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The Segal-Shale-Weil (oscillator) representation

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We treat oscillator representations as representations of the Lie *algebra* $\mathfrak{sl}_2$, rather than of the Lie *group* $SL_2(\mathbb{R})$ or coverings thereof, thus avoiding several complications that might obscure the main ideas.

The objects called *oscillator representations* are decades old, originally developed by physicists for their own purposes. For example, [Weyl 1928], [Stone 1930], [vonNeumann 1931]. Detailed study of the infinite-dimensional representation theory of small classical groups appeared first in [Wigner 1939], [Bargmann 1947]. A more modern treatment of oscillator representations began with [Shale 1962], [Segal 1963], and then [Weil 1964]. A relatively recent reference is [Li 2000]. [1]

Some completely natural operators on functions (or distributions) on $\mathbb{R}^n$ interact with each other in a manner that leads easily but surprisingly to a *representation* of the Lie algebra $\mathfrak{sl}_2$. Further, the action of $\mathfrak{sl}_2$ commutes with the natural action of the orthogonal group $O(n)$, *and* interacts well with Fourier transform. We give this straightforward discussion in the first section. Specifically, we have the usual basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}_2$ with the usual defining relations

$$[x,y] = h \qquad [h,x] = 2x \qquad [h,y] = -2y$$

The computations below will show, quite directly, that

$$x \to \text{multiplication by } r^2/2 \qquad y \to \text{application of } -\Delta/2 \qquad h \to \frac{n}{2} + \sum_i x_i \frac{\partial}{\partial x_i}$$

is a *Lie algebra homomorphism*, with Δ the usual Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

and $r = \sqrt{\sum_i x_i^2}$ the usual radius or norm on $\mathbb{R}^n$. Then we look for *weight vectors* for h, that is, eigenvectors. (The eigenvalues are called *weights* in this context.) Then we look among the weight vectors for *highest weight vectors* (that is, x-annihilated weight vectors) or *lowest weight vectors* (that is, y-annihilated weight vectors).

[1] It was only in the 1960's that it started to become clear that representation theory would be helpful in the theory of *automorphic forms*, particular regarding the *theta series* studied in [Hecke 1940] and [Siegel 1935]. Less obvious is that these ideas also bear upon things like Maaß' special waveforms [Maaß 1949]. We postpone discussion of modern applications to automorphic forms.

We first consider these operators on *polynomials*, which relates harmonic polynomials to representations of $\mathfrak{sl}_2$. This is the *Fock model* of the oscillator representation. The far more general analogous case of $O(p,q)\times\mathfrak{sp}_n$ was treated by [Kashiwara Vergne 1978].

Next, we find $O(n)\times\mathfrak{sl}_2$ *subrepresentations* of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. We find *lowest-weight representations* (holomorphic discrete series). This explains why the automorphic forms attached to positive definite quadratic forms are *holomorphic*. [Enright Howe Wallach 1983] shows that this phenomenon is typical: for example, *all* representations of $\mathfrak{sp}_n$ obtained via *compact* orthogonal groups are *lowest-weight* representations. A small but important point is that we do *not* look for h-weight-vectors, but, rather, for weight vectors for a different operator covertly related to the *compact* subgroup $SO_2(\mathbb{R})$ of $SL_2(\mathbb{R})$, despite our scrupulous avoidance of worry about conversion of the $\mathfrak{sl}_2$ action to a *group* action.

Finally, combining these computations with Casselman's *subrepresentation theorem* gives decisive information on the irreducible *quotients* of $\mathscr{S}(\mathbb{R}^n)$ as an $O(n)\times\mathfrak{sl}_2$ representation. This complements the discussion of subrepresentations of $\mathscr{S}(\mathbb{R}^n)$, and is also applicable to indefinite-signature orthogonal groups $O(p,q)$. A simple application is to the $O(1,1)\times\mathfrak{sl}_2$ implicit in [Maaß 1949], allowing one to anticipate the Casimir-operator eigenvalues of *special waveforms* attached to großencharacters of real quadratic extensions of $\mathbb{Q}$. We review relevant ideas about principal series, induced representations, Frobenius reciprocity, and Jacquet modules.

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1. *Experiments to adjust constants*

The computations below are experiments to determine a normalization to match the structure of $\mathfrak{sl}_2$. Part of the point is that there is *no* compulsion to guess the normalization in advance, but, rather, that one can *find* the proper normalization. Further, as noted in greater detail in the remarks at the end of the section, that the computations turn out well is less surprising than it seems at first. That is, the innocent-seeming structures at hand determine things more strongly than we might anticipate. [2] We will prove

[1.0.1] Theorem: The map

$$x=\begin{pmatrix}0&1\\0&0\end{pmatrix}\to\text{multiplication by } r^2/2 \qquad y=\begin{pmatrix}0&0\\1&0\end{pmatrix}\to\text{application of } -\Delta/2$$

$$h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\to\frac{n}{2}+\sum_i x_i\frac{\partial}{\partial x_i}$$

is a Lie algebra homomorphism from $\mathfrak{sl}_2$ to operators on functions (and/or distributions) on $\mathbb{R}^n$.

Proof: Without trying to anticipate renormalizations, in *temporary* notation define two natural linear operators X,Y on Schwartz functions f,

$$Xf=r^2\cdot f \qquad Yf=\Delta\cdot f$$

Then

$$[X,Y]f=(XY-YX)f=r^2\,\Delta f-\Delta(r^2\,f)=r^2\,\Delta f-\left(2n\cdot f+2\sum_{i=1}^n 2x_i\,\frac{\partial f}{\partial x_i}+r^2\,\Delta f\right)$$

$$=-\left(2n+4\sum_{i=1}^n x_i\,\frac{\partial}{\partial x_i}\right)\,f$$

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[2] That innocent-seeming structures determine things more strongly than anticipated is one of the persistent charms of mathematics.

The minus sign is irritating, but this might not be sufficient reason to change the normalization. However, casting about for further justification for getting rid of the sign, we might recall that the two operations, multiplication by r^2 and application of Δ, are *not quite* transformed into each other by Fourier transform F but, rather, the relation has a *sign* (in addition to other *positive* constants). We take this as sufficient reason to **renormalize** by putting that sign into the definition of the operator Y, namely,

$$Xf = r^2 \cdot f \qquad Yf = -\Delta \cdot f$$

so now

$$[X,Y]f = (XY - YX)f = -r^2\,\Delta f + \Delta(r^2\,f) = \left(2n \cdot f + 4\sum_{j=1}^n x_j\,\frac{\partial}{\partial x_j}\right)\,f$$

The factor of 4 is ugly and burdensome, but we need more reason than that to renormalize it away. Thus, though we are are not done with the corrections to our normalization, temporarily consider that operator we just computed, namely

$$H = [X,Y] = XY - YX = 2n + 4\sum_{j=1}^n x_j\,\frac{\partial}{\partial x_j}$$

We want to compute $[H,X] = HX - XH$. Since the summand $2n$ in H (meaning multiplication by $2n$) certainly commutes with everything, we can drop this summand in doing the computation of the commutator. Further, for $i \neq j$ certainly the x_i and x_j terms commute, that is,

$$\left(x_i\,\frac{\partial}{\partial x_i}\right) \cdot x_j^2 \cdot f \;-\; x_j^2 \cdot \left(x_i\,\frac{\partial}{\partial x_i}\right) \cdot f = 0$$

That is, the only possible non-trivial summands in the commutator will arise from summands in H and X with the same index. This gives us a soothing conceptual simplification of the computation. [3] Let f_j denote the partial derivative of f with respect to its j^{th} argument.

$$[H,X]f = 4\sum_{i=j}^n \left(x_j\,\frac{\partial}{\partial x_j} \cdot (x_j^2 f) - x_j^2 \cdot x_j\,f_j\right) = 4\sum_{j=1}^n \big(x_j\,(2x_j\,f + x_j^2\,f_j) - x_j^2 \cdot x_j\,f_j\big) = 8r^2\,f = 8\,Xf$$

It looks as though that factor of 4 is dispensable. But, being patient, before doing any more renormalizing, compute similarly, again noting that differently-indexed terms commute,

$$[H,Y]f = 4\sum_{j=1}^n \left(x_j\,\frac{\partial}{\partial x_j}(-f_{jj}) + \frac{\partial^2}{\partial x_j^2}(x_j\,f_j)\right) = 4\sum_{j=1}^n \left(-x_j\,f_{jjj} + \frac{\partial}{\partial x_j}(f_j + x_j\,f_{jj})\right)$$

$$= 4\sum_{j=1}^n (-x_j\,f_{jjj} + (f_{jj} + f_{jj} + x_j\,f_{jjj})) = 8\,\Delta f = -8\,Yf$$

Again a suspiciously large constant. Thinking of the *standard* structural equations $[h,x] = 2x$ and $[h,y] = -2y$, it appears that we should get rid of the factor of 4 somehow, but we do not want to damage the structural relation $[X,Y] = H$. A reasonable fix is to divide our current H by 4, and to divide both our current X and Y by 2. Thus, a palatable normalization is

$$Xf = \frac{r^2}{2} \cdot f \qquad Yf = \frac{-\Delta}{2} f \qquad Hf = \left(\frac{n}{2} + \sum_{i=1}^n x_i\frac{\partial}{\partial x_i}\right)\,f$$

in which case we do have

$$[X,Y] = H \qquad [H,X] = 2X \qquad [H,Y] = -2Y$$

[3] Still, computation of $[H,X]$ and $[H,Y]$ can be done in an entirely naive fashion without much harm.

matching the relations in $\mathfrak{sl}(2)$. ///

[1.0.2] Remark: Let the standard orthogonal group $O(n)$ have its usual linear action on $\mathbb{R}^n$, and thereby on *functions* on $\mathbb{R}^n$. The operators $f \to \mathbf{r}^2 \cdot f$ and $f \to \Delta f$ both *commute* with $O(n)$, so their bracket will commute. Thus the image of $\mathfrak{sl}_2$ commutes with the action of $O(n)$.

