

1. Rigged Hilbert spaces from pairs

A pair of symmetric, semi-bounded operators $S,T$ on a Hilbert space $\mathfrak{B}^0$ gives rise to a \textit{rigged Hilbert space} structure when the operators have a common domain $D = D_S = D_T$ dense in $\mathfrak{B}^0$ stabilized by them, that is, when $S(D) \subset D$ and $T(D) \subset D$, as follows.

Without loss of generality, suppose that $S,T$ are non-negative and $S + T \geq 1$, in the sense that

$$
\langle Sv, v \rangle_{\mathfrak{B}^0} \geq 0 \quad \langle Tv, v \rangle_{\mathfrak{B}^0} \geq 0 \quad \langle (S + T)v, v \rangle_{\mathfrak{B}^0} \geq \langle v, v \rangle_{\mathfrak{B}^0}
$$

(for all $v \in D$)

The $\mathfrak{B}^1$-norm relative to $S,T$ is

$$
\langle v, w \rangle_{\mathfrak{B}^1} = \langle (S + T)v, w \rangle_{\mathfrak{B}^0}
$$

and $\mathfrak{B}^1$ is the completion of $D$ with respect to this norm. The $\mathfrak{B}^k$-norm is described inductively:

$$
\langle v, w \rangle_{\mathfrak{B}^k} = \langle Sv, Sw \rangle_{\mathfrak{B}^{k-2}} + \langle Tv, Tw \rangle_{\mathfrak{B}^{k-2}}
$$

(for $v, w \in D$ and $k \geq 2$)

and $\mathfrak{B}^k$ is the Hilbert-space completion. Let $\mathfrak{B}^{+\infty}$ be the projective limit. The maps $\mathfrak{B}^k \to \mathfrak{B}^{k-1}$ induced by the denseness of $D$ in every $\mathfrak{B}^k$ are continuous injections with dense images, thus giving a \textit{rigged Hilbert-space}

$$
\ldots \to \mathfrak{B}^k \to \mathfrak{B}^{k-1} \to \ldots \to \mathfrak{B}^2 \to \mathfrak{B}^1 \to \mathfrak{B}^0 = V
$$

By design, $S$ and $T$ are continuous $D \to D$ with $\mathfrak{B}^k$-topology on the source and $\mathfrak{B}^{k-2}$-topology on the target:

$$
|Sv|_{\mathfrak{B}^{k-2}}^2 \leq |(Sv)|_{\mathfrak{B}^{k-2}}^2 + |Tv|_{\mathfrak{B}^{k-2}}^2 = |v|_{\mathfrak{B}^k}^2
$$

(for $v \in D$)

and similarly for $T$. Thus, $S,T$ extend by continuity to continuous maps $S^+, T^+ : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for all $k \geq 2$, and, then, to continuous maps $\mathfrak{B}^{+\infty} \to \mathfrak{B}^{+\infty}$. The triangle inequality shows continuity of $S + T$:

$$
|(S + T)v|_{\mathfrak{B}^{k-2}} \leq |Sv|_{\mathfrak{B}^{k-2}} + |Tv|_{\mathfrak{B}^{k-2}} \leq 2|v|_{\mathfrak{B}^k}
$$

(for $v \in D$)

so $S + T$ likewise extends by continuity to $(S + T)^+ : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for all $k \geq 2$, and then to $\mathfrak{B}^{+\infty} \to \mathfrak{B}^{+\infty}$.

Non-commutative polynomials in $S,T$ are to be understood as having domain $D$. Non-commutative monomials $Q$ of total degree $d$ are proven continuous $\mathfrak{B}^k \to \mathfrak{B}^{k-d}$ by induction on $d$, for $Q$ of degree $d$ giving a continuous linear map $\mathfrak{B}^k \to \mathfrak{B}^{k-d}$ for all $k \geq d$,

$$
|(Q \cdot v)|_{\mathfrak{B}^{k-d-1}}^2 = |(Qv)|_{\mathfrak{B}^{k-d-1}}^2 \ll Q |Sv|_{\mathfrak{B}^{k-1}}^2 \leq |v|_{\mathfrak{B}^k}^2
$$

(for $v \in D$)

and similarly for $Q \cdot T$. Symmetry of $S,T$ shows that this induction gives the same outcome as induction by adding factors on the left. The triangle inequality gives an induction on the number of summands in $Q$ to prove a similar continuity for all non-commutative polynomials: for a polynomial $Q$ of total degree $d$, and $M$ a monomial of total degree at most $d$,

$$
|(Q + M)v|_{\mathfrak{B}^{k-d}} \leq |Qv|_{\mathfrak{B}^{k-d}} + |Mv|_{\mathfrak{B}^{k-d}} \ll Q,M |v|_{\mathfrak{B}^k}
$$

(for $v \in D$)
Thus, all polynomials \( Q \) in \( S, T \) of total degree at most \( d \) extend by continuity to \( Q^\# : \mathcal{B}^k \rightarrow \mathcal{B}^{k-d} \), and to continuous maps of \( \mathcal{B}^{+\infty} \) to itself.

