1. Quiz

Problem 1. Draw a picture of the equation \( a(\vec{r} + \vec{s}) = a\vec{r} + a\vec{s} \), using \( a = -2 \), \( \vec{r} = (1, 1) \) and \( \vec{s} = (2, -1) \).

Problem 2. Use the dot product to show that any vector that lies in the plane \( 2x + y - z = -1 \) is perpendicular to the vector \( (2, 1, -1) \).

2. Linear combinations and matrix multiplication

One of the most important concepts in linear algebra is the linear combination, which can always be expressed as some sort of matrix multiplication. Let’s start with the simplest situation, where we multiply a matrix with a single row times a matrix with a single column, as in

\[
\begin{pmatrix}
x & y & z
\end{pmatrix}
\begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix} = 3x - y + 2z.
\]

Here are some things to notice:

- This matrix multiplication is the same as a dot product \( (x, y, z) \cdot (3, -1, 2) \). All matrix multiplication consists of taking dot products between the rows of the matrix on the left and the columns of the matrix on the right.
- A matrix consisting of a single row is also called a row vector, and a matrix with a single column can be called a column vector. The transpose of a row vector is a column vector. So if we want to write the above matrix multiplication within the text without using up so much space, we can write it this way: \( (x, y, z)(3, -1, 2)^T \).
- The expression \( 3x - y + 2z \) represents a linear combination. In fact, it represents two different linear combinations. First, it is the linear combination of the variables \( x, y, z \) using the coefficients \( 3, -1, 2 \), that is, it is 3 times \( x \) plus \(-1\) times \( y \) plus 2 times \( z \). Second, it is the linear combination of the real numbers \( 3, -1, 2 \) using the variable coefficients \( x, y, z \), or \( x \) times 3 plus \( y \) times \(-1\) plus \( z \) times 2, which we might write as \( x3 + y(-1) + z2 \), to emphasize that \( x, y, z \) are the coefficients...
Both points of view are important to us, so try to get used to this and try not to be confused.

- If we set our matrix multiplication equal to a constant, we get the equation for a plane. For example,

$$(x \ y \ z) \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 5.$$ 

is the equation for the plane which would be more usually expressed as $3x - y + 2z = 5$. Any point $(x, y, z)$ whose coordinates satisfy this equation is on the plane. If we think of the point $(x, y, z)$ as a vector going from the origin to the point, then the tip of the vector is on the plane.

- But we need to be careful not to get confused. Vectors only have length and direction; they can be located anywhere. If we say that a vector lies within a plane, then we usually mean that the vector has a direction that is parallel to the plane, which would allow us to locate the vector so that it lies within the plane. For example, the point $(x, y, z) = (1, 0, 1)$ is on the plane $3x - y + 2z = 5$. So if we locate the vector $(1, 0, 1)$ so that its tail is at the origin, then its tip will be on the plane. But we cannot locate this vector so that it lies within the plane. An example of a vector that lies within the plane is $(1, 1, -1)$. You can see that this vector is parallel to the plane because the dot product $(1, 1, -1) \cdot (3, -1, 2) = 0$ shows us that it is perpendicular to the vector $(3, -1, 2)$, which we know is perpendicular to the plane.

Let’s go back to our matrix product and add a second column to the matrix on the right:

$$(x \ y \ z) \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} = (3x - y + 2z, y + 2z).$$

The result of the matrix multiplication is a row vector with two elements. The first element is $3x - y + 2z$, which we have already seen. The second element is $y + 2z$, which is the result of doing the matrix multiplication $(x, y, z)(0, 1, 2)^T$.

Let’s write the right side a different way:

$$(x \ y \ z) \begin{pmatrix} 3 & 0 \\ -1 & 1 \\ 2 & 2 \end{pmatrix} = x(3, 0) + y(-1, 1) + z(2, 2).$$
Please convince yourself that this is the same as what we had in the previous equation. Then notice that it is a linear combination of the rows of the matrix
\[
\begin{pmatrix}
3 & 0 \\
-1 & 1 \\
2 & 2
\end{pmatrix}
\]
using the variable coefficients \(x, y, z\). This is a very important fact.

Here is another example:
\[
\begin{pmatrix}
x & y & z
\end{pmatrix}
\begin{pmatrix}
3 & 0 & 5 \\
-1 & 1 & -2 \\
2 & 2 & 0
\end{pmatrix}
= x(3,0,5) + y(-1,1,-2) + z(2,2,0).
\]
Again we see (check this!) that the result of the matrix multiplication is a linear combination of the rows of the second matrix, using the elements of the first matrix (row vector) as coefficients.