[1.0.3] Remark: There is less whimsy in distinguishing this representation of $\mathfrak{sl}_2$ than meets the eye. First, there are few $O(n)$-invariant homogeneous polynomials of degree 2, and few $O(n)$-invariant (pure) second-order differential operators, and X and Y are just those. Second, since the second-order operator part of the commutator will vanish, the bracket $[X, Y]$ will be a first-order (or lower) $O(n)$-invariant differential operator. There are few $O(n)$-invariant first-order differential operators. Then, in the commutator of H and X, the highest-order (first-order) terms cancel, leaving an $O(n)$-invariant polynomial of degree at most 2. Similarly, in the commutator of H and Y the highest-order (cubic) terms cancel, leaving an $O(n)$-invariant second-order operator. The only slightly serious aspect not resolved in advance is the normalization of constants, which was addressed by carrying out the obvious experiment-computations above.

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2. *Fock model: spaces of polynomials*

As a simple test case we consider the action of our operators on polynomials $\mathbb{C}[x_1, \ldots, x_n]$. For intelligibility, we review some results about the representation theory of $\mathfrak{sl}_2$, whose proofs are given in the appendix. The representation of $O(n) \times \mathfrak{sl}_2$ on *polynomials* $\mathbb{C}[x_1, \ldots, x_n]$ on $\mathbb{R}^n$ is the **Fock model**.

First, we determine *weight vectors* (that is, *eigenvectors*)[4] for the operator

$$h = \frac{n}{2} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

on polynomials $\mathbb{C}[x_1, \ldots, x_n]$, and then find *lowest-weight* vectors (annihilated by y) and *highest-weight* vectors (annihilated by x) among these polynomial weight vectors for h.

Euler's identity

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} \;=\; d \cdot f \qquad \text{(for } f \text{ homogeneous of degree } d\text{)}$$

shows that homogeneous degree d polynomials $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ are weight vectors for h with weight $\frac{n}{2} + d$. Since every polynomial is uniquely expressible[5] as a sum of *homogeneous* polynomials, we have successfully *decomposed* the vector space of *all* polynomials into *weight* spaces

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]^{(d)} = \bigoplus_{d=0}^{\infty} (\frac{n}{2} + d)\text{-weight-space}$$

[2.0.1] Claim: The operator x maps h's weight-vectors with weight λ to weight-vectors with weight $\lambda + 2$. The operator y maps weight-vectors with weight λ to weight-vectors with weight $\lambda - 2$.

[4] In this context, *weight vectors* are just *eigenvectors* for the operator H. They receive a different name to emphasize that they, as eigenvectors for h, will be transformed by the *other* operators x and y to eigenvectors for h with (systematically) *altered* eigenvalues. This mechanism is central to the representation theory of Lie algebras, and is already well illustrated in the case of $\mathfrak{sl}_2$.

[5] The fact that every polynomial is uniquely expressible as a sum of homogeneous polynomials is a misleadingly easy decomposition result. By contrast, in most situations the analogous decomposition is at best non-trivial to prove.

Proof: This is an archetype for computations in Lie algebra representations. Computing directly,

$$h \cdot (x \cdot v) = hx \cdot v = (hx - xh + xh) \cdot v = ([h,x] + xh) \cdot v = (2x + xh) \cdot v$$

$$= 2x \cdot v + xh \cdot v = 2x \cdot v + x \cdot \lambda v = (2+\lambda) x \cdot v$$

That is, x maps the λ weight space to the $\lambda + 2$ weight space. Similarly,

$$h \cdot (y \cdot v) = hy \cdot v = (hy - yh + yh) \cdot v = ([h,y] + yh) \cdot v = (-2y + yh) \cdot v$$

$$= -2y \cdot v + yh \cdot v = -2y \cdot v + y \cdot \lambda v = (-2+\lambda) y \cdot v$$

as claimed. ///

[2.0.2] Remark: Indeed, in the case at hand, multiplication by $r^2/2$ certainly *increases* the degree of a homogeneous polynomial by 2, and application of Δ certainly *decreases* degree by 2, but the argument of the claim shows that the mechanism is universal.

Next, we look for weight vectors annihilated by either x or y. Since x literally *raises* the weight (by 2), a vector annihilated by x would be a **highest weight** vector. Similarly, since y *lowers* the weight by 2 a vector annihilated by y would be a **lowest weight** vector.

[2.0.3] Claim: There are *no* x-annihilated vectors for $\mathfrak{sl}_2$ in $\mathbb{C}[x_1, \ldots, x_n]$. The y-annihilated vectors for $\mathfrak{sl}_2$ in $\mathbb{C}]x_1, \ldots, x_n]$ are *homogeneous harmonic polynomials.*

Proof: Since *functions* [6] are *never* annihilated by x, that is, by multiplication by $r^2/2$, there are no x-annihilated vectors here. A y-annihilated vector is one annihilated by application of $-\Delta/2$. ///

To identify $\mathfrak{sl}_2$-submodules of $\mathbb{C}[x_1, \ldots, x_n]$, start with the *lowest-weight* submodules, that is, those submodules generated by y-annihilated vectors.

Let [7]

$$U(\mathfrak{sl}_2) = \text{universal enveloping algebra of } \mathfrak{sl}_2$$

From Poincaré-Birkhoff-Witt, [8] since y acts by 0 and h acts by $\frac{n}{2} + d$ on lowest-weight vectors,

$$\mathfrak{sl}_2\text{-submodule generated by harmonic } P \text{ of degree } d \;= U(\mathfrak{sl}_2) \cdot P$$

$$= \sum_{a,b,c} \mathbb{C} \cdot x^a h^b y^c \cdot P \;=\; \sum_{a=0}^{\infty} \mathbb{C} \cdot x^a \cdot P \;=\; \sum_{a=0}^{\infty} \mathbb{C} \cdot r^{2a}\, P$$

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[6] At least *classical* functions cannot have support consisting of a single point, 0. Of course, (non-classical) *generalized functions* may have this feature, and we can and will profitably pursue this shortly.

[7] In general, the *universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is an *associative* algebra with a *Lie algebra* map $\mathfrak{g} \to U(\mathfrak{g})$ with the universal property that any Lie algebra map $\mathfrak{g} \to A$ to an associative algebra A factors through $\mathfrak{g} \to U(\mathfrak{g})$. In this definition we use the natural and obvious Lie algebra structure $\mathrm{Lie}(A)$ that can be put on any associative algebra A by $[a,b] = ab - ba$. This is needed to make sense of the Lie homomorphism requirement on $\mathfrak{g} \to U(\mathfrak{g})$ as well as on the map $\mathfrak{g} \to A$. One can also describe the *functor* $\mathfrak{g} \rightsquigarrow U(\mathfrak{g})$ as a *left adjoint* to the functor $A \rightsquigarrow \mathrm{Lie}(A)$, in the sense that $\mathrm{Hom}_{\mathrm{assoc}}(U(\mathfrak{g}), A) \approx \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A))$.

[8] Recall that the Poincaré-Birkhoff-Witt asserts in this case that the universal enveloping algebra U of $\mathfrak{sl}_2$ has a basis consisting of monomials $x^a h^b y^c$ in $U(\mathfrak{g})$ (with non-negative integers a, b, c).

