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The snake lemma and extensions of functionals

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

The snake lemma and the existence of a long exact sequence attached to a short exact sequence have many applications beyond those in the archetypical first encounter in basic algebraic topology. The application here is to existence and uniqueness of *extensions* of maps. [1]

1. *Hadamard's example*

[Hadamard 1932] considered the behavior of functionals

$$\int_\varepsilon^1 \frac{f(x)}{x^{3/2}}\,dx$$

as $\varepsilon \to 0^+$. For f continuous with $f(0) \neq 0$, this expression blows up as $\varepsilon \to 0^+$. Nevertheless, Hadamard attached meaning to the integral as follows.

Before letting $\varepsilon \to 0^+$, integrate by parts:

$$\int_\varepsilon^1 \frac{f(x)}{x^{3/2}}\,dx = [\frac{-2f(x)}{x^{1/2}}]_\varepsilon^1 + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx$$

$$= -2f(1) + \frac{2f(\varepsilon)}{\varepsilon^{1/2}} + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx = -2f(1) + \frac{2(f(\varepsilon)-f(0))}{\varepsilon^{1/2}} + \frac{2f(0)}{\varepsilon^{1/2}} + 2\int_\varepsilon^1 \frac{f'(x)}{x^{1/2}}\,dx$$

Of the four summands, only $-2f(0)/\varepsilon^{1/2}$ blows up as $\varepsilon \to 0^+$. In fact, assuming that f is at least once continuously differentiable, the term $2(f(\varepsilon)-f(0))/\varepsilon^{1/2}$ goes to 0.

Hadamard's surprising insight was to *drop* entirely the term $2f(0)/\varepsilon^{1/2}$, calling what remained the *partie finie* ('finite part') of the integral, denoted

$$\text{p.f.} \int_0^1 \frac{f(x)}{x^{3/2}}\,dx = -2f(1) + 2\int_0^1 \frac{f'(x)}{x^{1/2}}\,dx$$

This appears to be a scandalous lapse, not justifiable or purposeful. Nevertheless, Hadamard successfully applied this idea to hyperbolic partial differential equations.

A few years later [M.Riesz 1938/40] showed that *partie finie* functionals are *meromorphic continuations* of convergent integrals, as developed later at length in [Schwartz 1950-1] and [Gelfand-Shilov 1958]. In the example above, consider

$$u_s(f) = \int_0^1 f(x)\,x^s\,dx$$

[1] I first saw this use of the snake lemma in [Casselman 1993] in a discussion of an extended notion of *automorphic form*, specifically, to understand an argument for Maass-Selberg identities. There, it is observed that such considerations are reminiscent of Hadamard's *partie finie* [Hadamard 1932] in the context of hyperbolic partial differential equations. Casselman notes that [Zagier 1982] raises similar issues. In the spirit of [Gelfand-Shilov 1958], in effect following [M. Riesz 1938/40] and [M. Riesz 1949], and [Schwartz 1950], *partie finie* functionals are meromorphic continuations in a natural auxiliary parameter, not results of an *ad hoc* classical construction as in [Hadamard 1932]. Nevertheless, one may view meromorphic continuation as *ad hoc* itself.

for f at least once continuously differentiable, and for $\mathrm{Re}(s) > -1$. Integration by parts gives

$$u_s(f) = [\frac{f(x)x^{s+1}}{s+1}]_0^1 - \frac{1}{s+1}\int_0^1 f'(x)\,x^{s+1}\,dx = \frac{f(1)}{s+1} - \frac{1}{s+1}\,u_{s+1}(f')$$

Iteration of this gives a meromorphic continuation of u_s to $\mathbb{C}$ with $-1, -2, -3, \ldots$ removed. In particular, there is no pole at $s = -3/2$, and the latter equation gives

$$u_{-3/2}(f) = \frac{f(1)}{(-3/2)+1} - \frac{1}{(-3/2)+1}\int_0^1 f'(x)\,x^{(-3/2)+1}\,dx = -2f(1) + 2\int_0^1 \frac{f'(x)}{x^{1/2}}\,dx$$

It is striking that meromorphic continuation recovers Hadamard's formula. While this makes Hadamard's *partie finie* less suspect, it illustrates that extensions of functionals by meromorphic continuation may be counter-intuitive.

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2. *Extensions and the snake lemma*

The snake lemma, and the long exact sequence in (co-) homology, has an interesting application to some very small complexes, which appear in the proof of the following.

