1. Introduction

Each week, I will hand out notes to accompany my lectures for FM 5001. The purpose of this course is to prepare students for the mathematics that they will encounter in later FM courses. The main topics that we will discuss in FM 5001 are:

- First year calculus, including the basics of differentiation, integration, and power series. We will review the basic rules and techniques of differentiation and integration, but will not be concerned with doing lots of calculations. It will be more important for us to understand why these rules work, how they relate to each other, and what they tell us about the world around us. Power series will be seen as a fundamental tool for making functions simpler.
- Linear algebra, including vectors, coordinate systems, matrices, Gaussian elimination, linear transformations, eigenvalues and eigenvectors, diagonalization, matrix decomposition, principal component analysis, linear programming. We will learn the calculations that are involved in doing linear algebra, but we will usually rely on Matlab to do the tedious work. We will also see how linear algebra can help us understand some simple option pricing models in financial mathematics.
- Ordinary differential equations, focusing on first-order linear differential equations. We want to understand this basic case inside and out! It is essential to understanding Black-Scholes theory.

The main focus of FM 5002 will be multivariable calculus, elementary probability theory, together with a small amount of partial differential equations. In both semesters, we will include some introductory material from financial mathematics, particularly concerning the pricing of options in simple cases.

There is no textbook for FM 5001. There will be a textbook for the probability material in FM 5002. Many of you will want to get books for calculus and linear algebra review. Just about any book will do. If you have such books from previous courses, that will be fine. Otherwise, it would be more than sufficient to get the Schaum’s Outline books on
Calculus and Linear Algebra. There are lots of other Calculus review texts that you can get at amazon. The most popular one seems to be *Calculus for Dummies*, but it only covers single-variable calculus. Two possibilities that would including both single-variable and multi-variable calculus are *Calculus Know-It-All* by Stan Gibilisco and *How to Ace Calculus: the streetwise guide* plus *How to Ace the Rest of Calculus*, by Colin Adams, Abigail Thompson and Joel Hass.

2. Calculus prerequisites for FM 5001

In this section, I will give you a list of things you should have learned about in previous single-variable calculus courses. Please note that I do not expect that you remember all of this stuff. My main requirement is that most of this does not sound completely unknown to you.

2.1. Functions. There is a basic list of real-valued functions that will constantly be important to us.

- **Linear functions and affine linear functions.** Here are some typical linear functions of one and several variables, expressed in different ways that are commonly encountered:

  \[ f(x) = 5x \quad y = -\frac{3}{2}x \quad h(x, y, z) = 4x - y + z \quad z = ax + by . \]

  If you add a constant to a linear function, you get an affine linear function. Many people (including me) get a little sloppy, and often refer to affine linear functions as linear functions:

  \[ y = mx + b \quad g(x, y) = -2x - 5y + 10 \quad z = \frac{x - y + 1}{5}. \]

  Make sure that you understand these examples, so that you can distinguish linear functions from affine linear functions, and can correctly identify when a function is neither linear nor affine-linear.

- **Quadratic functions.**

  \[ f(x) = -x^2 + 3x - 1 \quad y = x^2 \quad h(x, y) = x^2 - \frac{xy}{2} \quad w = (x+y+z)^2 + 6. \]

  A quadratic function of a single variable can always be written as \( f(x) = ax^2 + bx + c \). If \( a = 0 \) in this expression, then we get an affine linear function, which can be thought of as a (degenerate) special case of quadratic functions. Here is a general expression for a quadratic function of three variables:

  \[ h(x, y, z) = ax^2 + bxy + cxz + dy^2 + eyz + f z^2 + gx + hy + jz + k, \]
where $a, b, c, d, e, f, g, h, j, k$ are constants. We will eventually see why such a complicated expression is not needed. In linear algebra, we will learn that there is an affine linear change of variables that can be used to transform this expression into the much simpler $u^2 + v^2 + w^2 + C$, where $u, v, w$, are the new variables and $C$ is a constant. For quadratic functions of a single variable, this relates to the familiar idea of “completing the square”:

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}.$$  

Notice that if you set this last expression equal to 0 and solve for $x$, you get the quadratic formula!