The classical theory of *spherical harmonics*[9] gives an *ad hoc* argument for the fact that every homogeneous polynomial $P(x)$ is expressible (uniquely) in the form

$$P(x) = f_o(x) + r^2 \cdot f_2(x) + r^4 \cdot f_4(x) + \ldots \qquad \text{(finite sum with harmonic } f_i)$$

But we can make this classical fact be a corollary of natural aspects of the representation theory of $\mathfrak{sl}_2$. The following two useful displayed isomorphisms are proven in an appendix. In the decomposition of $\mathbb{C}[x_1, \ldots, x_n]$, we now know that the isomorphism class of $U \cdot P$ depends only upon the *degree* of the harmonic homogeneous polynomial P. Let M_λ be the $\mathfrak{sl}_2$-module with lowest weight λ, and $\mathfrak{H}^{(d)}$ the space of homogeneous harmonic polynomials of degree d. Then

$$\mathbb{C}[x_1, \ldots, x_n] \;\approx\; \bigoplus_{d \geq 0} (\dim_{\mathbb{C}} \mathfrak{H}^{(d)}) \cdot M_{\frac{n}{2}+d} \qquad \text{(as } \mathfrak{sl}_2\text{-representation)}$$

Thinking of $\mathfrak{H}^{(d)}$ as a representation of the orthogonal group $O(n)$, we have a definitive conclusion

$$\mathbb{C}[x_1, \ldots, x_n] \;\approx\; \bigoplus_{d \geq 0} \mathfrak{H}^{(d)} \otimes M_{\frac{n}{2}+d} \qquad \text{(as } O(n) \times \mathfrak{sl}_2\text{-representation)}$$

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3. *Subrepresentations of* $\mathscr{S}(\mathbb{R}^n)$

Now we find $O(n) \times \mathfrak{sl}_2$ *subrepresentations* of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. These are all *extreme-weight representations*. This phenomenon is discussed in [Enright Howe Wallach 1983] in a much broader context. We do *not* look for h-weight-vectors, but, rather, for weight vectors for a different operator, explained below.

For several reasons, we should *not* expect to find weight vectors for

$$h \sim \frac{n}{2} + \sum_{j=1}^{n} x_j \, \frac{\partial}{\partial x_j}$$

in $\mathscr{S}(\mathbb{R}^n)$. An immediate reason is that differentiable functions which are eigenvectors for this operator are *positive homogeneous*, which is a tough condition for Schwartz functions to meet.

In any case, the action of Lie algebra elements imagined to arise from *unitary group actions* should be *skew-hermitian*, while the images of x, y are *symmetric*, with respect to the hermitian product $\langle , \varphi \rangle = \int_{\mathbb{R}^n} f \, \overline{\varphi}$.[10] Thus, rather than $r^2/2$ and $-\Delta/2$, we should take something like

$$x \to i \cdot \frac{r^2}{2} \qquad y \to -i \cdot \frac{-\Delta}{2} = \frac{i\Delta}{2} \qquad h \to \frac{n}{2} + \sum_j x_j \, \frac{\partial}{\partial x_j}$$

with the signs on the i's chosen to leave $[x, y] = h$ unchanged. However, eigenvectors for h would still be homogeneous.

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[9] The *theory of spherical harmonics* refers to the relatively elementary theory of harmonic polynomials, using little or no representation theory. Parts of this theory go back at least to Laplace in his analysis of the solar system. Sometimes expansion of polynomials or other functions on spheres in terms of harmonic polynomials are called *Laplace expansions*.

[10] The skew-hermitian-ness is natural. Let v, w be in a Hilbert space on which a Lie group G acts unitarily, meaning that $\langle gv, gw \rangle = \langle v, w \rangle$ for $g \in G$. Let $x \to e^x$ be the Lie exponential map. Then for all $t \in \mathbb{R}$, the unitary condition is $\langle v, w \rangle = \langle e^{tx} v, e^{tx} w \rangle$. Differentiating both sides in t and taking $t = 0$ gives, by Leibniz' rule, $0 = \langle xv, w \rangle + \langle v, xw \rangle$, the desired skew-hermitian-ness.

Another reason to look for weight vectors for an operator other than this obvious h is that it is more reasonable to look for eigenvectors of operators arising from *compact* groups, rather than from *non-compact* ones. That is, *imagining* that $\mathfrak{sl}_2$ is (the complexification of) the Lie algebra of $SL_2(\mathbb{R})$, with compact subgroup $SO(2)$, we have (complexified) Lie algebra

$$\mathfrak{so}_2 = \{\text{2-by-2 matrices } \alpha \text{ with } \alpha + \alpha^\top = 0\}$$

and decide to look for weight vectors for

$$\theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathfrak{so}_2$$

With renormalized $x \to ir^2/2$ and $y \to i\Delta/2$

$$\theta = x - y \to \frac{i}{2}(r^2 - \Delta)$$

Instead of the eigenvectors x, y for h, we need eigenvectors in $\mathfrak{sl}_2$ for θ. To find eigenvectors in $\mathfrak{sl}_2$ for the action

$$[\theta, -] \;:\; \alpha \longrightarrow \theta\alpha - \alpha\theta = [\theta, \alpha] = (\mathrm{ad}\theta)(\alpha)$$

it is convenient to diagonalize [11] the *matrix* θ

$$\theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1}$$

The eigenspaces of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ on $\mathfrak{sl}_2$ are easy to see:

$$\text{0-eigenspace} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \qquad 2i\text{-eigenspace} = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \qquad -2i\text{-eigenspace} = \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}$$

Thus, define the **raising** and **lowering** operators R, L by

$$R = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \qquad L = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$$

We have

$$[\theta, R] = 2i \cdot R \qquad [\theta, L] = -2i \cdot L \qquad [R, L] = -i \cdot \theta$$

[3.0.1] Remark: If we were to replace θ by $-i\theta$ then R, L, θ would be in exactly the same relation as x, y, h.

Thus, as with x, y, h, for a θ-weight-vector v with weight $i\lambda$, the vector Rv has weight $i\lambda + 2i$ and Lv has weight $i\lambda - 2i$.

[3.0.2] Claim: A weight vector $f \in \mathscr{S}(R^n)$ for θ has weight $i\lambda$ with $\lambda > 0$.

Proof: The point is that the operator $f \to (r^2 - \Delta)f$ is *positive*, since

$$\int_{\mathbb{R}^n} (r^2 - \Delta)f \cdot \overline{f}\, dx = \int_{\mathbb{R}^n} r^2\, f \cdot \overline{f}\, dx + \int_{\mathbb{R}^n} \nabla f \cdot \overline{\nabla f}\, dx$$

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[11] The matrix $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}/\sqrt{2}$ is the *Cayley* transform, and is a convenient normalization of an element in $SL_2(\mathbb{C})$ mapping the disk to the upper complex half-plane.

by integrating by parts. ///

[3.0.3] Corollary: For a weight vector $f \in \mathscr{S}(R^n)$ for θ there is $0 < m \in \mathbb{Z}$ such that

$$L^m \cdot f = 0$$

Proof: From the claim, the weight of f is $i\lambda$ with $\lambda > 0$. Then $L^j \cdot f$ is still a θ-weight-vector, with weight $i(\lambda - 2j)$. If $L^j f \neq 0$ then $\lambda - 2j > 0$. Thus, for large-enough j, certainly $L^j f = 0$. ///

A θ-weight vector annihilated by L is a **lowest-weight** vector. A representation of $\mathfrak{sl}_2$ generated by a lowest-weight vector in this sense is a **lowest-weight representation**.

[3.0.4] Remark: The appendix shows that, with weight vector f of weight $i\lambda$ with $L^m f = 0$ and $\lambda \geq 2m+1$, the $\mathfrak{sl}_2$ representation generated by f is a finite direct sum of lowest-weight representations.

We can exhibit many lowest-weight representations inside $\mathscr{S}(\mathbb{R}^n)$.

[3.0.5] Claim: A function $P(v)\, e^{-|v|^2/2}$ on $\mathbb{R}^n$, with P a homogeneous harmonic polynomial of degree d, is a lowest weight vector for θ, with weight

$$i(\frac{n}{2} + d) = \text{lowest weight of rep'n generated by } P(v)\, e^{-|v|^2/2} \qquad (P \text{ deg } d \text{ harmonic homog.})$$

Proof: First, letting $g(v) = e^{-|v|^2/2}$, compute

$$(r^2 - \Delta)(P(v)\, g(v)) = r^2\, P(v)\, g(v) - \sum_j \frac{\partial}{\partial x_j}(P_j(v)\, g(v) - x_j\, P(v)\, g(v))$$

$$= r^2\, P(v)\, g(v) - \sum_j (P_{jj}(v)\, g(v) - P(v)\, g(v) - 2x_j\, P_j(v)\, g(v) + x_j^2\, P(v)\, g(v))$$

$$= r^2\, P(v)\, g(v) - \Delta P(v)\, g(v) + n\, P(v)\, g(v) + 2d\, P(v)\, g(v) - r^2\, P(v)\, g(v) = (n+2d)\, P(v)\, g(v)$$

invoking Euler's theorem [12] along the way. Since $\theta = \frac{i}{2}(r^2 - \Delta)$ we obtain the claimed weight.

