Spectral Decompositions, Eisenstein series, L-functions

Paul Garrett

garrett@math.umn.edu
http://www.math.umn.edu/~garrett/
Fourier series *decompose* periodic functions $f(x)$ as sums of simpler periodic functions

$$f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{2\pi \imath nx}$$

There’s a handy formula for the *decomposition coefficients* $c_n$:

$$c_n = \int_0^1 f(x) e^{-2\pi \imath nx} \, dx$$

This decomposition is *discrete* in the sense that the atomic pieces, the exponential functions, are themselves in the space of functions being decomposed.

For future reference, we note that it may be better to consider periodic functions $f(x + 1) = f(x)$ as living on the circle $\mathbb{R}/\mathbb{Z}$ rather than on the unit interval.
By contrast, Fourier transforms give a continuous decomposition

\[ f(x) \sim \int_{-\infty}^{+\infty} \hat{f}(t) e^{2\pi i t x} \, dt \]

of \( L^2(\mathbb{R}) \), with a good formula for \( \hat{f} \):

\[ \hat{f}(t) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i t x} \, dx \]

This deserves to be called continuous since it is an integral (rather than a sum), and since the exponential functions are not in \( L^2(\mathbb{R}) \), so are themselves are not in the space being decomposed. (This is a little strange.)

The exponential functions’ not being in \( L^2 \) creates trouble both in defining \( f \rightarrow \hat{f} \) in the first place, and in the reconstitution of \( f \) from \( \hat{f} \), since the integrals generally don’t converge.
The Fourier series decomposition of $L^2(0, 1)$ lets us write the same thing in two different ways.

As a first example, the Fourier coefficients of $f(x) = x - \frac{1}{2}$ are easily computed

$$c_n = \begin{cases} 
-1/2\pi i n & \text{ (for } n \neq 0) \\
0 & \text{ (for } n = 0) 
\end{cases}$$

The Plancherel formula

$$\int_0^1 |f(x)|^2 \, dx = \sum_{n \in \mathbb{Z}} |c_n|^2$$

applied to $f(x) = x - \frac{1}{2}$ gives

$$\frac{1}{12} = \frac{1}{(2\pi)^2} \cdot \sum_{n \neq 0} \frac{1}{n^2}$$

giving the popular formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$
Though the Fourier series

\[ x - \frac{1}{2} \sim \frac{1}{-2\pi i} \sum_{n \neq 0} \frac{e^{2\pi inx}}{n} \]

for \( f(x) = x - \frac{1}{2} \) doesn’t converge absolutely, it does converge for \( 0 < x < 1 \). For example,

\[ \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = -f\left(\frac{1}{4}\right) + f\left(\frac{1}{4}\right) \]

\[ = \sum_{n \neq 0} \frac{-1}{2\pi in} \left( -e^{2\pi in(1/4)} + e^{2\pi in(3/4)} \right) \]

\[ = 2 \cdot \sum_{1 \leq n \text{ odd}} \frac{-1}{2\pi in} \left( -i^n + i^{-n} \right) \]

which simplifies to the ever-popular

\[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots \]
A slightly less trivial application of the Fourier series

\[ f(x) = x - \frac{1}{2} \sim \frac{1}{-2\pi i} \sum_{n \neq 0} \frac{e^{2\pi i n x}}{n} \]

is the example

\[-\frac{1}{3} = f\left(\frac{1}{3}\right) - f\left(\frac{2}{3}\right) = 2 \cdot \frac{-1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n} \cdot (\omega^n - \omega^{2n})\]

(letting \(\omega = e^{2\pi i/3}\) and grouping \(\pm n\) terms). This simplifies to

\[ \frac{4\pi}{3\sqrt{3}} = \frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \ldots \]
Similarly, for \( f(x) = \frac{1}{2}x(x - 1) + \frac{1}{12} \)

\[
\frac{1}{2}x(x - 1) + \frac{1}{12} \sim \frac{1}{4\pi^2} \sum_{n \neq 0} \frac{e^{2\pi i nx}}{n^2}
\]

and as an example of an application, letting \( \omega = e^{2\pi i/5} \) and grouping \( \pm n \) terms,

\[
\frac{2}{25} = f\left(\frac{1}{5}\right) - f\left(\frac{2}{5}\right) - f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right)
\]

\[
= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\omega^n - \omega^{2n} - \omega^{3n} + \omega^{4n}}{n^2}
\]