- **Polynomials.** These include quadratic, linear, and affine linear functions as special cases. Here are some polynomials in a single variable:

$$y = 5x^3 - x + 16 \quad f(x) = \frac{(3x - 2)^5}{10} \quad g(x) = ax^4 + bx^3 + cx^2 + dx + e.$$  

We will primarily be interested only in the single-variable case for polynomials of degree higher than two, but here is one example of a third degree polynomial in two variables:

$$z = x^3 - 3x^2y + 2xy^2 - 5y^3 + 10x^2 - xy + y^2 + x + 2y - 6.$$  

The first four terms in this expression are the “cubic” or “third degree” terms. After that, the next three terms are the “quadratic” or “second degree” terms, followed by the two linear terms, and then finally the constant term. The last three terms could be called the “affine linear” terms.

- **Power functions of a single variable.** These are very simple: $y = ax^p$, where $a$ is a real coefficient and $p$ is the power. The most common example is $y = \sqrt{x}$, for which $p = 1/2$.

- **Exponential functions of a single variable.**

$$f(x) = 3e^{-2x} \quad y = 2^x \quad g(x) = \frac{2}{10^{2x}}.$$  

All of these can be expressed in the form

$$y = ae^{bx}$$  

by choosing the constant $a, b$ appropriately. For example,

$$\frac{2}{10^{2x}} = 2 \cdot 10^{-2x} = 2(e^{\log 10})^{-2x} = 2e^{-2(\log 10)x}.$$
Throughout these notes, we will write log for the natural logarithm, rather than ln. If we want to write a logarithm with base 10, we will write \( \log_{10} \), and similarly for other bases.

- **Logarithmic functions of a single variable.**
  
  \[ f(x) = 5 \log(6x) \quad y = \log_2 15x. \]

  One reason these are important is that they are the inverses of the exponential functions. This fact was used in our brief discussion of exponential functions.

- **Rational functions.** A rational function is simply the ratio of two polynomials, such as
  
  \[ h(x, y) = \frac{x^2 + y^2}{2x}. \]

  They arise naturally when you do arithmetic with polynomials. One should not get too worried about all the possibilities that can arise with rational functions. The main thing to remember about them is that they often have values of \( x \) where they “blow up” because of a 0 in the denominator. For example, the function \( f(x) = (3x^2 - x + 1)/(x + 3) \) has a problem at \( x = -3 \).

- **Trigonometric functions of a single variable.**
  
  \( \sin x \quad \cos x \quad \tan x \)

  are the three most important ones, and these can of course appear with coefficients, such as \( y = 6 \cos(-3\pi x) \).

- **Inverse trig functions of a single variable.** The most important are

  \( \arcsin x \quad \arctan x \),

  which will also often appear with coefficients.

- **Sums, differences, products, quotients, and compositions of the above.** For example, a very important function will be

  \[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2} = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}, \]

  the standard normal probability density function, whose graph is often called the “bell-shaped curve”. You can think of this function as the composition of an exponential function and a quadratic function. Composition is also the most common way that we get exponential, logarithmic, and trigonometric functions of several variables, such as \( \cos(x + y) \), which is the composition of \( \cos \) with a linear function of \( x, y \), and
exp(-(x^2 + y^2)), which is the composition of an exponential with a quadratic.

It will be good for you to review this list often during the semester. The most important functions on the list are:

\[ x \quad x^2 \quad \sqrt{x} \quad \frac{1}{x} \quad e^x \quad \log x \quad \sin x \quad e^{-x^2} \]

You should become very familiar with these functions and their graphs, and then learn how to transform them into the other functions on the list.

2.2. Differentiation. In some previous course, you should have learned the basic rules of differentiation (see below), and using those rules, you should be able to differentiate all the single-variable functions that are listed in the previous sub-section. Here are the rules. We will spend some time talking about them, so that you can see why they work.

