Euler products from the start. Even the rationality of the Euler factors is sometimes part of the definition, it is just as likely that in another circumstance the issue of proving rationality may be fundamental.

6. L-functions defined via local data

Many L-functions and zeta functions fit into the following fairly elementary description, even if the ramifications are unclear.

In particular, this definition does not give any hints as to how to prove that the function so-defined has an analytic continuation, functional equation, and so on.

Fix a positive integer $N$, and fix a finite set $S$ of ‘bad’ primes. For a prime $p$ not in the bad set $S$, let $\Psi_p$ be an $N$-by-$N$ invertible matrix. Then we have an L-function attached to the local data $\{\Psi_p\}$ defined by

$$L(s, \{\Psi_p\}) = \prod_{p \notin S} \frac{1}{\det(1_N - p^{-s}\Psi_p)}$$

where ‘det’ denotes determinant and $1_N$ is the $N$-by-$N$ identity matrix.

Since $\Psi_p$ only enters via its determinant, we could be a little coy about things and say just that we have an assignment of conjugacy classes $p \to \langle \Psi_p \rangle$, rather than specific matrices.

Note that in this definition the factorization over primes is certainly built in, and the rationality of the Euler factors (as functions of $p^{-s}$) is also built in. But without any further information we cannot even be confident that this series converges in a half-plane, much less that it has an analytic continuation, etc.

This definition gives no indication where one could expect the ‘local data’ to come from, nor what might make a prime fall into the collection $S$ of ‘bad’ primes. In the case of Riemann’s zeta, the set $S$ is empty, and for every prime $p$, the local data at $p$ is just $\Psi_p = 1$. In the case of Dirichlet L-functions for a Dirichlet character $\chi$ modulo $F$, we take $S$ to be the set of primes dividing $F$, and the local data at all other primes is $\Psi_p = \chi(p)$.

An analogous definition of L-function can be given in which rational primes as above are replaced by the collection of all primes in a number field. However, such ‘generalization’ can be subsumed in the present set-up.
An issue of some consequence is that of defining Euler factors for infinite or archimedean primes. That is, ideally the assignment of gamma factors should proceed in perfect analogy with construction of all the other Euler factors.

7. Factoring unitary representations of adele groups

To define automorphic L-functions, we want to figure out how to attach local data $\Psi_v$ (in the above sense) to an irreducible unitary representation $\pi$ of a reductive adele group $G_A$. For the definition we will give, it will not matter whether $\pi$ is automorphic or not! This section takes the first of two steps in associating ‘local data’ to irreducible representations of these adele groups.

Since the local groups $G_v$ (v running over primes of the global field $k$ over which $G$ is defined) are known to be Type I, any irreducible unitary representation $\pi$ of $G_A$ factors over primes into a completed restricted tensor product:

$$\pi \approx \bigotimes_v \pi_v$$

where $\pi_v$ is an irreducible unitary representation of $G_v$ uniquely determined up to isomorphism. In particular, this factorization certainly applies to an irreducible unitary representation $\pi_f$ generated by a square-integrable cuspform $f$.

Let $K_v$ be a ‘good’ maximal compact subgroup of $G_v$. Recall that an irreducible unitary representation of $G_v$ is a spherical representation if it has a non-zero $K_v$-fixed vector, in which case there is a one-dimensional space of $K_v$-fixed vectors in the representation space.

With this terminology we can note that, further, for all but finitely-many primes $v$, the ‘local’ representations $\pi_v$ are spherical. Let the set $S$ of bad primes be at least large-enough to contain the finite set of primes $v$ for which $\pi_v$ is non-spherical.

Thus, we have turned the problem of acquisition of ‘local data’ into the problem of meaningfully associating ‘local data’ to spherical representations of the ‘local’ groups $G_v$. That is, we need to attach ‘numerical invariants’ to spherical representations. This is done in the next section.

8. Spherical representations and Satake parameters

Now we begin to see how to attach ‘local data’ $\Phi_v$ to spherical representations $\pi_v$ of the ‘local’ groups $G_v$. To do so, we must introduce the Satake parameters, which arise in the Satake transform.

For simplicity of notation, we will suppress all the subscripts referring to the prime. This ought not create undue confusion, since all the considerations of this section are local. Also, in this context we do not need to think of groups as in any way being functors, so we can simplify our way of talking about them: rather than ‘$k$-valued points of the $k$-group-scheme $G$’, we will just say ‘the group $G$’. And all subgroups will be presumed to be defined over whatever the base field is. In any case, in the basic example of $GL(n, \mathbb{Q}_p)$ these issues can be minimized.

