Characterization of differential operators

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Differential operators obviously do not increase support when applied to test functions. The converse is certainly not clear. [Peetre 1959,60] proved this, incorporating corrections from L. Carleson. We follow [Helgason 1984] pp 236-238, who adapts the argument from [Narasimhan 1968].

\[ 0.0.1 \] Theorem: Let \( V \) be a smooth manifold. A not-necessarily-continuous linear map \( D : C_c^\infty(V) \to C_c^\infty(V) \) that does not increase supports is a differential operator with smooth coefficients.

Proof: First, claim that the non-increase of support property implies that, for a test function \( f \) and a point \( x \), for any test function \( \varphi \) identically 1 on a neighborhood of \( x \), suitable truncation does not affect \( D \), in the sense that

\[
(Df)(x) = (D(\varphi f))(x)
\]

Indeed, \( f = \varphi f + (1 - \varphi)f \), and \( D \) is linear, so

\[
Df = D(\varphi f) + D((1 - \varphi)f)
\]

The non-increase of support implies that \( D((1 - \varphi)f)(x) = 0 \), yielding the claim.

This truncation property immediately allows us to consider the corresponding local problem, of operators on open subsets of Euclidean spaces, without loss of generality.

Next, the non-increase of support allows an extension of \( D \) to all smooth functions on \( V \) by using cut-off functions: given smooth \( f \) and a point \( x \), let \( \varphi \) be a test function identically 1 on a neighborhood of \( x \), and define \( Df(x) = D(\varphi f)(x) \). The latter is well-defined by the previous claim.

Let \( \|f\|_{U,m} \) be the sup on \( U \) of sups of the derivatives of \( f \) of orders \( \leq m \).

Next, claim that for \( f \) smooth on \( U \) with derivatives of order \( \leq m \) vanishing at 0, for every \( \varepsilon > 0 \) there is a smooth function \( g \) vanishing identically in a neighborhood of 0, coinciding exactly with \( f \) outside a larger neighborhood of 0, such that \( |f - g|_{U,m} < \varepsilon \). Let \( \varphi \) be a smooth function identically 0 on \( |x| \leq \frac{1}{2} \), identically 1 for \( |x| \geq 1 \), and \( 0 \leq \varphi \leq 1 \) everywhere. Then consider the family of modifications of \( f \) given by

\[
g_\delta(x) = \varphi(x/\delta) \cdot f(x) \quad \text{(for } \delta > 0 \text{ small)}
\]

Each \( g_\delta \) agrees with \( f \) outside the \( \delta \)-ball \( B_\delta \) at 0. It would suffice to prove

\[
\lim_{\delta \to 0} \|f - g_\delta\|_{B_{\delta},m} = 0
\]

Since \( f \) vanishes to order \( m \) at 0,

\[
\lim_{\delta \to 0} \|f\|_{B_{\delta},m} = 0
\]

so we must prove that

\[
\lim_{\delta \to 0} \|g_\delta\|_{B_{\delta},m} = 0
\]

For multi-index \( \alpha \), apply Leibniz’ rule to the \( \alpha \)th derivative of \( g_\delta \):

\[
g_\delta^{(\alpha)}(x) = \sum_{\beta + \gamma = \alpha} \binom{\alpha}{\beta} \delta^{-|\alpha|} \varphi^{(\beta)}(x/\delta) f^{(\gamma)}(x)
\]

Thus,

\[
|g_\delta^{(\alpha)}(x)| \ll \sum_{\beta + \gamma = \alpha} \delta^{-|\beta|} \|f^{(\gamma)}(x)\| \quad \text{(with } x \in B_\delta)\]

1
with implied constant independent of \( f \) and \( \delta \). The derivative \( f^{(\gamma)} \) vanishes to order \( m - |\gamma| \) at 0, so, from the Taylor expansion of \( f \) at 0,

\[
\sup_{B_\delta} |f^{(\gamma)}| = o(\delta^{m-|\gamma|})
\]

Thus,

\[
\sup_{B_\delta} |g_\delta^{(\alpha)}(x)| = o\left( \sum_{\beta + \gamma = \alpha} \delta^{m-|\beta|-|\gamma|} \right) = o(\delta^{m-|\alpha|})
\]

Thus, as claimed, \( |f - g_\delta|_{B_\delta,m} \to 0 \).

Next, claim a somewhat weaker continuity assertion than the theorem, namely, that for every point \( x_0 \) there is a sufficiently small neighborhood \( U \) of \( x_0 \), integer \( m \), such that

\[
|Df|_{U,0} \ll |f|_{U,m} \quad (\text{for } f \in C^\infty_c(U - \{x_0\}))
\]

with the implied constant independent of \( f \). This follows by a diagonal argument: if this failed at some \( x_0 \), then for given compact-closure neighborhood \( U_0 \) of \( x_0 \) there is \( f_1 \in C^\infty_c(U_0 - \{x_0\}) \) such that

\[
|Df_1|_0 \geq 2^2 \cdot |f_1|
\]

Let \( U_1 \) be the zero-set of \( f_1 \), so \( U_0 - \overline{U}_1 \) is a neighborhood of \( x_0 \), and there is \( f_2 \in C^\infty_c(U_0 - \overline{U}_1 - \{x_0\}) \) such that

\[
|Df_2|_0 \geq 2^4 \cdot |f_2|
\]