- **Linearity.** If \( f \) and \( g \) are differentiable functions, then \( (c_1 f(x) + c_2 g(x))' = c_1 f'(x) + c_2 g'(x) \) for all constants \( c_1, c_2 \).

- **Product rule.** If \( f \) and \( g \) are differentiable functions, then \( (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \).

- **Power rule.** \( (x^p)' = px^{p-1} \) for any power \( p \).

- **Quotient rule.** If \( f \) and \( g \) are differentiable and \( g(x) \neq 0 \), then
  \[
  \left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
  \]

- **Chain rule.** \( (f(g(x)))' = f'(g(x))g'(x) \).

Can you prove the quotient rule by using the other rules?

One you have these rules, you should be able to differentiate all single variable polynomials and rational functions, and once you also know the derivatives of the basic exponential, logarithmic, and trigonometric functions, then these rules make it relatively easy to differentiate all the functions that are of interest to us. Here are the derivatives you need:

\[ (\exp(x))' = \exp(x) \quad (\log x)' = \frac{1}{x} \quad (\sin x)' = \cos x \quad (\cos x)' = -\sin x. \]

2.3. Integration. In some previous course, you should have learned the basic rules of integration, the indefinite integrals of single variable polynomials, exponential functions, and basic trigonometric functions. You should have also learned about two important techniques: integration by substitution (which is the reverse of the chain rule), and integration by parts (which is the reverse of the product rule). You
should be aware that there is a difference between an indefinite integral and a definite integral. And finally, you should have learned the relationship between integration and differentiation (the Fundamental Theorem of Calculus). It is likely that you have forgotten much of this, but you should have been exposed to it at some time in your past.

Here are the basic rules of integration:

- **Linearity.**
  \[
  \int_a^b (c_1 f(x) + c_2 g(x)) \, dx = c_1 \int_a^b f(x) \, dx + c_2 \int_a^b g(x) \, dx.
  \]

- **Combining intervals of integration.**
  \[
  \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx.
  \]

- **The constant of integration.** If \(F(x)\) and \(G(x)\) are both indefinite integrals of \(f(x)\) then \(F(x) - G(x)\) is a constant. That’s why you often see something like \(\int x \, dx = x^2/2 + C\) in a formula for an indefinite integral. The “constant of integration” \(C\) cancels out when a definite integral is evaluated. For example, if you use the function \(x^2/2\) as the indefinite integral of \(x\) and I use the function \(1 + x^2/2\) as the indefinite integral of \(x\), we still both get the same answer if we calculate a definite integral of \(x\):
  \[
  \int_{-1}^2 x \, dx = \left. \frac{x^2}{2} \right|_{-1}^{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}
  \]
  and
  \[
  \int_{-1}^2 x \, dx = \left. (1 + \frac{x^2}{2}) \right|_{-1}^{2} = (1 + \frac{4}{2}) - (1 + \frac{1}{2}) = \frac{3}{2}
  \]

- **Fundamental Theorem of Calculus.** If \(F(x)\) is an indefinite integral of \(f(x)\), then \(f(x) = F'(x)\). This fact helps us figure out a lot of indefinite integrals. For example, we know that \((\sin x)' = \cos x\), so we conclude from the Fundamental Theorem of Calculus that \(\int \cos x \, dx = \sin x + C\).

Here are the two techniques of integration that you should have known at one time:

- **Substitution.** If \(x = g(u)\), then
  \[
  \int_a^b f(g(u))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
  \]
The trick in this rule is to “see” an appropriate function \( g(x) \) in your integrand. For example, in the following integral, we notice an obvious possibility:

\[
\int_a^b x \exp(-\frac{x^2}{2}) \, dx = \int_{a^2/2}^{b^2/2} \sqrt{2u} \exp(-u) \frac{1}{\sqrt{2u}} \, du = \int_{a^2/2}^{b^2/2} \exp(-u) \, du.
\]

- Integration by parts.

\[
\int_a^b f(x)g'(x) \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x) \, dx.
\]

The trick here is to “see” which part of the integrand should be \( f(x) \) and which part should be \( g'(x) \). In the following example, a good possibility is \( f(x) = x \). This is a very common choice.

\[
\int_a^b x \sin x \, dx = b \sin b - a \sin a - \int_a^b (-\cos x) \, dx.
\]

The point of this is that the integral on the right is an easy one, compared to the original integral:

\[
\int_a^b (-\cos x) \, dx = -(\sin b - \sin a).
\]

- There are lots of other techniques of integration, such as “partial fractions”, “trig substitution”, and so on. Do not worry about these. Most of the time, we will be happy to let Matlab do hard integrals.