So, let $G$ be a reductive linear group over an ultrametric local field $k$. Let $P$ be a minimal parabolic subgroup of $G$, with unipotent radical $N$ and choice of Levi component $M$. When $G = GL(n, \mathbb{Q}_p)$, we take $P$ to be upper-triangular matrices, $N$ to be elements of $P$ with 1’s on the diagonal, and $M$ just the diagonal matrices. Let $K$ be a ‘good’ maximal compact subgroup of $G$. Let $\mathcal{H}_{G,K}$ denote the spherical Hecke algebra defined as

$$\mathcal{H}_{G,K} = \text{left and right } K\text{-invariant complex-valued functions on } G$$

Give $G$ a right Haar measure so that the measure of $K$ is 1. This is also a left Haar measure. Then $\mathcal{H}_{G,K}$ is a convolution algebra with the convolution product

$$(\eta \ast \varphi)(g) = \int_G \eta(gh^{-1}) \varphi(h) \, dh$$
where $dh$ refers to the Haar measure.

A crucial fact from elementary representation theory is that the isomorphism class of a spherical representation is completely determined by the representation of the spherical Hecke algebra on the one-dimensional space of spherical vectors in the representation. Since the space of spherical vectors is just one-dimensional, automorphisms are just complex scalars. Thus, to describe a spherical representation $\pi$ is to describe a $\mathbb{C}$-algebra homomorphism

$$\mathcal{H}_{G,K} \to \mathbb{C}$$

To give such a description, we first describe the structure of the spherical Hecke algebra itself.

Let $\delta_P$ be the modular function of $P$, meaning that for a right Haar measure $\mu_P$ on $P$ we have

$$\mu_P(pE) = \mu_P(pE^{-1}) = \delta_P(p) \cdot \mu_P(E)$$

for any $p \in P$ and measurable $E \subset P$. The Satake transform $S\eta$ of a function $\eta \in \mathcal{H}_{G,K}$ is defined by the integral formula

$$(S\eta)(m) = \delta_P(m)^{-1/2} \int_N \eta(mn) \, dn = \delta_P(m)^{1/2} \int_N \eta(mn) \, dn$$

where $dn$ denotes a Haar measure on $N$ normalized so that

characteristic function of $K \to$ characteristic function of $K \cap M$

Let $\mathcal{H}_{M,K \cap M}$ be the spherical Hecke algebra of the reductive group $M$ (a fixed Levi component of the parabolic $P$). The Weyl group of $M$ in $G$ is

$$W = \text{normalizer of } M \text{ in } G/\text{centralizer of } M$$

Satake’s theorem is that the Satake transform $S$ gives an isomorphism from the spherical Hecke algebra $\mathcal{H}_{G,K}$ of $G$ to the Weyl-group-invariant elements $\mathcal{H}_{W}^{M,K \cap M}$ of the spherical Hecke algebra $\mathcal{H}_{M,K \cap M}$ of $M$.

The isomorphism is called the Satake isomorphism.

Now generally the spherical Hecke algebra $\mathcal{H}_{M,K \cap M}$ of $M$ is of the form

$$\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

for indeterminates $z_i$, where $n$ is the split rank of $G$. For example, for $GL(n, \mathbb{Q}_p)$ this rank is indeed $n$.

The crucial point is that the spherical Hecke algebra of the Levi component of a minimal parabolic is a commutative Noetherian ring.

Next, the Weyl group $W$ acts in such manner that the full spherical Hecke algebra of $M$ is integral over the $W$-invariant elements. Therefore, always

$$\mathcal{H}_{W}^{M,K \cap M} = \text{commutative Noetherian}$$

Thus, in particular, the spherical Hecke algebra $\mathcal{H}_{G,K}$ is a commutative Noetherian ring.

For example, in the case of $GL(n, \mathbb{Q}_p)$, the Weyl group $W$ is the group of permutations of the indices $1, 2, \ldots, n$, so acts upon

$$\mathcal{H}_{M,K \cap M} \approx \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}]$$

by permutation of these variables $z_i$ (and inverses, correspondingly). Thus, the $W$-fixed subalgebra is

$$\mathcal{H}_{W}^{M,K \cap M} \approx \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1} \ldots, z_n, z_n^{-1}]^W \approx \mathbb{C}[s_1, s_2, s_3, \ldots, s_n, s_n^{-1}]$$

