1. Logistics

I will be available for office hours for this class for the coming week only. The times are

- Friday 3:45 to 5:30.
- Monday 3:30 to 6:00
- Tuesday 3:30 to 5:00

Since these are not my normal office hours, I request that you contact me by email if you plan to come to my office. If I don’t hear from anyone I will probably leave by 4:30 on those days. In addition to these extra hours, I have my normal office hours for FM 5001. You may come to my office during those times, but students from FM 5001 will have priority:

- Tuesday 5:00 to 6:00
- Wednesday 10:00 to 11:00 and 4:00 to 5:00
- Thursday 4:00 to 5:00

My email address is gray@math.umn.edu and my office is Vincent 239, phone number 612 624-4813.

2. Brownian motion

In this section, we want to try to gain a basic understanding of Brownian motion, which is a stochastic process that we denote by $B_t$, $t \geq 0$.

In this notation, $t$ is the time variable and $B_t$ is the “position” of the Brownian motion at time $t$. In the background, there is a probability space $(\Omega, \mathcal{F}, P)$, and for each $t$, $B_t$ is a random variable, which means that it is a measurable function $B_t : \Omega \to \mathbb{R}$. Occasionally this is made more explicit by writing $B_t(\omega)$ for the random position of the Brownian motion at time $t$. Let’s look at three different ways to describe Brownian motion.

2.1. Qualitative description of Brownian motion. The stochastic process $B_t, t \geq 0$ is a standard Brownian motion if it satisfies the following requirements:

1. $B_0(\omega) = 0$ for all $\omega \in \Omega$. 

(2) For each \( \omega \in \Omega \), the function \( t \mapsto B_t(\omega) \) is continuous for \( t \geq 0 \).

(3) The increments \( B_t - B_s \) are independent on non-overlapping time intervals \([s, t)\).

(4) The distribution of the increment \( B_t - B_s \) depends only on \( t - s \).

(5) \( E(B_1) = 0 \) and \( E(B_1^2) = \text{Var}(B_1) = 1 \).

These requirements imply the following:

(6) The random variables \( B_t, t \geq 0 \) are jointly Gaussian, or in other words, Brownian motion is a Gaussian process.

(7) \( E(B_t) = 0 \) for all \( t \geq 0 \) and \( E(B_sB_t) = \text{Cov}(B_s, B_t) = \min\{s, t\} \) for all \( s, t \geq 0 \).

2.2. Brownian motion as a Gaussian process. Another way to define the standard Brownian motion is to say that it is a stochastic process \( B_t, t \geq 0 \) such that (2), (6), and (7) are satisfied. These three requirements completely determine all of the probabilities associated with Brownian motion, and in particular, they imply the remaining requirements (1), (3), (4), and (5). This is a useful point of view because it provides us with a joint probability density function for any finite number of the random variables \( B_t \).

2.3. Brownian motion as a limit of random walks. Let \( Y_1, Y_2, Y_3, \ldots \) be independent identically distributed random variables with mean 0 and variance 1. Define \( R_0 = 0 \) and

\[
R_k = Y_1 + Y_2 + \cdots + Y_k
\]

for \( k = 1, 2, 3, \ldots \). Then for each \( n = 1, 2, 3, \ldots \), define

\[
S_n(t) = \begin{cases} 
R_{nt}/\sqrt{n} & \text{if } t \text{ is a multiple of } 1/n \\
\text{by linear interpolation} & \text{otherwise.}
\end{cases}
\]

Note that for each \( n \), this process satisfies requirements (1), (2), and (5), and that it satisfies (3), (4) and (7) as long as the times are restricted to multiples of \( 1/n \). For large \( n \), it also approximately satisfies requirement (6) (because of the Central Limit Theorem), and if we choose the common distribution of the random variables \( Y_1, Y_2, Y_3, \ldots \) to be Gaussian, then (6) is fully satisfied. So we would expect that the stochastic process \( S_n(t), t \geq 0 \) converges in some sense as \( n \to \infty \) to a Brownian motion. This is indeed the case, but it requires a lot of advanced mathematics to state it precisely and to prove it.
2.4. *Some properties of Brownian motion.*

2.4.1. *Self-similarity.* Let’s define a new process. Fix $T > 0$ and let

$$
\tilde{B}_t = B(Tt)/\sqrt{T}, t \geq 0.
$$

It is not hard to show that this process satisfies (2), (6), and (7), so it is a standard Brownian motion. This relationship is sometimes called a *self-similarity* property of Brownian motion.

2.4.2. *Roughness.* Even though Brownian motion is continuous as a function of time, its graph is very rough. This fact is a consequence of the following:

$$
\frac{B_{t+\Delta t} - B_t}{\sqrt{\Delta t}} = \frac{\Delta B_t}{\sqrt{\Delta t}} \sim \mathcal{N}(0,1).
$$

This means that a typical increment of Brownian motion over a time interval of length $\Delta t$ is on the order of $\sqrt{\Delta t}$, which is much, much larger than $\Delta t$ if $\Delta t$ is small. This obviously causes a problem if we try to differentiate Brownian motion with respect to time. In fact, it can be shown that a Brownian motion path is *nowhere differentiable*, and that the graph of Brownian motion, as a curve, has infinite length over any time interval $[s,t]$ for $t > s$.

