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Tiniest example of (co-) homology

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For context, recall that in a category of G-modules for group G, the fixed-vector functor [1]

$$M \;\to\; M^G = \{m \in M : g \cdot m = m \text{ for all } g \in G\}$$

and cofixed-vector functor [2]

$$M \;\to\; M_G = \text{largest quotient of } M \text{ on which } G \text{ acts trivially}$$

are mutual adjoints: $\mathrm{Hom}_G(M_G, N) \approx \mathrm{Hom}_G(M, N^G)$. Because left adjoints are right-exact, the G-cofixed-vector functor $M \to M_G$ is right-exact, so has left-derived functors, called *group homology*. Similarly, because right adjoints are left-exact, the G-fixed-vector $M \to M^G$ is left-exact, so has right-derived functors, called *group cohomology*. [3]

In the situation where G is a real Lie group, taking smooth vectors V^∞ in G-representations gives representations of the Lie algebra $\mathfrak{g}$ of G. Then G-fixed becomes $\mathfrak{g}$-annihilation, and G-cofixed becomes $\mathfrak{g}$-coannihilation. Lie algebra homology consists of the left-derived functors of the $\mathfrak{g}$-coannihilation functor, and Lie algebra cohomology consists of the right-derived functors of the $\mathfrak{g}$-annihilation functor. Many other classical (co-) homologies are also derived functors of fairly trivial functors, with homologies and cohomologies of mutually adjoint functors having further relationships.

In the category of $\mathbb{C}[x]$-modules, similarly, the two simplest functors are $M \to M^x$ (the x-annihilated submodule) and $M \to M_x$ (the x-annihilated quotient). Of course, this is just saying "$\ker x$" and "M/xM" in a fancier way. These are mutual adjoints: $\mathrm{Hom}(M_x, N) \approx \mathrm{Hom}(M, N^x)$. Thus, M_x is a 0^{th} homology we might denote by $H_0(M, x)$, and M^x is a 0^{th} cohomology $H^0(M, x)$.

To determine the higher (co-)homologies, from a short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \times x} & & \downarrow{\scriptstyle \times x} & & \downarrow{\scriptstyle \times x} & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

[1] This familiar description is not the categorically-best characterization, but, rather, is a construction. The best description is that M^G is the largest subobject of M on which G acts trivially.

[2] This is a categorically correct description of the cofixed quotient. In contrast to the fixed-vector characterization/construction, the construction of cofixed-vector modules depends more delicately on the ambient category. For example, in the category of G-modules, M_G is the quotient of M by the submodule generated by all elements $m - g \cdot m$ for $m \in M$ and $g \in G$. In the category of topological vector spaces with continuous G actions, the quotient must be by the topological closure of the subspace generated by such elements, so that the quotient is Hausdorff.

[3] In some sources, group (co-) homology is defined *ad hoc* by specifying a particular (injective) projective resolution, without comment about the larger homological context.

the Snake Lemma gives a long exact sequence

$$0 \longrightarrow A^x \longrightarrow B^x \longrightarrow C^x \xrightarrow{\eta} A_x \longrightarrow B_x \longrightarrow C_x \longrightarrow 0$$

That is, A^x can be viewed as $H_1(A, x)$, with all higher homologies $H_i(A, x)$ vanishing, and A_x can be viewed as $H^1(A, x)$, with all higher cohomologies $H^i(A, x)$ vanishing. The short-long exact sequence is either/both a long (co-)homology sequence.