That is, it is generated by the elementary symmetric polynomials $s_1, \ldots, s_n$ in the $z_i$, together with the single item $s_n^{-1}$...
Finally, for essentially elementary reasons the integrality assures that any algebra homomorphism
\[ \lambda : \mathcal{H}_{G,K} \approx \mathcal{H}_{M,K \cap M} \to \mathbb{C} \]
extends to an algebra homomorphism
\[ \tilde{\lambda} : \mathcal{H}_{M,K \cap M} \to \mathbb{C} \]
Thus, if we have an identification
\[ \mathcal{H}_{M,K \cap M} \approx \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}] \]
then the images
\[ \tilde{\lambda}(z_1), \tilde{\lambda}(z_2), \ldots, \tilde{\lambda}(z_n) \]
are the Satake parameters associated to \( \tilde{\lambda} \).
Thus, in summary, to a spherical representation we attach a one-dimensional representation of the spherical Hecke algebra of \( G \), to which we associate a one-dimensional representation of the \( W \)-invariant spherical Hecke algebra of \( M \), which we extend to a one-dimensional representation of the whole spherical Hecke algebra of \( M \), which is completely determined by the images of the generators (denoted by \( z_1, z_2, \ldots, z_n \) above).

9. Local data, L-groups, higher L-functions

So far, to any irreducible unitary representation \( \pi \) of a reductive adele group \( G_A \), whether or not it arises from an automorphic form, we associate the list of ‘local’ representations \( \pi_v \) of the ‘local’ groups \( G_v \). All but finitely-many of these local representations are spherical, so are specified by Satake parameters.

For each prime \( v \) of the global field we arrange the Satake parameters into a suitable diagonal matrix \( \Phi_v \): in the case of \( GL(n, \mathbb{Q}_p) \) with Satake parameters \( \tilde{\lambda}(z_1), \ldots, \tilde{\lambda}(z_n) \) (in the notation of the previous section) we simply take
\[ \Phi_p = \begin{pmatrix} \tilde{\lambda}(z_1) \\ \tilde{\lambda}(z_2) \\ \vdots \\ \tilde{\lambda}(z_n) \end{pmatrix} \]
Then the standard L-function attached to the local data \( \Phi_v \) coming from the Satake parameters is
\[ L(s, \rho, \{ \Phi_v \}) = \prod_{v \in S} \frac{1}{\det(1 - q_v^{-s} \Phi_v)} \]
where \( q_v \) is the order of the residue class field for \( v \), where the ‘1’ denotes the identity matrix of appropriate size, and where \( S \) is the finite set of primes where the ‘local representation’ is not spherical. This also omits archimedean primes.

Further, once we have the ‘local data’ \( \Phi_v \) coming from the Satake parameters, we can create higher L-functions by ‘local date’ as follows. Suppose that we have arranged so that all the \( \Phi_v \) lie inside a group \( L G_v^o \) of matrices, for all possible spherical representations. Let
\[ \rho : L G_v^o \to GL(N, \mathbb{C}) \]
be a finite-dimensional representation of \( L G_v^o \), not depending upon \( v \). Then as ‘local data’ we might use
\[ \Psi_v = \rho(\Phi_v) \]
and form a higher L-function

\[ L(s, \rho, \{\Phi_v\}) = \prod_{v \notin S} \frac{1}{\det(1 - q_v^{-s} \rho(\Phi_v))} \]

where \( q_v \) is the order of the residue class field for \( v \), where the ‘1’ denotes the identity matrix of appropriate size, and where \( S \) is the finite set of primes where the ‘local representation’ is not spherical. This omits archimedean primes.

In the case of \( GL(n, \mathbb{Q}_p) \), the L-group is just \( GL(n, \mathbb{C}) \), and the auxiliary representation \( \rho \) is just a finite-dimensional representation

\[ \rho : GL(n, \mathbb{C}) \to GL(N, \mathbb{C}) \]

In even more general situations, there is nevertheless a very systematic general prescription for arranging the Satake parameters in a diagonal matrix \( \Phi_v \). Further, this is arranged so that all possible local data \( \Phi_v \) lie inside a group \( L^G_G \) depending upon the ‘local group’ \( G_v \). The group \( L^G_G \) is the (connected component of the) L-group attached to \( G \). The L-group idea can also be made to incorporate Galois groups and their representations.

It should be emphasized that the ‘L-group formalism’ is mostly just that, a formalism, and does not really circumvent fundamental issues, but mostly gives a unifying notation and language helpful in the general case. Anyway, in summary, the Satake parameters are used to specify a conjugacy class of semi-simple (i.e., diagonal) elements in the appropriate ‘L-group’. For \( GL(n, \mathbb{Q}_p) \) there is little to be gained, at the outset at least, from worrying about fancier definitions.

And, finally, if the local representations \( \pi_v \) are the tensor factors in an irreducible unitary representation \( \pi \) occurring inside a space \( L^2(Z_A \backslash G_k, \omega) \) of square-integrable cusps (for some \( \omega \)), then the associated L-functions constructed as above are called automorphic cuspidal L-functions.