Similarly, if we consider the matrix multiplication
\[
\begin{pmatrix}
3 & 0 \\
-1 & 1 \\
2 & 2
\end{pmatrix}
\begin{pmatrix}
y_1 & y_2
\end{pmatrix}
= y_1 \begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix}
+ y_2 \begin{pmatrix}
0 \\
1 \\
2
\end{pmatrix}.
\]
we see that the right side of the equation is a linear combination of the columns of the first matrix, with \(y_1, y_2\) as coefficients.

Here is the general situation:

**Proposition 1.**

(i) Let
\[
A = \begin{pmatrix}
\leftarrow & \vec{r}_1 & \rightarrow \\
\leftarrow & \vec{r}_2 & \rightarrow \\
\vdots & \vdots & \vdots \\
\leftarrow & \vec{r}_m & \rightarrow
\end{pmatrix}
\]
be a matrix whose rows are the row vectors \(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m\) and let \(\vec{a} = (x_1 \ x_2 \ \ldots \ x_m)\) be another row vector. Then
\[
\begin{pmatrix}
x_1 & x_2 & \ldots & x_m
\end{pmatrix}
A = x_1 \vec{r}_1 + x_2 \vec{r}_2 + \cdots + x_m \vec{r}_m.
\]

(ii) Let
\[
B = \begin{pmatrix}
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\end{pmatrix}
\begin{pmatrix}
\vec{s}_1 & \vec{s}_2 & \ldots & \vec{s}_n
\end{pmatrix}
\]
be a matrix whose columns are the column vectors $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n$ and let $\vec{y} = (y_1 \ y_2 \ \cdots \ y_n)^T$ be another column vector. Then

$$B\vec{y} = y_1\vec{s}_1 + y_2\vec{s}_2 + \cdots + y_n\vec{s}_n.$$  

Be sure that you understand what this proposition says, because it is really important.

**Exercise 1.** Calculate the following matrix products:

$$\begin{pmatrix} 3 & -2 & 0 \\ 5 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -4 \end{pmatrix}.$$  

**Exercise 2.** Express the vector $(2, 2, 2)$ as a linear combination of the vectors $(1, 0, -1)$ and $(5, 1, -3)$, and then write this linear combination in two ways: as the product of a row vector and a $2 \times 3$ matrix, and as a product of a $3 \times 2$ matrix and a column vector.

**Exercise 3.** Show that the vector $(1, 2, 2)$ cannot be expressed as a linear combination of the vectors $(1, 0, -1)$ and $(4, 3, 2)$.

3. **Matrices as linear transformations**

Let $A$ be an $m \times n$ matrix, which is to say that $A$ has $m$ rows and $n$ columns. We have seen that if we multiply $A$ on the left by a row vector with $m$ elements, then we get a row vector with $n$ elements. So we can think of $A$ as a function that turns row vectors in $\mathbb{R}^m$ into row vectors in $\mathbb{R}^n$. Similarly, $A$ can be thought of as a function that turns column vectors in $\mathbb{R}^n$ to column vectors in $\mathbb{R}^m$. For example:

$$(2 \ 1) \begin{pmatrix} 4 & 0 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 4 & 0 & 3 & -1 \end{pmatrix} + 1 \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 1 & 7 & -1 \end{pmatrix}$$

shows how a $2 \times 4$ matrix turns a row vector in $\mathbb{R}^2$ into a row vector in $\mathbb{R}^4$, and

$$\begin{pmatrix} 4 & 0 & 3 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

shows $A$ turning a column vector in $\mathbb{R}^4$ into a column vector in $\mathbb{R}^2$.

Because of the distributive law for matrix multiplication, we know that the functions we have just described are linear. For example $A(c_1\vec{y}_1 + c_2\vec{y}_2) = c_1A\vec{y}_1 + c_2A\vec{y}_2$ for scalars $c_1, c_2$ and column vectors $\vec{y}_1, \vec{y}_2$. So an $m \times n$ matrix represents both a linear function from $\mathbb{R}^m$
to $\mathbb{R}^n$ and a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. For some reason, these functions are called \textit{transformations} instead of functions.

Let’s focus first on thinking of an $m \times n$ matrix $A$ as a linear transformation that turns row vectors in $\mathbb{R}^m$ into row vectors in $\mathbb{R}^n$. More precisely, we say that $A$ \textit{represents} a linear transformation $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$. If $\vec{x}$ is a row vector in $\mathbb{R}^m$, then

$$L(\vec{x}) = \vec{x}A.$$ 

The domain of $L$ is $\mathbb{R}^m$ and the range of $L$ is $\mathbb{R}^n$.