This simplifies (less trivially than before) to

\[
\frac{4\pi^2}{25\sqrt{5}} = \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \ldots
\]
Indeed, there are polynomials $B_0(x), B_1(x), B_2(x), \ldots$ (essentially Bernouilli polynomials) with coefficients in $\mathbb{Q}$ so that

$$(2\pi i)^\ell \cdot B_\ell(x) \sim \sum_{n \neq 0} \frac{e^{2\pi i n x}}{n^\ell}$$

From this, for an $N$-periodic $\mathbb{Q}$-valued function $\chi$ on $\mathbb{Z}$ with a parity condition

$\chi(a + N) = \chi(a)$ and $\chi(-n) = (-1)^\ell \chi(n)$

we can conclude that

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^\ell} \in \pi^\ell \cdot \overline{\mathbb{Q}}$$

More precisely, as was secretly the case in the examples above, if $\chi$ is a primitive Dirichlet character modulo $N$, and $\chi(-1) = (-1)^\ell$, then

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^\ell} = \frac{2\pi^\ell}{g(\chi)} \cdot \sum_{a \mod N} \chi(a) \cdot B_\ell\left(\frac{a}{N}\right)$$

where $g(\chi)$ is a Gauss sum

$$g(\chi) = \sum_{a \mod N} \bar{\chi}(a) e^{-2\pi i a/N}$$
For both the circle and the line, there are several possible viewpoints about what we decompose with respect to.

Those decompositions do expand functions in terms of eigenfunctions of the (translation invariant) differential operator $d^2/dx^2$.

Also, the exponential functions are exactly the functions on $\mathbb{R}/\mathbb{Z}$ or on $\mathbb{R}$ which are homomorphisms to $\mathbb{C}^\times$. 

Also, Fourier series decompose the representation space $L^2(\mathbb{R}/\mathbb{Z})$ for the translation action $R_t f(x) = f(x+t)$ into irreducible one-dimensional representations $\mathbb{C} \cdot e^{2\pi i n x}$ with $n \in \mathbb{Z}$. (Similarly for Fourier transforms.)

Fourier series expand functions in eigenfunction expansions for the adjoint-closed algebra of compact operators

$$R_\eta f(x) = \int \limits_\mathbb{R} \eta(t) f(x + t) \, dt$$

for test functions $\eta \in C_c^\infty(\mathbb{R})$. (By contrast, on $L^2(\mathbb{R})$ these operators are not compact.)
It is more interesting to look at decomposition of functions on more complicated objects than the circle (for Fourier series), or the line (for Fourier transforms).

For example, consider the linear fractional (Möbius) transformation action of \( \Gamma = SL(2, \mathbb{Z}) \) on the complex upper half-plane \( \mathcal{H} \):

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  z
\end{pmatrix} = \frac{az + b}{cz + d}
\]

The space we want is the quotient

\[ X = \Gamma \backslash \mathcal{H} \approx \text{sphere } \mathbb{P}^1 \text{ with } \infty \text{ removed} \]

The metric on the \( \mathbb{P}^1 \) is inherited from the \( SL(2, \mathbb{R}) \)-invariant metric \( \frac{dx \, dy}{y^2} \) on \( \mathcal{H} \). It is not the usual sphere metric, but instead makes the missing point \textit{infinitely far away}.

For \( \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \) in \( SL(2, \mathbb{R}) \) there are formulas

\[
\frac{d}{dz} \left( \frac{az + b}{cz + d} \right) = \frac{1}{(cz + d)^2}
\]

\[
\text{Im} \left( \frac{az + b}{cz + d} \right) = \frac{\text{Im}(z)}{|cz + d|^2}
\]
If we want to look at **holomorphic** functions on \( \mathcal{H} \) with some sort of nice behavior under the action of \( \Gamma \), it turns out that actual **invariance** is too restrictive.