Now we must see that L *annihilates* such a function. Expanding in terms of x, y, h, we have

$$L = \frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} = \frac{1}{2}(ih + x + y) = \frac{1}{2}\left(\frac{in}{2} + i\sum_j x_j \frac{\partial}{\partial x_j} + \frac{ir^2}{2} + \frac{i\Delta}{2}\right)$$

Thus, removing a factor of $i/4$, we want to show that

$$\left(n + 2\sum_j x_j \frac{\partial}{\partial x_j} + r^2 + \Delta\right) P(v)g(v) = 0$$

We can shorten the computation a little by noting that we already computed that

$$(r^2 - \Delta)\, P(v)g(v) = (n+2d)\, P(g)g(v)$$

so

$$\Delta\, P(v)g(v) = (r^2 - n - 2d)\, P(v)g(v)$$

[12] The theorem of Euler's we invoke is that $\sum_j x_j \frac{\partial}{\partial x_j} P = d \cdot P$ for a homogeneous function P of degree d.

Thus, computing,

$$\left(n + 2\sum_j x_j \frac{\partial}{\partial x_j} + r^2 + \Delta\right) Pg = \left(n + 2\sum_j x_j \frac{\partial}{\partial x_j} + r^2 + r^2 - n - 2d\right) Pg$$

$$= \left(2\sum_j x_j \frac{\partial}{\partial x_j} + 2r^2 - 2d\right) Pg = 2\sum_j x_j (P_j\, g - x_j P\, g) + (2r^2 - 2d)\, Pg$$

$$= 2dPg - 2r^2 Pg + (2r^2 - 2d)\, Pg = 0$$

This shows that these functions are indeed annihilated by the lowering operator L. ///

[3.0.6] Remark: The results of the appendix show that a lowest-weight representation with lowest weight $i\lambda$ with $\lambda \geq 1$ is *irreducible.*

Thus, inside $\mathscr{S}(\mathbb{R}^n)$ we have

$$\bigoplus_{d=0}^{\infty} \mathfrak{H}^{(d)} \otimes M_{\frac{n}{2}+d} \qquad \text{(as } O(n) \times \mathfrak{sl}_2\text{-representation)}$$

where $\mathfrak{H}^{(d)}$ is the space of homogeneous degree d harmonic polynomials and $M_{\frac{n}{2}+d}$ is *the*[13] lowest-weight representation of $\mathfrak{sl}_2$ with lowest weight $\frac{n}{2} + d$.

[3.0.7] Remark: The subsequent discussion of *quotients* of $\mathscr{S}(\mathbb{R}^n)$ will suggest that the above examples are *all* the subrepresentations of $\mathfrak{sl}_2$.

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4. *Maps to principal series: subrepresentation theorem*

To study irreducibles of $\mathfrak{sl}(2)$ obtainable as quotients of $\mathscr{S}(\mathbb{R}^n)$, we will invoke Casselman's *subrepresentation theorem* [Casselman Milicic 1982], explained below. [14]

Instead of simply $\mathfrak{sl}_2$ representations, we need a bit more structure. This was one of Harish-Chandra's fundamental insights.

Let $K = SO_2(\mathbb{R})$ be the usual *special orthogonal* subgroup of $SL_2(\mathbb{R})$, defined to be

$$K = SO_2(\mathbb{R}) = \{g \in SL_2(\mathbb{R}) : g^\top g = 1_2\}$$

Letting $G = SL_2(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}_2$, a $(\mathfrak{g}, K)$-**module** or **Harish-Chandra module**[15] is a $\mathfrak{g}$-representation space V (a complex vector space) with an *additional* structure of K-representation, with the (natural) compatibilities[16] that

$$k \cdot (x \cdot (k^{-1} \cdot v)) = (kxk^{-1}) \cdot v \qquad \text{(for } v \in V,\ x \in \mathfrak{g}, \text{ and } k \in K)$$

[13] Again, for lowest weight $i\lambda$ with $\lambda \geq 1$ we know that the isomorphism class of the representation is uniquely determined by the lowest weight. Thus, for $n > 1$ or $d > 0$ we have this uniqueness.

[14] A much simpler proof of the subrepresentation theorem, applicable to $\mathfrak{sl}_2$ appears in [Casselman Osborne 1978].

[15] Sometimes an additional and implicit condition is imposed in the definition of $(\mathfrak{g}, K)$-module,, namely, that the multiplicity of each irreducible of K in V is *finite*. This is the condition of (K-) **admissibility**.

[16] These compatibilities are the ones that would arise if the representation space were the *smooth vectors* in a $SL_2(\mathbb{R})$-representation.

where $x \to kxk^{-1}$ is the adjoint action of $k \in K$ on $x \in \mathfrak{g}$, and that the action of the Lie algebra $\mathfrak{k}$ of K exponentiates to the action of K, that is, that

$$\frac{d}{dt}\Big|_{t=0} e^{t\cdot y}\cdot v = y\cdot v \qquad \text{(for } v\in V,\ y\in\mathfrak{k},\text{ with } e^{ty}\in K)$$

The notion of $(\mathfrak{g}, K)$-module turns out (with much hindsight) to be even better than the notion of $\mathfrak{g}$-module.

The import of the subrepresentation theorem recalled below is that every *irreducible* $(\mathfrak{g}, K)$-module imbeds into an explicit family of representations, the *principal series*, parametrized simply by representations of $\mathbb{R}^\times$, and described as follows.

First, we grant ourselves that all the (continuous, and actually *differentiable* representations of $\mathbb{R}^\times$ fall into two families

$$a \to \chi_s^+(a) = |a|^s \qquad (s\in\mathbb{C})$$

and

$$a \to \chi_s^-(a) = \operatorname{sgn}(a)\cdot|a|^s \qquad (s\in\mathbb{C})$$

where $\operatorname{sgn}(a) = \pm 1$ is the *sign* of $a\in\mathbb{R}^\times$. The first family consists of *even* representations, the second of *odd*. For each $\chi_s^\pm$ we will construct a useful $(\mathfrak{g}, K)$-module.

With $G = SL_2(\mathbb{R})$, as usual we have a few further important subgroups in addition to $K = SO_2(\mathbb{R})$, namely

$$\begin{aligned} P &= \text{upper-triangular} &&= \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \\ N &= \text{upper-triangular unipotent} &&= \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \\ M &= \text{diagonal} &&= \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \end{aligned}$$

Given a character χ of $\mathbb{R}^\times$, we have a corresponding character of M given by

$$\chi\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \chi(a)$$

Extend such a character *trivially* to N, thus giving a character on P, by

$$\chi\begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix} = \chi(a)$$

The **principal series representation** [17] with character χ is

$$I_\chi^\infty = \{\text{smooth } \mathbb{C}\text{-valued functions } f \text{ on } G : f(pg) = \chi(p)\cdot f(g) \quad \text{for } p\in P \text{ and } g\in G\}$$

The *group* $G = SL_2(\mathbb{R})$ acts on this space, by right translations

$$(g\cdot f)(h) = f(hg)$$

What is this I_χ^∞? There are many things that can be said about this construction. The very first one is that the construction is easy to *describe*, however ineffable it may be otherwise. Second, the construction is

[17] The word *series* is not clearly singular or plural. Since there is a *family* of these representations, one could say that there is a *series* of them. The minor confusion arises when the word *representation* is dropped, and reference is made to *the principal series* I_χ, for example.

an example of *induction* of a representation of a subgroup to a representation of the whole group. But this construction is not whimsical: a more substantial point is that the *induction functor* $\chi \to I_\chi^\infty$ is approximately an *adjoint functor* to the forgetful functor of restricting from G to P. We will elaborate on a variation of this feature below. Finally, the fact of the subrepresentation theorem makes the construction important, by its relevance to systematic modeling of irreducible representations.

But the vectorspace I_χ^∞ is a little too large for our present purposes.

The *Iwasawa decomposition* asserts that

$$G = P \cdot K$$

Thus, every function in the space I_χ^∞ is completely determined by its restriction to K, and the only condition on this restriction is left equivariance by

$$P \cap K = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

In particular, smooth functions on G restrict to smooth functions on $K = SO_2(\mathbb{R})$, which have (literal!) Fourier series expansions. While smooth functions do have nicely convergent Fourier series, we want to *avoid* issues of convergence. With hindsight, this is a good thing to do, although for fairly subtle reasons. A function f on G is K-**finite** if the collection of finite linear combinations

$$g \to \sum_i c_i \cdot f(gk_i)$$

of right translates $g \to f(gk_i)$ of f by elements k_i of K is *finite-dimensional*. Then, we look at the smaller subspace [18]

$$I_\chi = I_\chi^{K-fin} = \{f \in I_\chi^\infty : f \text{ is } K\text{-finite}\}$$

A bad effect of this shrinking of the space is that G no longer can act, since for typical $h \in G$ and K-finite f the function $g \to f(gh)$ will no longer be K-finite. However, we have *not* disrupted the action of $\mathfrak{g}$:

[4.0.1] Claim: The (right) action of the Lie algebra $\mathfrak{g}$ preserves (right) K-finiteness of functions on G.

[4.0.2] Remark: This is a more general fact than merely concerning principal series, and follows essentially because $\mathfrak{g}$ is finite-dimensional. The result readily admits further generalization.