You can use any row vector in $\mathbb{R}^m$ as an input to the linear transformation $L$, but we cannot say in general that every row vector in $\mathbb{R}^n$ is a possible output of $L$. Let’s take another look at this equation, from Proposition 1:

$$(x_1 \ x_2 \ \ldots \ x_m) \ A = x_1 \vec{r}_1 + x_2 \vec{r}_2 + \cdots + x_m \vec{r}_m.$$ 

What we see is that the outputs of $L$, which we call the \textit{image} of $L$, consists of all possible linear combinations of the rows of $A$. The language for the set of all possible linear combinations is \textit{linear span}.

So

The image of $L$ is the linear span of the rows of $A$.

Similarly, if we look at column vectors in $\mathbb{R}^n$ and focus on how multiplication by $A$ turns them into column vectors in $\mathbb{R}^m$, we see that $A$ represents a linear transformation $R : \mathbb{R}^n \rightarrow \mathbb{R}^m$, defined by

$$R(\vec{y}) = A\vec{y}$$ 

for column vectors $\vec{y} \in \mathbb{R}^n$. The domain of $R$ is $\mathbb{R}^n$, the range of $R$ is $\mathbb{R}^m$, and

The image of $R$ is the linear span of the columns of $A$.

\textbf{Exercise 4.} Suppose that $\vec{x}$ and $\vec{y}$ are both in the linear span of the vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$. Show that any linear combination of $\vec{x}$ and $\vec{y}$ is also in the linear span of $\vec{r}_1, \vec{r}_2, \vec{r}_3$.

\textbf{Exercise 5.} Let $A$ be a $4 \times 3$ matrix and let $B$ be a $4 \times 4$ matrix. (Note that the product $BA$ is a $4 \times 3$ matrix.) Let $L$ be the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by $L(\vec{x}) = \vec{x}A$, and let $M$ be the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by $M(\vec{x}) = \vec{x}BA$. Show that the image of $M$ is contained in the image of $L$.

\textbf{Exercise 6.} Let $A, B, L, \text{ and } M$ be as in the previous exercise, and assume that $B$ is an invertible matrix. That means that there exists a $4 \times 4$ matrix $C$ such that $CB = BC = I$, where $I$ is the identity
matrix:

\[ I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Show that the image of \( L \) is the same as the image of \( M \).

4. Application to financial mathematics

Here is a situation that illustrates why these ideas are important in financial mathematics. Suppose I have 3 different investment opportunities to choose from, with corresponding net profit vectors \((-10, 10, -20, 10), (0, -30, 50, 10), (-20, 10, 30, -30)\). These vectors show the net profit that I can get from one unit of investment, under four different possible outcomes in the future. For example, if the first outcome actually happens, then one unit of the first investment gives me a profit of \(-10\), I don’t get any profit or loss from the second investment, and one unit of the third investment gives me a profit of \(-20\). Let \( A \) be the matrix whose rows are these three row vectors. I can form a portfolio from the three investment opportunities by taking linear combinations of the rows of \( A \). We can express this in matrix form:

\[
\vec{x} A = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -10 & 10 & -20 & 10 \\ 0 & -30 & 50 & 10 \\ -20 & 10 & 30 & -30 \end{pmatrix}.
\]

This represents a portfolio in which I have \( x_1 \) units of the first investment, \( x_2 \) units of the second investment, and \( x_3 \) units of the third investment. That’s because we know that this matrix product is the same as the linear combination

\[ x_1 (-10 \ 10 \ -20 \ 10) + x_2 (0 \ -30 \ 50 \ 10) + x_3 (-20 \ 10 \ 30 \ -30). \]

Now the question is: what are the possible net profit vectors for the portfolios that I can create? This is the same as asking: What is the linear span of the rows of the matrix \( A \)? Answering this kind of question is important when you are trying to hedge against certain outcomes. For example, suppose you want to create a portfolio that will give you a net profit of 10 if the fourth outcome occurs, and otherwise has a net profit of \(-5\). Can you do it? That’s the same as trying to find a solution to the following:

\[
\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -10 & 10 & -20 & 10 \\ 0 & -30 & 50 & 10 \\ -20 & 10 & 30 & -30 \end{pmatrix} = \begin{pmatrix} -5 & -5 & -5 & 10 \end{pmatrix}.
\]
Equivalently, we look for a solution to the following system of 4 equations in 3 unknowns:

\[
\begin{align*}
-10x_1 - 20x_3 &= -5 \\
10x_1 - 30x_2 + 10x_3 &= -5 \\
-20x_1 + 50x_2 + 30x_3 &= -5 \\
10x_1 + 10x_2 - 30x_3 &= 10
\end{align*}
\]

Using the elimination method, or Matlab, we can quickly see that this system of equations has no solution.