A holomorphic **modular form** of weight \( 2k \) is a holomorphic function \( f \) on \( \mathcal{H} \)

- Which is of **moderate growth**: for some \( N \)

\[
f(x + iy) = O(y^N) \text{ as } y \to +\infty
\]

- Which for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) satisfies

\[
f\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \right) = (cz + d)^{2k} \cdot f(z)
\]

The last condition is equivalent to the **invariance** of the form

\[
f(z) \, dz^k
\]

under the action of \( \Gamma \). That is, holomorphic modular forms are \( \Gamma \)-invariant sections of a line bundle, rather than being \( \Gamma \)-invariant **functions**.
The simplest examples of modular forms are **holomorphic Eisenstein series**

\[ E_{2k}(z) = \sum_{m,n}^{'} \frac{1}{(cz + d)^{2k}} \quad (m, n \text{ not both } 0) \]

The Eisenstein series \( g_2 = E_4 \) and \( g_3 = E_6 \) occur in the Weierstrasss equation

\[ \varphi'(u)^2 = 4\varphi(u)^3 - 60g_2\varphi(u) - 140g_3 \]

for the elliptic function

\[ \varphi(z) = \sum_{m,n}^{'} \left( \frac{1}{(u - (mz + n))^2} - \frac{1}{(mz + n)^2} \right) \]

It is **elementary** to verify that Eisenstein series are modular forms.

The other simple kind of holomorphic modular forms (though with a somewhat smaller \( \Gamma \) in place of \( SL(2, \mathbb{Z}) \)) is a **theta series**

\[ \theta_Q(z) = \sum_{v \in \mathbb{Z}^n} e^{2\pi iQ[v] \cdot z} \]

where \( Q \) is a \( \mathbb{Z} \)-valued positive-definite quadratic form on \( \mathbb{R}^n \). It is **not** entirely elementary to verify that theta series are modular forms.
Note that $\Gamma = \text{sl}(2, \mathbb{Z})$ contains the subgroup
\[ N_{\mathbb{Z}} = \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \} \]
whose elements act on $\mathcal{H}$ by integer translations
\[ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} (x + iy) = (x + n) + iy \]
Also, for such matrices the corresponding $cz + d$ is simply 1. Thus, modular forms $f(z)$ have **Fourier expansions**
\[ f(z) = \sum_{n \in \mathbb{Z}} c_n(y) e^{2\pi inx} \]
Since $f(z)$ is holomorphic, necessarily $c_n(y)$ is a constant multiple of $e^{-2\pi iny}$. Further, the moderate growth condition excludes negative-index Fourier terms, so actually the Fourier expansion looks like
\[ f(z) = \sum_{n \geq 0} c_n e^{2\pi inz} \]
That is, we can **decompose** modular forms by the action of translations. Such $f(z)$ is called a **cuspform** if $c_0 = 0$, in which case $f(z)$ is actually of **rapid decay** as $y \to +\infty$. 
The Fourier expansions of holomorphic Eisenstein series can be directly computed, using residue calculus from complex analysis and using a little Fourier analysis.

\[ E_{2k}(z) = 2 \cdot \zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k - 1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i n z} \]

where as usual

\[ \sigma_{2k-1}(n) = \sum_{1 \leq d \mid n} d^{2k-1} \quad \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \]

The higher Fourier coefficients are elementary, while by contrast the constant term \( 2 \cdot \zeta(2k) \) is not.
An elementary complex analysis argument shows (much like Liouville’s theorem) that the only weight-zero modular forms are constants, and for low weights

weight 0 modular forms $= \mathbb{C}$
weight 2 modular forms $= \{0\}$
weight 4 modular forms $= \mathbb{C} \cdot E_4$
weight 6 modular forms $= \mathbb{C} \cdot E_6$
weight 8 modular forms $= \mathbb{C} \cdot E_8$
weight 10 modular forms $= \mathbb{C} \cdot E_{10}$
weight 12 modular forms $= \mathbb{C} \cdot E_{12} \oplus \mathbb{C} \cdot \Delta$

where Ramanujan’s

$$\Delta = \frac{1}{1728} \left( \left( \frac{E_4}{2\zeta(4)} \right)^3 - \left( \frac{E_6}{2\zeta(6)} \right)^2 \right)$$

$$= e^{2\pi i z} \cdot \left( \prod_{n=1}^{\infty} \left( 1 - e^{2\pi inz} \right) \right)^{24}$$

is the lowest-weight cuspform.
Restrictions on existence of modular forms imply relations. Some senseless but possibly slightly charming examples are some relations among the sum-of-powers-of-divisors functions $\sigma_{2k-1}$ that occur in the higher Fourier coefficients of Eisenstein series:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m) \sigma_3(n - m)$$