Proof: Let f be a right K-finite function on G generating a finite-dimensional K-representation V of functions on G. The bilinear map

$$\mathfrak{g} \times V \to \{\text{functions on } G\}$$

by

$$x \times \varphi \to x \cdot \varphi$$

of course gives rise to a *linear* map

$$\mathfrak{g} \otimes V \to \{\text{functions on } G\}$$

The latter linear map is a K-homomorphism:

$$((kxk^{-1} \otimes (k \cdot \varphi))(g) = \frac{d}{dt}\Big|_{t=0} \varphi(g \cdot e^{t\,kxk^{-1}} \cdot k) = \frac{d}{dt}\Big|_{t=0} \varphi(g \cdot k \cdot e^{t\,x}) = k \cdot (x \otimes \varphi)(g)$$

Thus, the K-representation space generated by $x \cdot \varphi$ is of dimension at most the product of $\dim \mathfrak{g}$ and $\dim V$. ///

[18] From the Iwasawa decomposition, in fact one can show that the K-finite vectors are inevitably smooth.

Therefore, the K-finite vectors I_χ form a $(\mathfrak{g}, K)$-module. Proof of the following appears in [Casselman Miličić 1982].

[4.0.3] Theorem: (Casselman) An irreducible $(\mathfrak{g}, K)$ module imbeds in *some* principal series I_χ. ///

[4.0.4] Remark: Thus, we have a slightly imprecise parametrization of irreducibles of $(\mathfrak{g}, K)$, namely, by the principal series into which they imbed. So we give a character $\chi = \chi_s^\pm$ of $M \approx \mathbb{R}^\times$, which is specified by the complex number s and sign $\pm$.

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5. *Adjointness: Frobenius reciprocity*

The defining property of the principal series $(\mathfrak{g}, K)$-module I_χ is its *adjointness* in relation to the forgetful functor

$$\mathrm{Res}^{\mathfrak{g},K}_{\mathfrak{p},K_M} \;:\; (\mathfrak{g}, K)\text{-modules} \rightsquigarrow (\mathfrak{p}, K_M)\text{-modules} \qquad (\text{where } K_M = K \cap M = K \cap P = \{\pm 1_2\})$$

This adjointness is usually called **Frobenius reciprocity**, an example of an *adjunction* relationship between functors:

[5.0.1] Theorem: For all K-admissible $(\mathfrak{g}, K)$-modules V there is a natural isomorphism

$$i_{V,\chi} \;:\; \mathrm{Hom}_{\mathfrak{p},K_M}(\mathrm{Res}^{\mathfrak{g},K}_{\mathfrak{p},K_M} V,\, \chi) \;\approx\; \mathrm{Hom}_{\mathfrak{g},K}(V,\, I_\chi)$$

[5.0.2] Remark: The natural isomorphism is indeed the main point here, not the *construction* or the *formulaic* aspects of the isomorphism. The theorem can be construed as asserting the *existence* of an adjoint functor $\chi \rightsquigarrow I_\chi$ to the forgetful functor $\mathrm{Res}^{\mathfrak{g},K}_{\mathfrak{p},K_M}$, rather than asserting any particular construction of either I_χ or the isomorphism itself. Nevertheless, the *proof* of existence does proceed by *exhibiting* a map. In particular, with our earlier model of I_χ by functions on G, suppressing the bulky notation for the forgetful functor $\mathrm{Res}^{\mathfrak{g},K}_{\mathfrak{p},K_M}$, the map $i = i_{V,\chi}$ and its inverse *will be proven to be*

$$\begin{array}{rcll} (i\varphi)(v)(pk) &=& \chi(p)\cdot\varphi(k\cdot v) & (\text{with } v \in V,\ p \in P,\ k \in K, \text{ and } \varphi \in \mathrm{Hom}_{\mathfrak{p},K_M}(V,\, \chi)) \\ (i^{-1}\Phi)(v) &=& \Phi(v)(1) & (\text{with } v \in V,\ \Phi \in \mathrm{Hom}_{\mathfrak{g},K}(V,\, I_\chi)) \end{array}$$

Proof: Let $i = i_{V,\chi}$. Once the formulaic description of the alleged adjoint I_χ and adjunction i are given, the most significant remaining point is to note that the formula for $i\varphi$ gives a *well-defined* function on G, since for $h \in P \cap K$

$$\chi(ph)\,\varphi(h^{-1}k\cdot v) = \chi(p)\,\chi(h)\chi(h^{-1})\,\varphi(k\cdot v) = \chi(p)\,\varphi(k\cdot v)$$

because φ is a $(K \cap P)$-homomorphism. Thus, in fact, we have a well-defined function on $P \times K$. The smoothness of $i\varphi$ on G follows from the immediate smoothness of the function on $P \times K$.

That the formulas for i and i^{-1} are mutual inverses is readily verified:

$$i^{-1}(i\varphi)(v) = (i\varphi)(v)(1) = \chi(1)\,\varphi(1\cdot v) = \varphi(v) \qquad (\text{writing } 1 = 1\cdot 1 \in P\cdot K)$$

and

$$\begin{array}{rcll} i(i^{-1}\Phi)(v)(pk) &=& \chi(p)\,(i^{-1}\Phi)(k\cdot v) & (\text{definition of } i) \\ &=& \chi(p)\,\Phi(k\cdot v)(1) & (\text{definition } i^{-1}) \\ &=& \Phi(k\cdot v)(p\cdot 1) & (\text{since } \Phi(k\cdot v) \text{ is in } I_\chi) \\ &=& \Phi(v)(p\cdot k) & (\text{since } \Phi \text{ is a right } K\text{-map}) \end{array}$$

Finally, there is the issue of *naturality* of the isomorphism in both V and χ, which amounts to commutativity of relevant diagrams corresponding to $(\mathfrak{p}, K_M)$-maps $f : \chi \to \chi'$ and $(\mathfrak{g}, K)$-maps $F : V \to V'$ (suppressing restrictions), namely

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathfrak{p},K_M}(V',\chi) & \xrightarrow{-\circ F} & \mathrm{Hom}_{\mathfrak{p},K_M}(V,\chi) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathfrak{p},K_M}(V,\chi') \\ \downarrow{\scriptstyle i_{V',\chi}} & & \downarrow{\scriptstyle i_{V,\chi}} & & \downarrow{\scriptstyle i_{V,\chi'}} \\ \mathrm{Hom}_{\mathfrak{g},K}(V',I_\chi) & \xrightarrow{-\circ F} & \mathrm{Hom}_{\mathfrak{g},K}(V,I_\chi) & \xrightarrow{f\circ -} & \mathrm{Hom}_{\mathfrak{g},K}(V,I_{\chi'}) \end{array}$$

where we still write f for the induced map $I_\chi \to I_{\chi'}$, and $F : V \to V'$ for the map of $(\mathfrak{p}, K_M)$-modules (after forgetting part of the $(\mathfrak{g}, K)$ structure). It's straightforward to use the formulas for the maps i to see that every such diagram commutes. This completes the verification of existence of the object I_χ and maps as required by the adjunction. ///

[5.0.3] Remark: Note that the proof of Frobenius reciprocity in this form needed $(\mathfrak{p}, K_M)$-module structure, not merely $\mathfrak{p}$-module structure, despite the tininess of the group $K_M = \{\pm 1\}$. We will give a necessary variant below, in effect constructing an adjoint functor for slightly different categories.

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6. *Co-isotypes, Jacquet modules*

Since the one-dimensional representations χ of P are trivial on N, the general conclusion of Frobenius reciprocity can be sharpened, as follows.

On one-dimensional representations χ of P that are trivial on N, the Lie algebra

$$\mathfrak{n} = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} = \mathrm{Lie}(N)$$

acts by 0. For an $\mathfrak{n}$-module V, let $V_\mathfrak{n}$ be the **co-isotype** of the trivial representation of $\mathfrak{n}$, namely, $V_\mathfrak{n}$ is a quotient $q : V \to V_\mathfrak{n}$ through which every $\mathfrak{n}$-map to a trivial $\mathfrak{n}$ module factors. [19] The quotient $V \to V_\mathfrak{n}$ is also called the **Jacquet module** of V, by analogy with the p-adic case, and $V \rightsquigarrow V_\mathfrak{n}$ is the **Jacquet functor**.

[6.0.1] Claim: The co-isotype $V \to V_\mathfrak{n}$ exists.

[6.0.2] Remark: From the defining property it is unique up to unique isomorphism if it exists, as usual. The existence proof proceeds by constructing $V_\mathfrak{n}$ in an unsurprising fashion as the quotient

$$V_\mathfrak{n} = V/\mathfrak{n}V$$

Proof: Certainly for a trivial $\mathfrak{n}$-module W (meaning that $n \cdot w = 0$ for all $n \in \mathfrak{n}$ and $w \in W$) and an $\mathfrak{n}$-map $f : V \to W$,

$$f(n \cdot v) = n \cdot f(v) = 0 \qquad \text{(for all } v \in V \text{ and } n \in \mathfrak{n})$$

so all $\mathbb{C}$-linear combinations of elements $n \cdot v$ are in every such kernel. On the other hand, on the quotient $V/\mathfrak{n}V$ the Lie algebra $\mathfrak{n}$ acts trivially:

$$n \cdot (v + \mathfrak{n}V) = n \cdot v + n \cdot \mathfrak{n}V \subset \mathfrak{n}V \qquad \text{(for all } n \in \mathfrak{n} \text{ and } v \in V)$$

---

[19] This trivial co-isotype is also provably the largest *quotient* on which $\mathfrak{n}$ acts trivially. The *isotype* $V^\mathfrak{n}$ is perhaps the more intuitive object, consisting of $\mathfrak{n}$-fixed, that is, $\mathfrak{n}$-*annihilated* vectors in V. More properly, it is a subobject $i : V^\mathfrak{n} \to V$ through which all mapst *from* trivial $\mathfrak{n}$-modules factor.