Let’s add another possible investment opportunity: \((1, 1, 1, 1)\). This is called a “savings account”. We put 1 dollar in savings, and under all four possible outcomes, we still have 1 dollar in the future. (We are assuming that the savings account is guaranteed, and that there is no interest.) Can we now find a way to create the hedge \((-5, -5, -5, 10)\)? That is, can we solve the following?

\[
\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} -10 & 10 & -20 & 10 \\
0 & -30 & 50 & 10 \\
-20 & 10 & 30 & -30 \\
1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -5 & -5 & 10 \end{pmatrix}.
\]

The answer is “Yes”. We have 4 equations in 4 unknowns, and the equations are “independent”. In fact, no matter what we have on the right side, we can find a portfolio that will produce that vector.

What is the cost of such a portfolio? The first three profit vectors are net profit, so they don’t cost us anything. The fourth profit vector is our savings account, so its cost is equal to \(x_4\). This means that \(x_4\) is the cost of the portfolio.

Now let’s look at this another way. Consider the matrix equation

\[
\begin{pmatrix} -10 & 10 & -20 & 10 \\
0 & -30 & 50 & 10 \\
-20 & 10 & 30 & -30 \\
1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\
p_2 \\
p_3 \\
p_4 \end{pmatrix} = \begin{pmatrix} 0 \\
0 \\
0 \\
1 \end{pmatrix}.
\]

This system of 4 equations in 4 unknowns also has a solution. Let’s multiply on the left by our solution to the previous system of equations:

\[
\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} -10 & 10 & -20 & 10 \\
0 & -30 & 50 & 10 \\
-20 & 10 & 30 & -30 \\
1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\
p_2 \\
p_3 \\
p_4 \end{pmatrix} = x_4.
\]
This implies that
\[
\begin{pmatrix}
-5 & -5 & -5 & 10
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{pmatrix}
= x_4.
\]

So we can find the cost of our hedge portfolio (or any other portfolio) by solving Equation (1) for the vector \( \vec{p} = (p_1, p_2, p_3, p_4)^T \), and then calculating the dot product of our portfolio with \( \vec{p} \).

If it happens that all of the elements of \( \vec{p} \) are positive, then this gives us a probability vector, and taking the dot product with \( \vec{p} \) is the same as finding the expected value of the portfolio using these probabilities. If \( \vec{p} \) has any elements that are not positive, then we can create a portfolio with no negative elements and at least one positive element, and the portfolio costs nothing! This would be arbitrage. For example, if we discovered that our solution is \( \vec{p} = (-1, 1/3, 1/3, 4/3)^T \), then our costless risk-free money-making portfolio could be \( (1, 1, 1, 1/4) \). No matter which of the four outcomes actually occurred, we would make money, at no cost to ourselves. We have almost indicated how to prove the No Arbitrage Theorem from the last set of notes. Here it is again:

**Theorem 1** (No Arbitrage). Either there exists a portfolio vector \((x_1, \ldots, x_m)\) such that \( \nu_i \geq 0 \) for all \( i = 1, \ldots, n \) and \( \nu_i > 0 \) for at least one value of \( i \), or there exists a strictly positive probability vector \((p_1, \ldots, p_n)^T\) such that \( \mu_j = 0 \) for all \( j = 1, \ldots, m \).

Here is how the proof goes. Let’s suppose there exists a strictly positive probability vector \( \vec{p} \) as described, which we write as a column vector. Then
\[
A' \vec{p} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix},
\]
where \( A' \) is the matrix we get by adding a row of 1’s to the bottom of \( A \). If \( \vec{x} \) is the portfolio vector described in the theorem, let \( \vec{x}' \) be the vector you get by appending an extra element \( x_{m+1} \) onto \( \vec{x} \). This corresponds to the amount that we put into the savings account, so it is the cost of our portfolio. Note that \( \vec{x}' A' \vec{p} = x_{m+1} \) and that \( \vec{x} A \vec{p} = 0 \). Since \( \vec{p} \) has strictly positive elements, the net profit vector for the portfolio \( \vec{x} \), which is \( \vec{x} A \), must have at least one negative element, so there is no arbitrage.
(For the rest of the proof, see next week’s notes.)

Exercise 7. Consider the following three investment opportunities, showing the net profit under four possible outcomes: $(-20, 20, 20, 20)$, $(0, 20, -10, -10)$, $(30, -40, -20, -30)$. Show that these do not offer the possibility of arbitrage, and determine the price of the following investment opportunity: $(20, 0, 30, 0)$.

Exercise 8. Consider the following three investment opportunities: $(20, -40, 30, 30)$, $(0, 10, -10, 0)$, $(30, -10, 50, -10)$. Show that they offer the possibility of arbitrage, and find a nonzero portfolio consisting of these three investments that cannot lose money.