$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(n) \sigma_5(n - m)$$

which follow from

$$\left( \frac{E_4}{2\zeta(4)} \right)^2 = \frac{E_8}{2\zeta(8)}$$

and

$$\frac{E_4}{2\zeta(4)} \cdot \frac{E_6}{2\zeta(6)} = \frac{E_{10}}{2\zeta(10)}$$
Similarly, constraints on existence of modular forms yield relations between theta series and Eisenstein series. First, at the level of an amusement, one can prove the traditional

number of ways to express an odd \( n \) as sum of 8 squares \( n = x_1^2 + \ldots + x_8^2 \) with \( (x_i \in \mathbb{Z}) \)

\[ = 16 \sigma_3(n) \]

More seriously, the Siegel-Weil formula asserts that normalized Eisenstein series \( E_{4k}/2\zeta(4k) \) are rational-coefficient linear combinations of theta series.

Since Fourier coefficients of theta series are rational, and since all parts of these Eisenstein series are explicit except for the \( \zeta(2k) \),

\[ \zeta(2k) = \text{rational} \times \pi^{2k} \]

This argument works as well for zeta functions of any totally-real algebraic number field, such as \( \mathbb{Q}(\cos \, 2\pi/n) \) for integer \( n \).
The best of Riemann’s proofs of the analytic continuation of \( \zeta(s) - \sum 1/n^2 \) uses an **integral representation**

\[
\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty y^{s/2} \frac{\theta(iy) - 1}{2} \frac{dy}{y}
\]

with \( \theta \) defined on the upper half-plane \( \mathcal{H} \) by

\[
\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 z}
\]

The Poisson summation formula \( \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \) applied to the test function \( f(n) = e^{-\pi n^2 y} \) gives

\[
\theta(iy) = \frac{1}{\sqrt{y}} \theta(-1/iy)
\]

from which we obtain the analytic continuation:
Proof of analytic continuation of $\zeta(s)$

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_1^\infty + \int_0^1$$

$$= \text{entire} + \int_0^1 y^{s/2} \frac{\theta(iy)}{2} \frac{dy}{y} - \int_0^1 y^{s/2} \frac{dy}{2y}$$

$$= \text{entire} + \int_1^\infty y^{-s/2} \frac{\theta(-1/iy)}{2} \frac{dy}{y} - \frac{1}{s}$$

$$= \text{entire} + \int_1^\infty y^{-s/2} \frac{\theta(iy)}{2\sqrt{y}} \frac{dy}{y} - \frac{1}{s}$$

$$= \text{entire} + \int_1^\infty y^{(1-s)/2} \frac{\theta(iy) - 1}{2} \frac{dy}{y}$$

$$+ \int_1^\infty y^{(1-s)/2} \frac{dy}{2y} - \frac{1}{s}$$

$$= \text{entire}(s) + \text{entire}(1 - s) - \frac{1}{1 - s} - \frac{1}{s}$$

Symmetry of the last expressions gives the functional equation

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1 - s)$$
Hecke used holomorphic cuspforms
\( f(z) = \sum_{n>0} c_n e^{2\pi inz} \) to obtain analytic continuations and functional equations of \( L \)-functions

\[
L_f(s) = \sum_{n>0} \frac{c_n}{n^s}
\]

by an \textbf{integral representation} parallel to Riemann’s, but simpler

\[
(2\pi)^{-s} \Gamma(s) L_f(s) = \int_0^\infty y^s f(iy) \frac{dy}{y}
\]

\[
= \int_1^\infty + \int_0^1 = \text{entire} + \int_1^\infty y^s f(iy) \frac{dy}{y}
\]

\[
= \text{entire} + \int_1^\infty y^{-s} f(-1/iy) \frac{dy}{y}
\]

\[
= \text{entire} + (-1)^k \int_1^\infty y^{2k-s} f(iy) \frac{dy}{y}
\]

\[
= \text{entire}(s) + (-1)^k \text{entire}(2k - s)
\]
A less elementary family of modular forms consists of **Maass’ waveforms**, $\Gamma$-invariant moderate-growth functions on $\mathcal{H}$ which are eigenfunctions for the $SL(2, \mathbb{R})$-invariant Laplacian

$$
\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
$$

*We want a decomposition of $L^2(\Gamma \backslash \mathcal{H})$ by $\Delta$.\n
The simplest (but not square-integrable) waveform is another type of Eisenstein series