Thus, the quotient $q_V : V \to V/\mathfrak{n}V$ is a co-isotype for the trivial $\mathfrak{n}$-representation. One should also check the *naturality*, namely, that given an $\mathfrak{n}$-homomorphism $f : V \to V'$ we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle q_V}\downarrow & & \downarrow{\scriptstyle q_{V'}} \\ V_{\mathfrak{n}} & \xrightarrow{f_{\mathfrak{n}}} & V'_{\mathfrak{n}} \end{array}$$

with the obvious definition $f_{\mathfrak{n}}(v + \mathfrak{n}V) = f(v) + \mathfrak{n}V'$. This is immediate. ///

Let

$$\mathfrak{m} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} = \mathrm{Lie}(M)$$

Since $\mathfrak{m}$ normalizes $\mathfrak{n}$, for a $\mathfrak{p}$-module V the Jacquet module $V_{\mathfrak{n}}$ will have a natural $\mathfrak{m}$-module structure

$$m \cdot (v + \mathfrak{n}V) = \mathfrak{m} \cdot v + m \qquad\qquad (\text{for } m \in \mathfrak{m},\ v \in V)$$

Similarly, the Jacquet module $V_{\mathfrak{n}}$ attached to a $(\mathfrak{p}, K_M)$-module V will have a $(\mathfrak{m}, K_M)$-module structure.

7. *Convenient modifications*

A pervasive feature of our discussion is that we have avoided worrying about associating to the Lie algebra representation of $\mathfrak{sl}_2$ a *group* representation, whether of $SL_2(\mathbb{R})$ or of a covering. This avoidance is justified both by the ease of setting up the Lie algebra representation, and the decisiveness of the conclusions we can draw already at the level of Lie algebra representations.

Our intent is to invoke the subrepresentation theorem to approximately determine all irreducible quotients of $\mathscr{S}(\mathbb{R}^n)$, in the sense that we take the subrepresentation theorem as justification to simply look for principal series to which $\mathscr{S}(\mathbb{R}^n)$ has at least one non-trivial map. Then Frobenius reciprocity and application of the Jacquet functor would reduce the determination of maps from $\mathscr{S}(\mathbb{R}^n)$ to I_χ to the determination of $\mathfrak{n}$-fixed vectors in $(\mathfrak{m}, K_M)$-*quotients* of $\mathscr{S}(\mathbb{R}^n)$. In fact, we will dualize to look at *subobjects* of the space of tempered distributions.

However, there is a hitch, namely, that the subrepresentation theorem refers to $(\mathfrak{g}, K)$-modules, rather than $\mathfrak{g}$-modules. When looking at the oscillator representation, this is a non-trivial issue, since we have not described any Lie *group* representation corresponding to the Lie *algebra* representation of $\mathfrak{sl}_2$. [20]

Let $\mathfrak{p}$ be a subalgebra of a Lie algebra $\mathfrak{g}$, and let $\chi \rightsquigarrow I_\chi^{\mathrm{alg}}$ be a right adjoint functor to the forgetful functor $V \rightsquigarrow \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} V$ from $\mathfrak{g}$-modules to $\mathfrak{p}$-modules. [21] That is, we want natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{p}}(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} V,\ \chi) \approx \mathrm{Hom}_{\mathfrak{g}}(V,\ I_\chi^{\mathrm{alg}})$$

for $\mathfrak{g}$-modules V and $\mathfrak{m}$-modules χ extended trivially to $\mathfrak{n}$, thus giving χ on $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$.

[20] Indeed, for n *odd* the associated group representation is of a two-fold *covering* group of $SL_2(\mathbb{R})$, called the *metaplectic* group in [Weil 1964]. For n *even*, there is a corresponding representation of $SL_2(\mathbb{R})$, but its definition is not a trivial matter. Indeed, the relative ease of definition of the Lie algebra representation is in sharp contrast to the (interesting and important) complications involved in defining the group representation.

[21] We might *hope* that the earlier functor $\chi \to I_\chi$ might *also* be a right adjoint $\chi \rightsquigarrow I_\chi^{\mathrm{alg}}$, thus reducing worry about defining a K-structure on the oscillator representation. However, these functors have different targets and sources, so there are complications.

[7.0.1] Claim: The right adjoint $\chi \rightsquigarrow I^{\mathrm{alg}}_\chi$ *exists.*

[7.0.2] Remark: *Existence* of the right adjoint functor is the point, but, as usual, existence is proven by a construction. The proof will show that

$$I^{\mathrm{alg}}_\chi = \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}),\, \chi)$$

where we give this space of maps a left $U(\mathfrak{g})$-module structure by

$$(x \cdot \varphi)(y) = \varphi(x^\iota \cdot y) \qquad\qquad (\text{for } x, y \in U(\mathfrak{g}) \text{ and } \varphi \in \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}),\, \chi))$$

where ι is the (standard) involutive anti-automorphism on $U(\mathfrak{g})$ defined as follows. First, for x in the copy of $\mathfrak{g}$ inside $U(\mathfrak{g})$, define $x^\iota = -x$. Then inductively (on degree in $U(\mathfrak{g})$) define $(x \cdot y)^\iota = y^\iota \cdot x^\iota$. [22]

Proof: Having let slip the *object(s)* claimed to give the adjoint, we should also explicate the asserted isomorphism

$$i \;:\; \mathrm{Hom}_{\mathfrak{p}}(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} V,\, \chi) \approx \mathrm{Hom}_{\mathfrak{g}}(V,\, \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), \chi))$$

Unsurprisingly, it is given by

$$(i\varphi)(v)(x) = \varphi(x^\iota \cdot v) \qquad\qquad (\text{for } \varphi \in \mathrm{Hom}_{\mathfrak{p}}(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} V,\, \chi),\ v \in V, \text{ and } x \in U(\mathfrak{g}))$$

and

$$(i^{-1}\Phi)(v) = \Phi(v)(1_{U(\mathfrak{g})}) \qquad\qquad (\text{for } \Phi \in \mathrm{Hom}_{\mathfrak{g}}(V,\, I^{\mathrm{alg}}_\chi), \text{ and } v \in V)$$

Verification that $i\varphi$ is a $U(\mathfrak{g})$-linear map, and that these are mutual inverses is straightforward. ///

For a $(\mathfrak{g}, K)$-module V and $\mathfrak{m}$-representation χ extended to $\mathfrak{p}$ as usual, we have

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathfrak{g},K}(V, I_\chi) & \xrightarrow{\;\approx\;} & \mathrm{Hom}_{\mathfrak{p},K_M}(V, \chi) \\
{\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\
\mathrm{Hom}_{\mathfrak{g}}(V, I^{\mathrm{alg}}_\chi) & \xrightarrow{\;\approx\;} & \mathrm{Hom}_{\mathfrak{p}}(V, \chi)
\end{array}$$

with an induced injection on the left.

[7.0.3] Remark: Thus, in the happy case that $\mathrm{Hom}_{\mathfrak{g}}(V, I^{\mathrm{alg}}_\chi)$ is *small*, we can get a good estimate on $\mathrm{Hom}_{\mathfrak{g},K}(V, I_\chi)$ by computing via a slightly different adjointness relation. We take this as justification our suppression of concern about K-structures.

[7.0.4] Remark: Note that in the earlier $(\mathfrak{g}, K)$-module formulation the character χ was an $(\mathfrak{m}, K_M)$-module, not merely an $\mathfrak{m}$-module. In the case of $\mathfrak{sl}_2$ this only loses $K_M = \{\pm 1\}$, but this tiny group does have some impact: we can no longer keep track of the *parity* of χ. That is, we are inadvertently sticking together the two representations with the same complex parameter s but two different choices of *sign*. In fact, the $O(n)$ *does* contain ± 1, but we won't worry about capitalizing on that.

[22] This situation is typical of adjoints to forgetful functors between categories of modules. Especially for *commutative* rings $R \subset S$ with unit $1_R = 1_S$, we have $\mathrm{Hom}_R(V, W) \approx \mathrm{Hom}_S(V, \mathrm{Hom}_R(S, W))$ and $\mathrm{Hom}_R(W, V) \approx \mathrm{Hom}_S(S \otimes_R W, V)$ for R-module W and S-module V.