Here is some informal mathematics that shows an important consequence of the roughness of Brownian motion. Let $f$ be a nice function. Then from the second-order Taylor expansion, we have

$$
f(B_{t+\Delta}) - f(B_t) \approx f'(B_t) \Delta B_t + \frac{1}{2} f''(B_t) (\Delta B_t)^2.
$$

If we add up a sequence of terms like this over a time interval $[r,s]$, we get $f(B_s) - f(B_r)$ on the left side. If Brownian motion were differentiable, then we would get a Riemann sum for the integral

$$
\int_r^s f'(B_t) B'_t \, dt
$$

from the first term on the right, and the second term would typically be very small. But because $(\Delta B_t)^2$ is like $\Delta t$, we can’t differentiate $B_t$ with respect to $t$, and furthermore, the second term does not get small. Instead, it also gives us something looking like a Riemann sum, for the integral

$$
\int_r^s \frac{1}{2} f''(B_t) \, dt.
$$

One way to express this result is

$$
df(B_t) = f'(B_t) dB_t + \frac{1}{2} f''(B_t) \, dt.
$$
If \( B_t \) were not so rough, the equation would have been \( df(B_t) = f'(B_t) dB_t \), but the roughness produces an extra term. We have been highly informal here, but it turns out that this result is correct, in a sense that will be made precise in the next two weeks (Ito’s Lemma).

2.4.3. Functions of Brownian motion. In all of the following, \( B_t, t \geq 0 \) is a standard Brownian motion.

- **Brownian motion with drift** is the stochastic process \( \sigma B_t + \mu t \), where \( \sigma > 0 \) and \( \mu \in \mathbb{R} \). The constant \( \sigma \) is simply a scaling factor that can be thought of as indicating the strength of the fluctuations. The constant \( \mu \) is called the drift. It is easy to see that this process is a Gaussian process with mean \( \mu t \) at time \( t \), and covariance \( \sigma^2 \min\{s,t\} \).

- **Brown bridge** is the stochastic process \( B_t - tB_t, t \in [0,1] \). This is also a Gaussian process. It starts at 0 at time 0 and ends at 0 at time 1, but fluctuates a lot like standard Brownian motion in between.

- **Geometric Brownian motion** is the stochastic process \( \exp(\sigma B_t + \mu t) \).

  This is not a Gaussian process. It is very important in financial mathematics.

- Let \( M_t = \max\{B_s, 0 \leq s \leq t\} \).

  Since Brownian motion is continuous as a function of \( t \), this process is well-defined. I claim the following:

  \[
P(M_t \geq c) = 2P(B_t \geq c) \quad \text{for } c \geq 0.
  \]

  Here is the reasoning: if the Brownian motion gets above \( c \) at any time \( s < t \), then after that, half the time it will be above \( c \) at time \( t \), and half the time it won’t. So

  \[
P(B_t \geq c | M_t \geq c) = 1/2.
  \]

  But the event \( \{B_t \geq c\} \) is contained in the event \( \{M_t \geq c\} \), so

  \[
P(B_t \geq c | M_t \geq c) = \frac{P(B_t \geq c \text{ and } M_t \geq c)}{P(M_t \geq c)} = \frac{P(B_t \geq c)}{P(M_t \geq c)}.
  \]

  Setting this equal to \( 1/2 \) gives the desired result.

**Exercise 1.** Find the conditional density function of \( B_3 \) given \( B_1 \).

**Exercise 2.** We claimed that there is a sense in which the process \( S_n(t), t \geq 0 \) converges to a standard Brownian motion. Let’s focus just on a fixed value of \( t \). Discuss the sense in which \( S_n(t) \) converges to \( B_t \).
Does \( \lim_{n \to \infty} S_n(t) \) exist in the usual sense of limits of sequences of real numbers? If not, can you give a precise sense in which there is some type of convergence?

**Exercise 3.** Calculate the probability that \( \{B_1 \geq 1\} \cap \{B_2 \geq 1 + B_1\} \cap \{B_3 \geq 1 + B_2\} \).

**Exercise 4.** Prove that \( \tilde{B}_t, t \geq 0 \) is a standard Brownian motion.

**Exercise 5.** Calculate the covariance function of Brownian bridge, and calculate the probability that Brownian bridge is at least 1/4 at time \( t = 1/3 \) and less than \(-1/4\) at time \( t = 2/3 \).

**Exercise 6.** Let \( Y_t \) be geometric Brownian motion, with \( \sigma = \mu = 1 \). Calculate \( P(Y_s \geq 3 \text{ for some value of } s \leq 1) \).

3. **\( \sigma \)-fields as representations of information**

3.1. The information represented by an event \( A \).

3.2. The information represented by a finite collection of events.

3.3. The information represented by an infinite collection of events.

3.4. The information represented by a discrete random variable.

3.5. The information represented by a random variable.

3.6. The information represented by an infinite collection of random variables.

3.7. Conditional expectations with respect to discrete \( \sigma \)-fields.

3.8. Definition and rules for general conditional expectations.

3.9. Conditional expectations using conditional density functions.
3.10. Examples.

Exercise 7. Let $B_t, t \geq 0$ be a standard Brownian motion. Calculate
\[ E(B_2(B_4 - B_2) + B_1(B_2 - B_1)|\mathcal{F}_2), \]
where $\mathcal{F}_2 = \sigma(B_s, 0 \leq s \leq 2)$.

Exercise 8. Fill in the details in the calculation that is done in Example 1.4.15 in the text, starting on page 74 with the paragraph that begins “Taking conditional expectations, we obtain . . . ” In particular, indicate exactly where the rules for conditional expectations are used.

Exercise 9. Prove Rule 1 on page 70 of the text.

Exercise 10. Let $Y_1, Y_2$ be random variables that each take only three different values. Describe $\sigma(Y_1, Y_2)$ and $E(X|\sigma(Y_1, Y_2))$ in some detail.

Exercise 11. Let $X$ be a random variable such that
\[ P(X = 1) = 1/2 \quad P(X = 2) = 1/4 \quad P(X = -1) = 1/4. \]
Calculate $E(X | X^2)$. 