Algebra Prelim Written Exam *Spring 2017: discussion*

[This document is
http://www.math.umn.edu/~garrett/prelim_discussion/algebra/spring_2017_discussion.pdf

[1] Determine the abelian groups of order 900.
Discussion: Structure theorem for finitely-generated abelian groups, with some care about confidently listing *all*.

[2] Let p be the smallest prime dividing the order of a finite group G, and suppose G has a subgroup H of index p. Show that H is *normal*.
Discussion: Iconic.

[3] Show that the ideal I in $\mathbb{Z}[x]$ generated by 11 and x^2+1 is maximal.
Discussion: Show that the quotient is a field, which by an isomorphism theorem asks to show that x^2+1 is irreducible modulo 11, which is best confirmed by observing that x^2+1 is the fourth cyclotomic polynomial, while $\mathbb{Z}/11^\times$ is cyclic of order 10, so by Lagrange's theorem there cannot be any element of order 4.

[4] Let S,T be diagonalizable operators on a finite-dimensional complex vector space V. Suppose that $ST=TS$. Show that there is a basis for V consisting of simultaneous eigenvectors for all S,T.
Discussion: :)

[5] Show that $x^5+y^7+z^{11}$ is irreducible in $\mathbb{C}[x,y,z]$.
Discussion: Eisenstein's criterion plus Gauss' lemma, with some attention to $\mathbb{C}[x,y,z] \approx \mathbb{C}[x,y][z]$ and inclusions to $\mathbb{C}(x)[y][z]$, or similar.

[6] Exhibit a finite field with 32 elements.
Discussion: Best is to exhibit an irreducible quintic in $\mathbb{F}_2[x]$, but also saying "splitting field (in an algebraic closure of" $x^{32}-x$, or, also, the set of all zeros (in an algebraic closure) of $x^{32}-x$.

[7] Let T be a diagonalizable operator on a finite-dimensional complex vector space V. Given an eigenvalue λ of T on V, show that there is a polynomial $P \in \mathbb{C}[x]$ such that $P(T)$ is the projector to the λ-eigenspace.
Discussion: Lagrange interpolation!

[8] Explicitly determine all fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$, where ζ is a primitive 12^{th} root of unity.

Discussion: Obviously about Galois theory's (inclusion reversing) bijection between intermediate fields and subgroups of the Galois group. It is reasonable to produce elements of subfields by *averaging* and/or otherwise looking for things invariant under subgroups. However, a thing invariant under a subgroup may accidentally be invariant under a larger subgroup, so not reliably generate the anticipated intermediate field. In the example at hand, because of the restricted class of subgroups, this problem is easy to avoid, but needs mention.