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Solving $P(\frac{d}{dx})u = \delta$

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The solvability of such equations on $\mathbb{R}^n$ is the *Malgrange-Ehrenpreis theorem.* The one-dimensional case admits a simpler approach, due to the simpler nature of the polynomial ring in a single variable. Consider a one-dimensional constant-coefficient differential equation

$$P(\frac{1}{2\pi i}\frac{d}{dx})u \;=\; a_n u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_1 u' + a_0 u \;=\; \delta$$

with polynomial $P(x) \in \mathbb{C}[x]$. The inserted normalizing constant simplifies Fourier transform computations: normalize Fourier transform so that this set-up gives

$$P(x)\cdot \widehat{u} \;=\; 1$$

The extreme case where $P(x)$ has no real zeros is easy, but not interesting, since (integration against) $1/P(x)$ is a tempered distribution.

The nearly opposite extreme case where $P(x)$ has distinct, real zeros $\{x_1,\ldots,x_n\}$ is more interesting. The essential feature is the possibility of a simple partial fractions decomposition with explicit coefficients:

$$\frac{1}{P(x)} \;=\; \sum_j \frac{1}{P'(x_j)\cdot(x-x_j)}$$

Due to the failure of local integrability, it is not legitimate to say that $\widehat{u} = 1/P(x)$, nor that $\widehat{u}$ is equal to the partial fraction expansion. However, the distribution p such that $xp = 1$ is the *principal value* integral $PV\frac{1}{x}$ attached to $1/x$. This strongly suggests that

$$\widehat{u} \;=\; \sum_j \frac{1}{P'(x_j)} PV\frac{1}{x-x_j}$$

As tempered distributions, $(x-x_j)\cdot PV\frac{1}{x-x_j} = 1$. Thus, since polynomial multiplication is commutative, the j^{th} factor $x - x_j$ can act first on the j^{th} principal-valued distributions $PV\frac{1}{x-x_j}$, and

$$P(x)\cdot\sum_j \frac{1}{P'(x_j)} PV\frac{1}{x-x_j} \;=\; \sum_j \frac{1}{P'(x_j)}\prod_{k\neq j}(x-x_k)$$

We want to prove that this is identically 1, as an identity of polynomials. Indeed, evaluating at $x = x_\ell$, all but the ℓ^{th} product vanishes, and the ℓ^{th} gives $P'(x_\ell)$. Thus, the expression is 1 at all the zeros x_j of $P(x)$. The expression is a polynomial of degree $n-1$, so it is completely determined by its value at n distinct points. Thus, indeed, as tempered distributions,

$$P(x)\cdot\sum_j \frac{1}{P'(x_j)} PV\frac{1}{x-x_j} \;=\; 1$$

Next, the Fourier transform of $PV\frac{1}{x}$ is a constant multiple of $\mathrm{sgn}(x)$, from homogeneity and parity considerations. The constant is determined by application to $xe^{-\pi x^2}$, whose Fourier transform is $-i$ times itself: on one hand,

$$(PV\frac{1}{x})\widehat{\ }(xe^{-\pi x^2}) \;=\; (PV\frac{1}{x})\Big((xe^{-\pi x^2})\widehat{\ }\Big) \;=\; -i(PV\frac{1}{x})(xe^{-\pi x^2}) \;=\; -i\int_{\mathbb{R}} e^{-\pi x^2}\,dx \;=\; -i$$

On the other hand,

$$\int_{\mathbb{R}} \mathrm{sgn}(x) \cdot x e^{-\pi x^2}\, dx \;=\; 2\int_0^\infty x e^{-\pi x^2}\, dx \;=\; 2\int_0^\infty x^2 e^{-\pi x^2}\, \frac{dx}{x} \;=\; \int_0^\infty x e^{-\pi x}\, \frac{dx}{x} \;=\; \frac{1}{\pi}$$

Thus,

$$(PV\frac{1}{x})\widehat{\ } \;=\; -\pi i \cdot \mathrm{sgn}(x)$$

and

$$(PV\frac{1}{x - x_j})\widehat{\ } \;=\; e^{-2\pi i \xi x_j} \cdot (-\pi i) \cdot \mathrm{sgn}(x)$$

Having obtained the constant,

$$u \;=\; -\pi i \cdot \Big(\sum_j \frac{1}{P'(x_j)} \cdot e^{2\pi i \xi x_j} \Big) \mathrm{sgn}(x)$$

solves the differential equation $P(\frac{1}{2\pi i}\frac{d}{dx})u = \delta$ when $P(x)$ has real, distinct roots x_j. ///

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