[2.0.1] Proposition: Let A, B, C be R-modules over a (not necessarily commutative) $\mathbb{C}$-algebra R. Let

$$0 \to A \overset{j}{\to} B \overset{q}{\to} C \to 0$$

be a short exact sequence, and let T be an R-endomorphism of B which stabilizes A (as subobject of B), so induces an R-endomorphism on $C \approx B/A$ by

$$T(b + A) = Tb + A$$

Then we have a natural exact sequence

$$0 \to \ker_A T \to \ker_B T \to \ker_C T \to A/TA \to B/TB \to C/TC \to 0$$

Proof: This is the long exact homology sequence attached to the short exact sequence of complexes

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & A & \overset{j}{\longrightarrow} & B & \overset{q}{\longrightarrow} & C & \to 0 \\ & T\downarrow & & T\downarrow & & T\downarrow & \\ 0\to & A & \overset{j}{\longrightarrow} & B & \overset{q}{\longrightarrow} & C & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

with complexes

$$\mathfrak{A} \,:\, 0 \to A \overset{T}{\longrightarrow} A \to 0, \qquad \mathfrak{B} \,:\, 0 \to B \overset{T}{\longrightarrow} B \to 0, \qquad \mathfrak{C} \,:\, 0 \to C \overset{T}{\longrightarrow} C \to 0$$

That is, $H_o(\mathfrak{A}) = \ker_A T$, $H_1(\mathfrak{A}) = A/TA$, and similarly for B and C. ///

[2.0.2] Corollary: When $T : A \to A$ is a bijection, $\ker_B T \to \ker_C T$ is an isomorphism. ///

3. *Homogeneous distributions*

For clarity, we consider a variant of Hadamard's example that fits more simply into Schwartz' context.

Let

$$\langle f, u_s \rangle \;=\; u_s(f) \;=\; \int_{\mathbb{R}} f(x) \cdot |x|^s \, \frac{dx}{|x|} \qquad\qquad (\text{for } f \in \mathscr{S},\ \mathrm{Re}(s) > 0)$$

The measure is arranged to be invariant under dilations. The function u_s satisfies the differential equation

$$\Big(x \, \frac{d}{dx} - s\Big) \, u_s \;=\; 0 \qquad\qquad (\text{at least for } \mathrm{Re}(s) > 0)$$

Let V be the subspace of Schwartz functions $\mathscr{S}$ vanishing to infinite order at 0. There is a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathscr{S} \longrightarrow \{\text{Taylor expansions of smooth functions at } 0\} \longrightarrow 0$$

There is the short exact sequence of duals, as well,

$$0 \longrightarrow \{\text{distributions supported at } 0\} \longrightarrow \mathscr{S}' \longrightarrow V^* \longrightarrow 0$$

Let Z be the distributions supported at 0. By classification, we know that Z consists of finite linear combinations of δ and its derivatives.

Let v_s be the restriction of u_s to a functional on V. That is, $v_s \in V^*$. Certainly v_s still satisfies the same differential equation as u_s, but is better than u_s, since the integral for v_s converges for all $s \in \mathbb{C}$. That is, v_s is only integrated against functions vanishing to infinite order at 0.

Given v_s for arbitrary $s \in \mathbb{C}$, we would like to ask whether there *exists a unique* $u_s \in \mathscr{S}'$ extending v_s and satisfying the differential equation above. At the same time, it is essentially elementary to understand solutions of that differential equation in Z, since [2]

$$\Big(x \, \frac{d}{dx} + (\ell + 1)\Big) \, \Big(\frac{d^\ell}{dx^\ell} \delta\Big) \;=\; 0$$

Indeed, for s *not* a negative integer, the differential equation has *no* solution in Z.

Let $T = x \, \frac{d}{dx} - s$, and consider the three little complexes

$$0 \to Z \xrightarrow{T} Z \to 0 \qquad 0 \to \mathscr{S}' \xrightarrow{T} \mathscr{S}' \to 0 \qquad 0 \to \mathscr{S}_0' \xrightarrow{T} \mathscr{S}_0' \to 0$$

The snake lemma result says that

$$\ker_{\mathscr{S}'} T \;\approx\; \ker_{\mathscr{S}_0'} T \qquad\qquad (\text{for } s \text{ not a negative integer})$$

That is, unless s is a negative integer, the solution $v_s \in V^*$ to the differential equation extends, and extends *uniquely* to a solution u_s in $\mathscr{S}'$.

[3.0.1] Remark: The gamma function can be rewritten

$$\Gamma(s) \;=\; \int_0^\infty t^s \, e^{-t} \, \frac{dt}{t} \;=\; 2 \int_{\mathbb{R}} t^{2s} \, e^{-t^2} \, \frac{dt}{|t|} \;=\; u_{2s}\big(2e^{-t^2}\big)$$

[2] Since these distributions are compactly supported, at $\{0\}$, we can simplify computations concerning them by evaluating things on the smooth functions x^n.

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