$$
E_s(z) = \sum_{\gcd(m,n)=1} y^s \frac{1}{|mz + n|^{2s}}
$$

$$
= \sum_{\gamma \in \Gamma_{\infty} \backslash \Gamma} \text{Im}(\gamma(z))^s
$$

where $\Gamma_{\infty}$ consists of elements $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ in $\Gamma$. The $\Delta$-eigenvalue of $E_s$ is

$$
\lambda = \lambda_s = s(s - 1)
$$
Waveforms have Fourier expansions

\[ f(z) = \sum_{n} c_n(y) e^{2\pi i n z} \]

where the Fourier coefficient satisfies a differential equation

\[ u'' - \left( \frac{\lambda}{y^2} + 4\pi^2 n^2 \right) u = 0 \]

The Eisenstein series correctly suggests the entirely unobvious fact that an integral representation for a solution with eigenvalue \( \mu(\mu - 1) \) is

\[ u(y) = \int_{-\infty}^{+\infty} \frac{y^\mu}{(x^2 + y^2)^\mu} e^{-2\pi i n x} \, dx \]

This integral only converges for \( \text{Re}(\mu) > \frac{1}{2} \), but can be rearranged to a better form

\[ u(y) = \frac{\pi^\mu}{\Gamma(\mu)} \int_{0}^{\infty} e^{-\pi n^2 / t} e^{-\pi y^2 t} t^{\mu - \frac{1}{2}} \frac{dt}{t} \]

This solution is the only rapidly decreasing solution (as \( y \to +\infty \)).
Maass did for waveforms what Hecke did in the holomorphic case. For cuspidal waveform (zero-th Fourier coefficient 0) with normalized rapid-decay $u_n$ with eigenvalue $\mu(\mu - 1)$

$$f(z) = \sum_{n \neq 0} c_n \cdot u_n(y) e^{2\pi i n x}$$

the same integral transform is appropriate

$$\int_0^\infty y^s f(iy) \frac{dy}{y}$$

$$= \pi^{-s} \Gamma\left(\frac{s + \mu}{2}\right) \Gamma\left(\frac{s + 1 - \mu}{2}\right) \sum_{n \geq 1} \frac{c_n}{n^s}$$

The computation of the gamma factors is feasible when the integral representation of the $u_n$’s is known.

The same argument proves analytic continuation and functional equation.
What’s in $L^2(\Gamma \backslash \mathcal{H})$ besides cuspidal waveforms?

The orthogonal complement of cuspidal forms is spanned by functions

$$\theta_\varphi(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\text{Im}(\gamma z))$$

where $\varphi \in C_c^\infty(0, \infty)$, so decompose these into $\Delta$-eigenfunctions. Define $TF$ on $C_c^\infty(0, \infty)$

$$TF(s) = \int_{-\infty}^\infty F(r) r^{-s} \frac{dr}{r}$$

**Theorem:**

$$\theta_\varphi(g) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} T\varphi(s) E_s(g) \quad (\text{Re}(s) > 1)$$

The complement of cuspidal forms is integrals of Eisenstein series. However, we should push the line of integration to $\text{Re}(s) = 1/2$, despite the Eisenstein series only converging for $\text{Re}(s) > 1$. **Meromorphically continue the Eisenstein series.**
One can compute separately, much as in the holomorphic case

\[ E_s(z) = y^s + \frac{\xi(2s - 1)}{\xi(2s)} y^{1-s} + \text{higher-order terms} \]

where \( \xi \) is \( \zeta \) with its proper gamma factor

\[ \xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \]

Though the zeta function was understood prior to the Eisenstein series, Selberg noted that meromorphic continuation of \( E_s \) would prove meromorphic continuation of \( \zeta(s) \).

This idea was vastly extended by Langlands, who looked at constant terms of more complicated Eisenstein series on bigger groups, like \( GL(n) \), obtaining in 1960’s the *meromorphic continuation* of a large family of L-functions. Shahidi extended and refined Langlands’ idea to obtain *functional equations* as well. This is the *Langlands-Shahidi method* for obtaining integral representations.
The Rankin-Selberg-method is another type of integral representation which makes essential use of the meromorphic continuation of Eisenstein series.