3. **Affine linear changes of variables in \( \mathbb{R}^2 \)**

3.1. **An example.** A long time ago, in some math class (maybe High School Algebra 2 or Precalculus) you learned how to shift and scale graphs of functions by doing simple changes of variables. For example, if you start with the function \( y = x^2 \) and shift its graph upward 2 units, you get the function \( y = x^2 + 2 \). If you want to shrink this graph in the \( y \)-direction by a factor of 2, you get \( y = (x^2 + 2)/2 \). Now if you want to shift to the left 3 units, you get \( y = ((x + 3)^2 + 2)/2 \). And finally, if you want to expand in the \( x \)-direction by a factor of 5, you get the graph of the function

\[
y = \frac{(\frac{x}{5} + 3)^2 + 2}{2}.
\]

Other ways to write this relationship are

\[
(2y - 2) = (\frac{1}{5}x + 3)^2 \text{ and } 2(y - 1) = \left(\frac{x + 15}{5}\right)^2.
\]
If we let \( u = x/5 + 3 \) and \( v = 2y - 2 \), then we get \( v = u^2 \).

Here is another way to write the relationship between the variables \((x, y)\) and the variables \((u, v)\):

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}1/5 & 0 \\
0 & 2
\end{pmatrix} \begin{pmatrix}x \\
y
\end{pmatrix} + \begin{pmatrix}3 \\
-2
\end{pmatrix}.
\]

Or we can solve for \((x, y)\) in this equation:

\[
\begin{pmatrix}x \\
y
\end{pmatrix} = \begin{pmatrix}5 & 0 \\
0 & 1/2
\end{pmatrix} \begin{pmatrix}u \\
v
\end{pmatrix} + \begin{pmatrix}-15 \\
1
\end{pmatrix}.
\]

This matrix equation captures nicely what we need to do to get from \( v = u^2 \), whose graph is the standard parabola with vertex at the origin, to the graph of the function in (1): Expand by a factor of 5 in the horizontal direction, shrink by a factor of 2 in the vertical direction, shift left 3 units and up 2 units.

3.2. **Scaling and shifting.** Here is a more general situation: suppose we have some relationship between two variables \( u \) and \( v \) that can be expressed with an equation. For example, \( v \) might be a function of \( u \) as it was in our example, or there might be an equation involving both \( u \) and \( v \), such as \( u^2 + v^2 = 1 \) (this is the equation for the unit circle centered at the origin in the \( uv \)-plane).

Now suppose we want to alter the graph of this relationship by doing some scaling and shifting. This gives us a new graph, which we can think of as depicting the relationship between two variables \( x, y \). If we scale by a factor of \( a \) in the horizontal direction and \( b \) in the vertical direction, and if we shift \( c \) units in the horizontal direction and \( d \) units in the vertical direction, then in order to change coordinates from \((u, v)\) to \((x, y)\), we use the following:

\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}a & 0 \\
0 & b
\end{pmatrix} \begin{pmatrix}u \\
v
\end{pmatrix} + \begin{pmatrix}c \\
d
\end{pmatrix}.
\]

This is just a fancy matrix way to express the two equations \( x = au + c \) and \( y = bv + d \).

As long as neither \( a \) nor \( b \) is 0, we can solve for \( u, v \) in terms of \( x, y \):

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}1/a & 0 \\
0 & 1/b
\end{pmatrix} \begin{pmatrix}x \\
y
\end{pmatrix} + \begin{pmatrix}-c/a \\
-d/b
\end{pmatrix}.
\]

The corresponding equations are \( u = (x - c)/a \) and \( v = (y - d)/b \).