---

### 2. Large extensions of operators

For Hilbert spaces with a **complex conjugation** stabilizing \( D \), operators \( S, T \) commuting with the conjugation have **large extensions**, still denoted \( S^\#, T^\# \), to the dual of \( \mathcal{B}^{+\infty} \).

For \( k \geq 1 \), let \( \mathcal{B}^{-k} \) be the complex-linear Hilbert-space dual of \( \mathcal{B}^k \), with hermitian inner product \( \langle , \rangle_{\mathcal{B}^{-k}} \) coming from the norm

\[
|\lambda|_{-k} = \sup_{v \in \mathcal{B}^k : |v| \leq 1} |\lambda v| \quad \text{ (for } \lambda \in \mathcal{B}^{-k})
\]

The natural **complex-bilinear** pairing on \( \mathcal{B}^k \times \mathcal{B}^{-k} \) is

\[
\langle , \rangle_{\mathcal{B}^k \times \mathcal{B}^{-k}} : \mathcal{B}^k \times \mathcal{B}^{-k} \rightarrow \mathbb{C} \quad \text{ by } \quad \langle v, \lambda \rangle_{\mathcal{B}^k \times \mathcal{B}^{-k}} = \lambda(v) \quad \text{ (} v \in \mathcal{B}^k \text{ and } \lambda \in \mathcal{B}^{-k})
\]

The maps

\[
\ldots \rightarrow \mathcal{B}^k \rightarrow \mathcal{B}^{k-1} \rightarrow \ldots \rightarrow \mathcal{B}^2 \rightarrow \mathcal{B}^1 \rightarrow \mathcal{B}^0
\]

give Hilbert-space adjoints

\[
(\mathcal{B}^0)^\ast \rightarrow \mathcal{B}^{-1} \rightarrow \mathcal{B}^{-2} \rightarrow \ldots \rightarrow \mathcal{B}^{-(k-1)} \rightarrow \mathcal{B}^{-k} \rightarrow \ldots
\]

These two collections of maps can be spliced together, and the hermitian inner products compared with the complex-bilinear pairings, when when \( \mathcal{B}^0 \) has a complex-conjugate-linear conjugation map, as follows. The conjugation \( v \rightarrow \overline{v} \) should have expected properties: \( \overline{v} = v, \overline{\alpha v} = \overline{\alpha} \cdot \overline{v} \) for complex \( \alpha \), and \( \langle v, \overline{w} \rangle_{\mathcal{B}^0} = \overline{\langle w, v \rangle_{\mathcal{B}^0}} \). Suppose \( D \) is stabilized by \( v \rightarrow \overline{v} \), and that \( S \) and \( T \) commute with \( v \rightarrow \overline{v} \).

A compatible conjugation map is induced on \( \mathcal{B}^k \) and \( \mathcal{B}^{-k} \), and \( i : D \rightarrow \mathcal{B}^{+1} \) and \( j : \mathcal{B}^{+1} \rightarrow \mathcal{B}^0 \) commute with the conjugation.

Using the conjugation on \( \mathcal{B}^0 \), let \( \Lambda : \mathcal{B}^0 \rightarrow (\mathcal{B}^0)^\ast \) be the complex-linear isomorphism of \( \mathcal{B}^0 \) with its complex-linear dual by \( \Lambda(x)(y) = \langle y, \overline{x} \rangle_{\mathcal{B}^0} = \langle x, \overline{y} \rangle_{\mathcal{B}^0} \). The continuous injection \( j : \mathcal{B}^{+1} \rightarrow \mathcal{B}^0 \) dualizes to \( j^\ast : (\mathcal{B}^0)^\ast \rightarrow \mathcal{B}^{-1} \) by \( j^\ast \mu(x) = \mu(jx) \) for \( \mu \in (\mathcal{B}^0)^\ast \) and \( x \in \mathcal{B}^{+1} \), and we have the splicing

\[
\mathcal{B}^{+\infty} \rightarrow \mathcal{B}^{+2} \rightarrow \mathcal{B}^{+1} \rightarrow \mathcal{B}^0 \rightarrow \ldots \rightarrow \mathcal{B}^{-1} \rightarrow \ldots
\]

with \( \mathcal{B}^{+\infty} = \text{colim} \mathcal{B}^{-k} \) the strong dual of \( \mathcal{B}^{+\infty} \). [1]

