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Uniqueness of invariant distributions

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We give a very general uniqueness proof which gives as corollaries the uniqueness of G-invariant distributions on real Lie groups G and on totally disconnected groups G, in the respective senses. This argument also reproves uniqueness of Haar measures (assuming existence).

Recall that an *approximate identity* is a sequence $\{\psi_i\}$ of non-negative real-valued functions ψ_i such that $\int_G \psi_i = 1$ for all i (with right Haar measure), and such that given a neighborhood U of 1 in G there is i_o such that for $i \geq i_o$ the support of ψ_i is inside U.

Let R be the right translation action of G on functions f on G, given as usual by

$$R_g f(h) = f(hg)$$

And the usual left translation L is

$$L_g f(h) = f(g^{-1}h)$$

We consider various topological vector spaces V of complex-valued functions f on G such that

$$g \longrightarrow T_g f$$

$$g \longrightarrow L_g f$$

are continuous V-valued functions on G, for each $f \in V$. Assuming so, say that G *acts continuously on* V *by right and left translations.*

Theorem: Let G be a (separable, locally compact, Hausdorff) topological group. Let V be a quasi-complete topological vector space of complex-valued functions on G contained in the compactly supported continuous functions $C^0_c(G)$ on G, on which G acts continuously by right and left translations. Suppose that V contains a sequence $\{\varphi_i\}$ such that the sequence

$$\check{\varphi}_i(g) = \varphi_i(g^{-1})$$

is an approximate identity. Then there is a unique *right* G-invariant element of the dual space V^* (up to constant multiples), and it is (with *right* Haar measure)

$$f \longrightarrow \int_G f(g)\, dg$$

Remark: Granting the standard properties of weak (Gelfand-Pettis) integrals of continuous compactly-supported functions with values in a quasi-complete locally convex topological vector space, we can give an abstract proof which applies to the Lie group case and to the totally disconnected case, as well as to the product case and other situations.

Proof: Let F be a continuous V-valued function on G. In our application it will be

$$F(g) = R_g f$$

for f a function in a space V of $\mathbf{C}$-valued functions on G. For ψ lying in the space $C^o_c(G)$ of compactly-supported continuous $\mathbf{C}$-valued functions on G, and for $v \in V$ we have a Gelfand-Pettis integral

$$\int_G \psi(g)\, F(g)\, dg$$

with right Haar measure. By the hypothesis that V is quasi-complete the integral exists and is unique. The following result is standard in concept, if somewhat apocryphal in details.

Proposition: For an approximate identity ψ_i and a quasi-complete locally convex topological vectorspace V and a continuous V-valued function F on G, in the topology of V

$$\int_G \psi_i(g)\, F(g)\, dg \longrightarrow F(1)$$

Proof: Since the integral of ψ_i is 1, we can view $\psi_i\, dg$ as giving a probability measure on $\mathrm{spt}\,\psi_i$. And then by hypothesis $g \longrightarrow F(g)$ is a continuous V-valued function on this measure space. By properties of Gelfand-Pettis integrals, this integral lies in the closure of the convex hull of the values $F(h)$ for $h \in \mathrm{spt}\,\psi_i$. Given a convex neighborhood N of 0 in V, by the continuity in $g \in G$, there is a sufficiently small neighborhood U of 1 in G so that for $h \in U$ we have $F(h) \in F(1) + N/2$. For i large enough $\mathrm{spt}\,\psi_i \subset U$. By the local compactness of G, we can choose a smaller neighborhood U' of 1 in G so that the closure $\bar{U}'$ of U' is compact and contained in U. Then the image of $\bar{U}'$ under $h \to F(h)$ is compact and contained in $F(1) + N/2$. By convexity of $N/2$ the convex hull of the image is still contained in $F(1) + N/2$. Then the closure of the convex hull of the image is surely contained in $F(1) + N/2 + N/2 = F(1) + N$. This proves that the vectors $\int \psi_i f$ approximate $F(1)$ in V. ///

Now back to the main proof. Let φ_i be in V, such that $\check{\varphi}_i$ is an approximate identity. By the previous proposition applied to $F(g) = R_g f$, since V sits inside continuous compactly supported complex-valued functions on G,

$$\int_G \check{\varphi}_i(h)\, R_h f\, dh \longrightarrow f$$

in the topology on V. For an invariant element u of V^*, by the continuity of u on V

$$u(f) = \lim_i\, u\left(g \longrightarrow \int_G \check{\varphi}_i(h)\, f(hg)\, dh\right)$$

This is

$$u\left(g \longrightarrow \int_G f(hg)\, \varphi_i(h^{-1})\, dh\right) = u\left(g \longrightarrow \int_G f(h)\, \varphi_i(gh^{-1})\, dh\right)$$

by replacing h by hg^{-1}. By properties of Gelfand-Pettis integrals, and since f is guaranteed to be a compactly-supported continuous function, we can move the functional u inside the integral: the above becomes

$$\int_G f(h)\, u\left(g \longrightarrow \varphi_i(gh^{-1})\right)\, dh$$

Using the *right* G-invariance of u the evaluation of u with right translation by h^{-1} gives

$$\int_G f(h)\, u(g \longrightarrow \varphi_i(g))\, dh = u(\varphi_i) \cdot \int_G f(h)\, dh$$