8. *Maps from* $\mathscr{S}(\mathbb{R}^n)$ *to principal series*

Thinking of the subrepresentation theorem, we approximately determine the irreducible quotients of $\mathscr{S}(\mathbb{R}^n)$, in the sense that we look for maps from $\mathscr{S}(\mathbb{R}^n)$ to principal series I_χ. Then Frobenius reciprocity and application of the Jacquet functor reduce the determination of maps from $\mathscr{S}(\mathbb{R}^n)$ to I_χ to the determination of $\mathfrak{n}$-fixed vectors in *quotients* of $\mathscr{S}(\mathbb{R}^n)$. We dualize to look at $\mathfrak{n}$-fixed *subspaces* of the space of *tempered distributions.*

[8.0.1] Remark: In the last section we saw that the modification of forgetting an alleged compact-group structure and only remembering the $\mathfrak{g}$-structure does not disturb the computation suggested by the subrepresentation theorem. As above, I_χ^{alg} is *not* the usual unramified principal series, but, rather, is the right adjoint to the forgetful functor from $\mathfrak{g}$-modules to $\mathfrak{p}$-modules. Again, we have

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathfrak{g},K}(V,I_\chi) & \xrightarrow{\approx} & \mathrm{Hom}_{\mathfrak{p},K_M}(V,\chi) & & \\ \Big\downarrow{\scriptstyle \mathrm{inc}} & & \Big\downarrow{\scriptstyle \mathrm{inc}} & & \\ \mathrm{Hom}_{\mathfrak{g}}(V,I_\chi^{\mathrm{alg}}) & \xrightarrow{\approx} & \mathrm{Hom}_{\mathfrak{p}}(V,\chi) & \xrightarrow{\approx} & \mathrm{Hom}_{\mathfrak{m}}(V_{\mathfrak{n}},\chi) \end{array}$$

This motivates computing $\mathscr{S}(\mathbb{R}^n)_{\mathfrak{n}}$.

We revert to our notation x, y, h for the special triple of elements of $\mathfrak{sl}_2$.

We convert the question to one about *subobjects* rather than *quotients*. That is, given a short exact sequence

$$0 \to A \to B \to C \to 0$$

of locally convex topological vector spaces with A closed and C the quotient, we have a natural short exact sequence [23]

$$0 \longrightarrow C^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$$

Thus, with C a quotient of $B = \mathscr{S}(\mathbb{R}^n)$ by a closed subspace A,

$$0 \longrightarrow C^* \longrightarrow \mathscr{S}(\mathbb{R}^n)^* \longrightarrow A^* \longrightarrow 0$$

As usual, for $\gamma \in \mathfrak{sl}_2$ there is a natural *adjoint* action on distributions by

$$(\gamma \cdot u)(\varphi) = u(-\gamma \cdot \varphi)$$

for test functions φ.

Thus, we will look for *submodules* Q^* of the space $\mathscr{S}(\mathbb{R}^n)^*$ of tempered distributions with vectors annihilated by $\mathbb{C} \cdot x = \mathfrak{n}$. It is easy to find distributions annihilated by x, that is, by multiplication by $r^2/2$. To begin with,

[8.0.2] Claim: The tempered distributions $\mathscr{S}^*(\mathbb{R}^n)^{\mathfrak{n}}$ annihilated by $\mathfrak{n} = \mathbb{C} \cdot x$, that is, by multiplication by r^2, are supported at 0, so are linear combinations of derivatives of Dirac's δ.

Proof: For a test function φ the definition of $r^2 \cdot u$ is

$$(r^2 \cdot u)(\varphi) = u(r^2 \cdot \varphi)$$

[23] The surjectivity to A^*, follows from Hahn-Banach.

When φ has (closed!) support not containing 0, φ/r^2 is still a test function, and

$$0 = (r^2 \cdot u)(\varphi/r^2) = u(\varphi)$$

That is, the support of u is $\{0\}$. That any such u is a linear combination of derivatives of δ is essentially the theory of Maclaurin-Taylor expansions. ///

Normalize the Fourier transform as

$$Ff(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x}\, f(x)\, dx$$

Since (up to constants) F interchanges $x \sim \frac{r^2}{2}$ and $y \sim \frac{-\Delta}{2}$, its relation to h is straightforward:

$$FhF^{-1} = F[x,y]F^{-1} = [FxF^{-1}, FyF^{-1}] = [(2\pi i)^{-2}\, y, (-2\pi i)^2\, x] = [y,x] = -h$$

In particular, weight vectors for h are mapped to weight vectors by Fourier transform (with weights multiplied by -1). All the distributions supported at 0 are *tempered*, their Fourier transforms are polynomials, and vice-versa.

[8.0.3] Claim: The x-annihilated distributions $\mathscr{S}^*(\mathbb{R}^n)^{\mathfrak{n}}$ are of the form $u = FP$ with a *harmonic* polynomial P.

Proof: Applying the Fourier transform F, the condition $r^2 u = 0$ becomes

$$0 = F(0) = F(r^2 \cdot u) = \frac{-\Delta}{4\pi^2}(Fu)$$

so $\Delta(Fu) = 0$. We know that Fu is a polynomial, and this condition is that the polynomial Fu is harmonic. Then Fourier inversion expresses u as a Fourier transform of a harmonic polynomial. ///

In $\mathscr{S}(\mathbb{R}^n)^{\mathfrak{n}}$ *homogeneous* distributions are weight vectors for h, as before. Note that each of these spaces of distributions is stable under the action of $O(n)$.

[8.0.4] Claim: The space $\mathscr{S}(\mathbb{R}^n)^{\mathfrak{n}}$ of x-annihilated distributions decomposes as a direct sum of weight spaces for h, where

$$-(\tfrac{n}{2}+d)\text{-weightspace in } \mathscr{S}(\mathbb{R}^n)^{\mathfrak{n}} = \{u : u = FP, \text{ for homogeneous harmonic polynomial } P \text{ of degree } d\}$$

Proof: Euler's formula asserts exactly that $hP = (\frac{n}{2}+d)\cdot P$. Taking Fourier transform flips the sign. ///

Let

$$(F\mathfrak{H}^{(d)})^\perp = \{f \in \mathscr{S}(\mathbb{R}^n) : u(f) = 0, \text{ for all } u = FP, \text{ with } P \in \mathfrak{H}^{(d)}\}$$

Dualizing back to the Schwartz space,

[8.0.5] Corollary: The quotient

$$\mathscr{S}(\mathbb{R}^n)\,/\,(F\mathfrak{H}^{(d)})^\perp$$

is the $(\frac{n}{2}+d)$-weightspace inside the $\mathfrak{n}$-cofixed vectors $\mathscr{S}(\mathbb{R}^n)_{\mathfrak{n}}$ of $\mathscr{S}(\mathbb{R}^n)$. ///

[8.0.6] Corollary: The space of maps $\mathrm{Hom}_{\mathfrak{g}}(\mathscr{S}(\mathbb{R}^n), I_\chi^{\mathrm{alg}})$ is at most one-dimensional, and is one-dimensional exactly when $\chi(h) = \frac{n}{2}+d$. ///

[8.0.7] Remark: In fact, especially in light of the earlier computations about *subrepresentations* of $\mathscr{S}(\mathbb{R}^n)$, this determination of $\mathscr{S}(\mathbb{R}^n)_{\mathfrak{n}}$ as $\mathfrak{m}$-module suggests that the actual irreducible appearing is a *holomorphic discrete series*, at least for $n \in 2\mathbb{Z}$.

9. *Appendix:* $\mathfrak{sl}_2$ *representations*

We recall some basic abstract features of representations of $\mathfrak{sl}_2$, especially those with a *lowest weight* λ, particularly for $\lambda > \frac{1}{2}$. As usual, we take elements x, y, h in $\mathfrak{sl}_2$ such that

$$[x, y] = h \qquad [h, x] = 2x \qquad [h, y] = -2y$$

Since all the considerations here are abstract, it doesn't matter *which* triple x, y, h from $\mathfrak{sl}_2$ we use, as long as these relations hold.

The symmetric complex-bilinear pairing

$$\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha\, \beta)$$

on $\mathfrak{sl}_2$ is *non-degenerate*, meaning that if $\langle \alpha, \beta \rangle = 0$ for all β then $\alpha = 0$. [24]

Given a basis $\{e_i\}$ for $\mathfrak{sl}_2$, as usual a *dual basis* $\{e_i^*\}$ is another basis for $\mathfrak{sl}_2$ such that

$$\langle e_i, e_j^* \rangle = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

The non-degeneracy of $\langle , \rangle$ uniquely determines the basis dual to a given basis. Up to a normalizing constant, the *Casimir operator* [25] for $\mathfrak{sl}_2$ is

$$z = \sum_j e_j\, e_j^* \in U(\mathfrak{sl}_2)$$

This expression is provably independent of the choice of basis $\{e_i\}$, and the Casimir operator is in the *center* of the universal enveloping algebra $U(\mathfrak{g})$. [26]

One application of the centrality of the Casimir operator is that, given a z-eigenvector v with eigenvalue λ the Casimir operator z will act as the scalar λ on the whole $\mathfrak{sl}_2$-representation generated by v.