**[2.0.1] Note:** Thus, for \( k, \ell \geq 0 \), letting \( \varphi : \mathcal{B}^k \rightarrow \mathcal{B}^{-\ell} \) be the injective map induced by the identity map \( D \rightarrow D \), the comparison of hermitian and complex-bilinear forms is essentially described by

\[
\langle v, w \rangle_{\mathcal{B}^k} = \langle v, \overline{w} \rangle_{\mathcal{B}^k \times \mathcal{B}^{-k}} \quad \text{ (for } v, w \in \mathcal{B}^k \)
\]

**[2.0.2] Note:** Since \( D \) injects to \( \mathcal{B}^0 \) and is dense in \( \mathcal{B}^0 \), every \( \mathcal{B}^k \rightarrow \mathcal{B}^{k-1} \) for \( k \geq 1 \) is injective with dense image. The injectivity and dense image of \( \mathcal{B}^{+1} \rightarrow \mathcal{B}^0 \) give injective adjoint \( (\mathcal{B}^0)^\ast \rightarrow \mathcal{B}^{-1} \) with dense

---

[1] For general categorical reasons, \( \mathcal{B}^{+\infty} \) is the dual of \( \mathcal{B}^{-\infty} \), but \( (\mathcal{B}^{+\infty})^\ast = \mathcal{B}^{-\infty} \) needs the fact that a continuous linear map from a limit of Banach spaces to a normed space necessarily factors through a limit map.
image. Since $S$ is symmetric and commutes with conjugation, the extensions $S^\# , T^\#$ are compatible with the complex-linear identification $\Lambda : \mathfrak{B}^0 \to (\mathfrak{B}^0)^*$.

### [2.1] Large extensions of operators

The extended operators $S^\# , T^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for $k \geq 2$ have adjoints $(S^\#)^*$ and $(T^\#)^*$ mapping $\mathfrak{B}^{-(k-2)} \to \mathfrak{B}^{-k}$.

For even indices $k$, compatibility with conjugation and the complex-linear isomorphism $\Lambda : \mathfrak{B}^0 \approx (\mathfrak{B}^0)^*$ allows us to consider these adjoints as extensions of $S^\# , T^\#$, and denote them simply by the same symbols, $S^\#$ and $T^\#$.

To connect positive and negative odd indices $k$, the conjugation allows us to extend $S^\# , T^\#$ to maps $\mathfrak{B}^{+1} \to \mathfrak{B}^{-1}$, by

$$(S^\# x)(y) = \langle x, y \rangle_{\mathfrak{B}^1}, \quad (T^\# x)(y) = \langle x, y \rangle_{\mathfrak{B}^1}, \quad (x, y \in \mathfrak{B}^{+1})$$

Again, these extensions are indeed compatible with $\mathfrak{B}^{+1} \to \mathfrak{B}^0 \approx (\mathfrak{B}^0)^* \to \mathfrak{B}^{-1}$.

Thus, $S, T$ extend to $S^\# , T^\# : \mathfrak{B}^k \to \mathfrak{B}^{k-2}$ for all $k \in \mathbb{Z}$, inducing $S^\# , T^\# : \mathfrak{B}^+ \to \mathfrak{B}^+$ and the large extensions $S^\# , T^\# : \mathfrak{B}^- \to \mathfrak{B}^-$, denoted by the same symbols. [2]

Then non-commutative polynomials $Q$ in $S, T$ with real coefficients are likewise compatible with conjugation, so have large extensions $Q^\#$. Writing a non-commutative polynomial’s arguments as $x, y$, the compatibility of such polynomials with formation of large extensions is

$$Q(S, T)^\# = Q(S^\# , Q^\#)$$

---

[2] Laplacians on test functions give the archetype for $S^\# : \mathfrak{B}^+ \to \mathfrak{B}^+$, and the extension to distributional differentiation is the archetype for the large extension $S^\# : \mathfrak{B}^- \to \mathfrak{B}^-$. 

3