Real Analysis Prelim Written Exam *Spring 2017 discussion*

[1] Given $\varepsilon > 0$, construct an open set $U \subset \mathbb{R}$ containing $\mathbb{Q}$ and with Lebesgue measure less than ε.

Discussion: Keyword *countable subadditivity.*

[2] Give an example of a sequence $\{f_n\}$ of continuous functions on $[0, 1]$ such that $\lim_n f_n(x) = 0$ for all $x \in [0, 1]$, but $\int_0^1 f_n(t)\, dt = 1$ for all n.

Discussion: For example, narrowing tents centered at $1/n$, of width $2/n$, and height $n/2$.

[3] Suppose that $f \in L^1(\mathbb{R})$ and $\int_a^b f(x)\; dx = 0$ for all $a, b \in \mathbb{R}$. Show that $f(x) = 0$ almost everywhere.

Discussion: Key idea: outer and inner regularity of Lebesgue measure (cf. Riesz-Markov-Kakutani theorem).

[4] Let $f \in L^2[0, 1]$ be differentiable *almost everywhere*, with derivative $f' \in L^2[0, 1]$. Show that there is a constant C such that $|f(x) - f(y)| < C \cdot |x - y|^{\frac{1}{2}}$ for $x, y \in [0, 1]$.

Discussion: Yes, this question is a bit less traditional (!?) than most of the others.

Yes, if we can invoke the fundamental theorem of calculus, then the assertion follows immediately from the Cauchy-Schwarz-Bunyakowsky inequality.

Yet we know the Cantor's/devil's staircase as an example of failure of a version of the fundamental theorem of calculus. That is, we'd need *absolute continuity.*

Yet, evidently, the distributional derivative's being in L^2 has a subtle impact on the situation! One key point is that f' must be thought of distributionally. That is, a *pointwise* notion of derivative leads to nonsense: the derivative of the Cantor's staircase function is not in L^2, etc.

Rather, ideally, we should realize that $|f|^2_{L^2} + |f'|^2_{\mathrm{L}^2}$ is the norm-squared on the Hilbert space $H^1[0, 1]$ of Sobolev-index $+1$ functions on the interval: $H^1[0, 1]$ is the collection of *distributions* f that are given by integration against an L^2 function, and whose (distributional!) derivatives are given by integration against an L^2 function. Further, we should know that *smooth* functions are *dense* in this (or any good) space of functions.

Thus, it suffices to prove that there is a uniform constant C such that

$$\sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|} \;\leq\; |f|_{H^1}$$

for f *smooth*. This *does* follow from the fundamental theorem of calculus.

[5] Let $C = \{v = (v_1, v_2, \ldots) \in \ell^2 : |v_n| \leq \frac{1}{n}\} \subset \ell^2$. Show that C is compact.

Discussion: This set is the *Hilbert cube*, and this result is standard. Best to invoke the *total boundedness* criterion for (pre-) compactness rather than slogging through the *sequential* version of compactness! Discussions of this type of question are in my homework-example discussions.

[6] Let E be a Lebesgue measurable subset of $[0,1]$, and let $f(x) = \int_E \sin(tx)\; dt$. Show that $f(x)$ is continuous.

Discussion: I'd meant this to be a one-liner invoking Riemann-Lebesgue (that Fourer transforms of L^1 functions are continuous and going to 0 at infinity), but it is also possible to treat this issue directly, albeit not having a one-line proof.

[7] For $1 < p < \infty$, $f \in L^p(\mathbb{R})$, and $g \in L^1(\mathbb{R})$, show that $|f * g|_{L^p} \leq |f|_{L^p} \cdot |g|_{L^1}$.

Discussion: This is a simple version of *Young's inequality* (or *Hausdorff-Young*). Yes, it uses Hölder's inequality, but also some algebra tricks using $\frac{1}{p} + \frac{1}{q} = 1$. This is also in my homework-example discussions.

[8] Show that $C^1[a,b]$ with norm $|f|_{C^1} = \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)|$ is a Banach space.

Discussion: First, this should be an iconic issue. Yes, for Cauchy sequence f_n in this metric, the sup-norm limit f of f_n is continuous (by a standard argument), and the sup-norm limit g of f'_n is continuous. What is not obvious is that f is *differentiable* and has derivative g. It seems infeasible to prove this bare-handedly, but using the fundamental theorem of calculus makes it work.