By assumption the latter expressions approach $u(f)$ as $i \longrightarrow \infty$. For f so that the latter integral is non-zero, we see that the limit of the $u(\varphi_i)$ exists, and then we conclude that $u(f)$ is a constant multiple of the indicated integral with right Haar measure. ///

Corollary: Let G be a product of a totally disconnected group and a Lie group. Then there is a unique (up to constant multiples) right G-invariant continuous linear functional u on the space of test functions $C_c^\infty(G)$ on G, namely an integral

$$u(f) = \int_G f(x)\, dx$$

The corollary follows from the theorem upon verification that spaces of test functions (in both Lie and totally disconnected cases) are quasi-complete.

A small sample application:

Corollary: Let k be a p-adic field. There is a unique $k^\times$-invariant distribution u on k (up to constant multiples). Any such distribution is a constant multiple of evaluation at 0

$$u(f) = f(0)$$

Proof: When restricted to $S(k^\times) = S(k - 0)$ the distribution is a $k^\times$-invariant distribution on $k^\times$, so by the previous result this restriction is a constant multiple

$$u(f) = c \cdot \int_{k^\times} f(x)\, dx$$

of integration over $k^\times$ with respect to a multiplicative Haar measure. For a Schwartz function f and $t \in k^\times$ let tf be the function

$$tf(x) = f(tx)$$

In particular, take f to be the characteristic function of the local integers $\mathbf{o}$. Then

$$u(\text{ char fcn of } \mathbf{o}^\times) = u(f - \varpi^{-1}f) = u(f) - u(\varpi^{-1}f) = u(f) - u(f) = 0$$

by invariance. That is,

$$c \cdot \int_{k^\times} \text{char fcn of } \mathbf{o}^\times \, dx = 0$$

Since the measure of $\mathbf{o}^\times$ is positive, this implies that $c = 0$. That is, any $k^\times$-invariant functional on $S(k^\times)$ which extends to $S(k)$ is necessarily 0 on $S(k^\times)$.

By the exactness of the natural short exact sequence

$$0 \longrightarrow S(k^\times) \longrightarrow S(k) \longrightarrow S(\{0\}) \longrightarrow 0$$

(since $k^\times$ is open in k, and k is totally disconnected), any $k^\times$-invariant distribution on $S(k)$ factors through $S(\{0\})$. But every distribution on the one-dimensional space $S(\{0\})$ is just a multiple of evaluation at 0. This proves the claim. ///

Slightly less trivially:

Corollary: There is a unique $\mathbf{R}^\times$-invariant distribution u on $\mathbf{R}$ (up to constant multiples). Any such distribution is a constant multiple of evaluation at 0

$$u(f) = f(0)$$

Remark: In particular, the functional

$$v(f) = \int_{\mathbf{R}^\times} f(x)\, \frac{dx}{|x|}$$

defined on (for example) the space of test functions vanishing to infinite order at 0 (or even the smaller space of test functions vanishing identically in a neighborhood of 0) *cannot* be extended to an $\mathbf{R}^\times$-invariant functional on the whole space of test functions.

Proof: When restricted to $C_c^\infty(\mathbf{R}^\times)$ the functional u is a $\mathbf{R}^\times$-invariant distribution on $\mathbf{R}^\times$, so by the previous proposition this restriction is a constant multiple

$$u(f) = c \cdot \int_{\mathbf{R}^\times} f(x) \, \frac{dx}{|x|}$$

where we choose the convenient multiplicative Haar measure $dx/|x|$, where dx itself is the usual additive Haar (Lebesgue) measure. For a test function f and $t \in \mathbf{R}^\times$ let tf be the function

$$tf(x) = f(tx)$$

Take $f(x) \geq 0$ to be a test function identically 1 on some neighborhood of 1, and such that $f(x)$ is monotonically decreasing as $|x|$ increases. Then for any t the function $f - tf$ is in $C_c^\infty(\mathbf{R}^\times)$. And

$$0 = u(f) - u(tf) = u(f - tf) = c \cdot \int_{\mathbf{R}^\times} [f(x) - f(tx)] \, \frac{dx}{|x|}$$

for some constant c, by the uniqueness on $C_c^\infty(\mathbf{R}^\times)$. But for $t > 1$ the monotone decrease of f implies that $f - tf \geq 0$, and is strictly positive on a set of positive measure. Thus, the indicated integral of $f - tf$ is strictly positive. Thus, $c = 0$. That is, any $\mathbf{R}^\times$-invariant functional u on $C_c^\infty(\mathbf{R})$ restricts to 0 on $C_c^\infty(\mathbf{R}^\times)$. This proves that the support of u is 0. From the elementary theory of Taylor series expansions this implies that u is a finite linear combination of Dirac's delta functional and its (distributional) derivatives. All these derivatives are *homogeneous*, but except for delta itself are not actually $\mathbf{R}^\times$-invariant. Thus, u is some multiple of the Dirac delta. ///