By induction, obtain open sets \( U_i \) with \( U_i \cap U_j = \emptyset \) for \( i, j \geq 1 \), and test functions

\[
f_i \in C^\infty_c(U_0 - \overline{U}_1 - \ldots - \overline{U}_{i-1} - \{x_0\})
\]

with

\[
|Df_i|_0 \geq 2^{2i} \cdot |f_i|
\]

Then the sum

\[
\sum_i \frac{f_i}{2^i \cdot |f_i|}
\]

converges and gives a test function, equal to the \( i^{th} \) summand \( f_i/(2^i \cdot |f_i|) \) on \( U_i \). The linearity and non-increase of support of \( D \) imply that

\[
Df \bigg|_{U_i} = \frac{1}{2^i \cdot |f_i|} \cdot Df_i \bigg|_{U_i}
\]

Thus, there exists \( x_i \in U_i \) such that \( Df(x_i) > 2^i \). But \( f \) is continuous and compactly supported, so this is impossible, proving the claim.

Next, thinking in terms of that last weak continuity, we prove a local result: for a neighborhood \( U \) of a point \( x \), under the continuity hypothesis

\[
|Df|_{U,0} \ll |f|_{U,m}
\]

on a sufficiently small neighborhood of \( x \), \( D \) is a differential operator with smooth coefficients. For the proof of this, for each \( x \in U \) and multi-index \( \alpha \), let

\[
P_{x,\alpha}(y) = (x - y)^\alpha = (x_1 - y_1)^{\alpha_1} \cdots (x_n - y_n)^{\alpha_n}
\]

For \( f \in C^\infty_c(U) \) and fixed \( x \in U \), consider a subsum of the Taylor expansion of \( f \) near \( x \),

\[
F = f - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} f^{(\alpha)}(x) \cdot P_{\alpha,x}
\]
This $F$ vanishes to order $m$ at $x$. As shown above, given $\varepsilon > 0$ there is a test function $\Phi$, vanishing identically in a neighborhood of $x$ (depending upon $\varepsilon$), agreeing identically with $F$ outside a larger neighborhood of $x$ (depending upon $\varepsilon$), and with $|F - \Phi|_m \leq \varepsilon$. The continuity assumption gives $|D(F - \Phi)|_0 \to 0$ as $\varepsilon \to 0$. The non-increase of support implies that each $D\Phi$ vanishes identically near $x$. Thus, $|DF(x)| < \varepsilon$ for every $\varepsilon > 0$, so $DF(x) = 0$. Thus, for each $x \in U$,

$$Df(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} f^{(\alpha)}(x) \cdot DP_{\alpha,x}(x)$$

To understand $b_\alpha(x) = DP_{\alpha,x}(x)$, observe that it is a sum of terms $P_\beta(x) y^\beta$ with $P_\beta$ a polynomial. By linearity of $D$,

$$D(\sum_\beta P_\beta(x) \cdot y^\beta) = \sum_\beta P_\beta(x) \cdot D(y^\beta)$$

By hypothesis $D(y^\beta)$ is a test function, so the diagonal

$$DP_{\beta,x}(x) = \sum_\beta P_\beta(x) \cdot D(x^\beta)$$

is a finite sum of polynomial multiples of test functions, and is a test function itself. Thus, the expression for $Df(x)$ exhibits it as a differential operator with smooth coefficients on $U$.

Finally, we reduce the general question of expressibility of $D$ to the local one, essentially by a partition of unity argument. At each $x \in V$, let $U_x$ be a small-enough neighborhood of $x$, $m_x$ an integer, so that we have a continuity bound

$$|Df|_{U_x,0} \ll |f|_{U_x,m_x} \quad \text{(for } f \in C_c^\infty(U_x - \{x\}) \text{)}$$

with implied constant independent of $f$. For an open $U \subset V$ with compact closure $\overline{U} \subset V$, take a finite subcover $U_{x_1}, \ldots, U_{x_n}$ of the opens $U_x$. Let $\{\varphi_j\}$ be a partition of unity subordinate to the cover $U_{x_1}, \ldots, U_{x_n}$ and $V - \overline{U}$ of $V$. For $f$ a test function on the set

$$U' = U - \{x_1\} - \ldots - \{x_n\}$$

certainly

$$f = \sum_{j=1}^{n+1} \varphi_j \cdot f = \sum_{j=1}^{n} \varphi_j \cdot f$$

and each $\varphi_j f$ satisfies a corresponding continuity bound. Expanding the derivatives of $\varphi_j f$ by Leibniz, we find that $f$ itself satisfies such a continuity bound on $U_{x_j}$, and, therefore, satisfies a uniform continuity bound throughout $U'$. Thus, on $U'$, $D$ is a differential operator with smooth coefficients

$$Df(x) = \sum_j a_j(x) \cdot \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \quad \text{(for } x \in U', f \in C_c^\infty(U'))$$

In fact, the non-increase of support property allows us to extend the validity of this to $f \in C_c^\infty(U)$, at least for $x \in U'$: take $\varphi \in C_c^\infty(U')$ identically $1$ near $x$ and identically $0$ near every $x_i$. Then $\varphi f \in C_c^\infty(U')$, and the property $D(\varphi f)(x) = Df(x)$ observed earlier gives

$$Df(x) = \sum_j a_j(x) \cdot \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \quad \text{(for } x \in U', f \in C_c^\infty(U))$$

Finally, because both sides of the last equation are continuous in $x$, this equality holds not merely for $x \in U'$, but for $x \in U$. This holds for every $\overline{U} \subset V$, so is valid on $V$.  

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