For holomorphic cuspforms $f(z) = \sum a_n e^{2\pi i n z}$, $g(z) = \sum b_n e^{2\pi i n z}$ of weight $2k$

$$\int_{\Gamma \backslash \mathcal{H}} f(z) \overline{g(z)} E_s(z) y^{2k} \frac{dx \, dy}{y^2}$$

$$= (4\pi)^{-s} \Gamma(s) \sum_n \frac{a_n \overline{b_n}}{n^s}$$

The rapid decay of the cuspforms combined with the moderate growth (even of the meromorphically continued Eisenstein series) makes the integral converge away from the poles of the Eisenstein series.
Bernstein refined of one of Selberg’s methods for meromorphic continuation of Eisenstein series.

Consider the holomorphically parametrized system of equations

$$\begin{cases} 
\Delta u &= s(s-1) \cdot u \\
\left( y \frac{\partial}{\partial y} - (1 - s) \right) c(u) &= (2s - 1) y^s 
\end{cases}$$

For $u$ a nice function of moderate growth on $\Gamma \backslash \mathcal{H}$. Note that $E_s$ satisfies this system, and the trickier part of the constant term is masked.

**Theorem** (Bernstein) Suppose that for $s$ in a non-empty open subset of $\mathbb{C}$ the system has the unique solution $E_s$. Suppose that (locally everywhere) for some $n$ there is a holomorphic family of linear maps

$$h_s : \mathbb{C}^n \rightarrow \{ \text{moderate growth functions on } \Gamma \backslash \mathcal{H} \}$$

so that all solutions of the system lie inside the image $h_s(\mathbb{C}^n)$. Then $E_s$ has a meromorphic continuation to $s \in \mathbb{C}$.

The first condition is *uniqueness*, the second *finiteness*.
**Proof:** Fix $s \notin \mathbb{R}$ with $\text{Re}(s) \gg 0$. If a function $v_s$ other than $E_s$ were to satisfy the system, then

$$E_s - v_s = c \cdot y^{1-s} + \text{higher}$$

The *theory of the constant term* assures that the higher-order terms are of rapid decay, so this difference would be square-integrable. But the Laplacian $\Delta$ on $\mathcal{H}$ is essentially self-adjoint, so any $L^2(\Gamma \backslash \mathcal{H})$ eigenfunction must have a real eigenvalue. But $s(s-1) \notin \mathbb{R}$, contradiction. Thus, for $s$ in a non-empty open set $E_s$ is the unique solution to the system.

For large real $T$, let

$$y_T^s = \begin{cases} y^s & \text{(for } y \geq T) \\ 0 & \text{(else)} \end{cases}$$

And form special Eisenstein series by

$$E(y_T^s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y_T^s(\gamma z)$$

where $y_T^s$ is viewed as a function on $\mathcal{H}$. These converge for all $s$. 

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If \( v_s \) is a solution of \( \Delta v_x = s(s - 1) \), then by the theory of the constant term

\[ v_s = a E(y_T^s) + b E(y_T^{1-s}) + \text{rapid decay} \]

Thus, any solution of the original system is inside the space

\[ \mathbf{C} \cdot E(y_T^s) + \mathbf{C} \cdot E(y_T^{1-s}) + \{ \text{rapid decay} \} \]

In fact, solutions are inside a space

\[ \mathbf{C} \cdot E(y_T^s) + \mathbf{C} \cdot E(y_T^{1-s}) + \{ \text{constant term compactly supported} \} \]

**Lemma** (Selberg-Bernstein) Let \( V \) be a Banach space. Consider a parametrized family of linear maps \( T_s(v) = 0 \), with \( T_s : V \rightarrow W \), where \( W \) is a Banach space, and \( s \rightarrow T_s \) is holomorphic for the uniform-norm Banach-space topology. Suppose that for some fixed \( s_o \) the operator \( T_{s_o} \) has a left inverse modulo compact operators, that is, that there exists an operator \( A : W \rightarrow V \) so that

\[ A \circ \lambda_{s_o} = 1_V + (\text{compact operator}) \]

Then the finiteness condition holds.
But the theorem of Selberg, Gelfand, Piatetski-Shapiro, Langlands on the compactness of integral operators on modular forms with compact-support constant term meets the condition of the lemma!

The same type of argument applies at the very least to Eisenstein series made with cuspidal data on maximal proper parabolics. (The expanded version of my Tel Aviv talk (March 2001) gives details.)

A refinement of Bernstein’s criterion allows for discussion of poles in terms of unitariness of associated representations.

Via Langlands-Shahidi and Rankin-Selberg integral representations, knowledge of poles of Eisenstein series gives information about poles of L-functions.