By *weight vector* we will mean *eigenvector* for h. A *lowest* weight vector is a weight vector annihilated by y.

[9.0.1] Theorem: A representation of $\mathfrak{sl}_2$ generated by a lowest-weight vector v with weight $\lambda > 0$ is *irreducible*, with isomorphism class determined completely by λ. The eigenvalue of the Casimir operator (as

[24] In the case at hand, this non-degeneracy can be proven as follows. Note that $\mathfrak{sl}_2$ is stable under *conjugate-transpose* $\beta \to \beta^*$. With $\alpha = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, we have $\langle \alpha, \alpha^* \rangle = 2|a|^2 + |b|^2 + |c|^2$, which is positive definite. This proves non-degeneracy.

[25] This construction of a *Casimir operator* applies to any *semi-simple* Lie algebra $\mathfrak{g}$, creating the simplest non-trivial element in the *center* of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Without worrying about the general definition of *semi-simplicity*, in effect we grab a critical property of it (*Cartan's criterion for semi-simpliciy*), that there is a suitably $\mathfrak{g}$-invariant non-degenerate bilinear form $\langle , \rangle$ on $\mathfrak{g}$.

[26] The centrality and independence of expression can be proven as follows. Recall the natural isomorphism $\mathrm{End}_{\mathbb{C}}(\mathfrak{sl}_2) \approx \mathfrak{sl}_2 \otimes \mathfrak{sl}_2^*$. The pairing $\langle , \rangle$ has the invariance property $\langle gvg^{-1}, gwg^{-1} \rangle = \langle v, w \rangle$ for $g \in SL_2(\mathbb{C})$, so the identification $\mathfrak{sl}_2^* \approx \mathfrak{sl}_2$ via $\langle , \rangle$ is $SL_2(\mathbb{R})$-equivariant. Then we have $SL_2(\mathbb{C})$-equivariant

$$\mathrm{End}(\mathfrak{sl}_2) \approx \mathfrak{sl}_2 \otimes \mathfrak{sl}_2^* \approx \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \subset \bigotimes{}^\bullet \mathfrak{sl}_2 \to U(\mathfrak{sl}_2)$$

The *identity* map id in $\mathrm{End}_{\mathbb{C}}(\mathfrak{sl}_2)$ certainly commutes with the action of $SL_2(\mathbb{C})$. The equivariance implies that the image of id in $U(\mathfrak{sl}_2)$ commutes with $SL_2(\mathbb{C})$, so surely with $\mathfrak{sl}_2$. In coordinates, this definition yields the expression in the text.

normalized here) on this representation is $\frac{1}{2}\lambda^2 - \lambda$. The representation is the *direct* sum

$$\bigoplus_{j=0}^{\infty} \mathbb{C} \cdot x^j \cdot v$$

[9.0.2] Theorem: A representation V of $\mathfrak{sl}_2$ generated by a single weight-vector v (with weight λ) annihilated by some *power* y^{m+1} of y, with $\lambda > 2m$, is a finite *direct* sum of at most $m+1$ irreducible submodules each generated by a lowest weight vector. The lowest weights of these submodules are among $\lambda, \lambda-2, \lambda-4, \ldots, \lambda-2m$, each occurring at most once.

Proof: *(of first theorem)* This argument is an archetype. For basis x, y, h the dual basis is $x^* = y$, $y^* = x$, and $h^* = h/2$. The lowest-weight property makes the computation of the scalar have an especially explicit conclusion: for v with h-weight λ,

$$zv = (\frac{1}{2}h^2 + xy + yx) \cdot v = \frac{\lambda^2}{2}v + 0 + yx \cdot v = \frac{\lambda^2}{2}v + 0 + (yx - xy + xy) \cdot v$$

$$= \frac{\lambda^2}{2}v + 0 - hv + xyv = \frac{1}{2}\lambda(\lambda-2)v = \frac{1}{2}((\lambda-1)^2 - 1)v$$

Since v is annihilated by y and acted upon by h by a scalar, by the easy half of Poincaré-Birkhoff-Witt M is *spanned* by elements $x^\ell v$, with respective weights $\lambda + 2\ell$.

If M had a proper submodule N, then N could not contain v (which generates the whole). Let $w = \sum_{\ell \geq j} c_\ell \, x^\ell v$ (with finite sum) be an element in N. We can isolate the bottom term $c_j \, x^j v$ by use of h, as follows. For large t,

$$(h - (\lambda + 2j + 2))(h - (\lambda + 2j + 4)) \ldots (h - (\lambda + 2j + 2t)) \cdot w = (2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2t) \cdot c_j \cdot x^j v$$

Thus, a proper submodule N would contain a lowest-weight (y-annihilated) vector $x^j v$ with $0 < j \in \mathbb{Z}$. Then, $x^j v$ would have weight μ among $\lambda+2$, $\lambda+4$, and so on. Since z commutes with x^j, $x^j v$ is an z-eigenvector with with the same eigenvalue as v. That is,

$$\frac{1}{2}((\lambda-1)^2 - 1) = \frac{1}{2}((\mu-1)^2 - 1)$$

Since $\mu \geq \lambda+2$, for $\lambda > 0$ this is impossible, by elementary inequalities. Thus, M has no proper submodules.

Let I be the left ideal generated in the enveloping algebra by $h - \lambda$ and y. Then U/I is an $\mathfrak{sl}_2$-module with lowest weight vector $1 + I$ with weight λ. This module U/I is *universal* in the sense that it surjects to any $\mathfrak{sl}_2$-module with lowest weight λ. By what we just saw, U/I itself is *already* irreducible. Thus, $\lambda > 0$ completely determines the isomorphism class.

Last, since the *universal* representation U/I with lowest weight λ is already irreducible, invoke Poincaré-Birkhoff-Witt to see that it is the *direct* sum of subspaces $\mathbb{C} \cdot x^j \cdot v$

$$U(\mathfrak{sl}_2)/I = \bigoplus_{a \geq 0, b \geq 0, c \geq 0} \mathbb{C} \cdot x^a y^b h^c / I = \bigoplus_{a \geq 0} \mathbb{C} \cdot x^a \cdot v$$

as claimed. ///

Proof: *(of second theorem)* Let $m+1$ be the *least* non-negative integer such that $y^{m+1}v = 0$. We do induction on m. For $m = 0$, v is a lowest-weight vector, and we invoked the first theorem. By Poincaré-Birkhoff-Witt, the whole representation V is *spanned* by $x^a y^b \cdot v$ with $b \leq m$ and $a \geq 0$. We claim that there is a constant c such that

$$y^m \cdot (v - x^m y^m \, c) = 0$$

Once we know this, by induction $y - x^m y^m\, v$ generates a direct sum of at most m lowest-weight representations with weights among λ, $\lambda-2$, $\lambda-4$, ..., $\lambda-2(m-2)$, each occuring at most once. By the first theorem, these are all irreducible. And certainly $y^m v$ is a lowest-weight vector with weight $\lambda - 2m$, and it generates an irreducible. Thus,

$$U(\mathfrak{sl}_2)\cdot v = M_{\lambda-2m} + \bigoplus_\mu M_\mu$$

where M_ν is the (irreducible) representation with lowest weight ν, and the direct sum runs over irreducibles M_μ with lowest weights μ from among the set $\{\lambda, \lambda-2, \lambda-4, \ldots, \lambda-2(m-1)\}$. The issue is to be sure that the sum is entirely direct. The intersection of $M_{\lambda-2m}$ with the other (direct) sum is a submodule of $M_{\lambda-2m}$, which is either 0 or $M_{\lambda-2m}$, by the irreducibility of $M_{\lambda-2m}$. If the latter, there would exist a non-trivial map of $M_{\lambda-2m}$ to the direct sum, hence, to at least one of the summands. But all these irreducibles are mutually non-isomorphic, so this cannot happen. Thus, the sum is direct.

Last, we turn to proving existence of the constant c such that

$$y^m \cdot (v - x^m y^m\, c) = 0$$

Again by Poincaré-Birkhoff-Witt, the subspace $\mathbb{C}\cdot y^m v$ is the *only* subspace of the representation with h-weight $\lambda - 2m$. Thus, *if* $y^m x^m y^m \cdot v \neq 0$, this vector would be a non-zero scalar multiple of $y^m v$, and we'd be done. The submodule generated by $y^m v$ is a lowest-weight representation, is irreducible, and is $\bigoplus_{j\ge 0}\mathbb{C}\cdot x^j v$, by the first theorem here. Thus, $x^m \cdot y^m v \neq 0$. Then $y^m \cdot x^m y^m v = 0$ would imply that $y^j \cdot x^m y^m v = 0$ for some $j < m$, and we'd have a proper submodule of $U(\mathfrak{sl}_2)\cdot y^m v$, contradicting the irreducibility. Thus, there is such a constant c, and the induction argument succeeds. ///

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