If we plug these into our relationship between \( u \) and \( v \), we get the appropriate relationship between \( x \) and \( y \). For example, if the original
relationship was $u^2 + v^2 = 1$, then the new relationship is
\[(x - c)^2/a^2 + (y - d)^2/b^2 = 1,\]
which is the equation of an ellipse with horizontal radius $a$ and vertical radius $b$, centered at the point $(c, d)$. That’s exactly what we expected to get!

This method can be used to turn the function $y = x$ into any affine linear function, the function $y = x^2$ into any quadratic function, the function $y = e^x$ into any exponential function, the function $y = \log x$ into any logarithmic function, and the function $y = \sin x$ into any function of the form
\[y = d_1 \cos(ax + b) + d_2 \sin(ax + c).\]
This last one requires a little work if both $\cos$ and $\sin$ appear in the new relationship. Also, there are some degenerate cases, like turning $y = x$ into a horizontal line like $y = 3$ that need a little extra effort. But the overall message is that many of our functions are related to many other functions of the same type by an affine linear change of variables.

### 3.3. General affine linear changes of variables

But why use matrices to do this, when some simple equations will do? The reason is that we can do more with matrices than scaling and shifting. For example, the matrix
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
rotates the plane counterclockwise about the origin by the angle $\theta$. In general, any affine linear transformation of the $xy$-plane can be written as
\[A\vec{x} + \vec{b}\]
where $A$ is a $2 \times 2$ matrix and
\[\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},\]
The matrix $A$ accomplishes the linear part of the transformation, which might include scaling, rotation, and “shear”, and the vector $\vec{b}$ accomplishes the “affine” part of the transformation, which is always a shift. So let’s suppose we want to rotate the parabola $v = u^2$ counterclockwise through an angle of $\pi/4$. What is the corresponding change of
variables? It is
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
  \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix} \begin{pmatrix}
  u \\
  v
\end{pmatrix}.
\]

We can solve for \( u, v \) in terms of \( x, y \) by using the inverse of the rotation matrix, which is the matrix that rotates the plane through the angle \(-\pi/4\). We get
\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix} = \begin{pmatrix}
  \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
  -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix}.
\]

This gives us the following two equations:
\[
u = \frac{\sqrt{2}}{2} (x + y) \quad \text{and} \quad v = \frac{\sqrt{2}}{2} (-x + y).
\]

Plug those into the equation \( v = u^2 \) and we have
\[
\frac{\sqrt{2}}{2} (-x + y) = \frac{(x + y)^2}{2},
\]
which can be re-arranged as
\[
x^2 + \sqrt{2}x + y^2 - \sqrt{2}y + 2xy = 0.
\]

Did you know that it was that easy to find a relationship whose graph is a rotated parabola?

The same method can be used to apply lots of other affine linear transformations to graphs. It works best when the matrix \( A \) is invertible. Start with a relationship of the form \( f(u, v) = 0 \). Decide on an affine linear transformation, with matrix \( A \) and shift \( \vec{b} \). Then apply this to the variables \( u, v \):
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = A \begin{pmatrix}
  u \\
  v
\end{pmatrix} + \begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix}.
\]

Then invert:
\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix} = A^{-1} \begin{pmatrix}
  x \\
  y
\end{pmatrix} - A^{-1} \begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix}.
\]

This gives you equations for \( u \) and \( v \) in terms of \( x \) and \( y \). Plug those into the relationship \( f(u, v) = 0 \) to get a new relationship between \( x \) and \( y \). The graph of this new relationship is what you get if you apply the affine linear transformation to the graph of the original relationship \( f(u, v) = 0 \).
4. ASSIGNMENT FOR THIS WEEK

Review, review, review!

(i) Get some review books for calculus and linear algebra if you don't already have some.

(ii) Familiarize yourself with the graphs of the following types of functions of a single variable: affine linear, quadratic, power, exponential, logarithmic, trigonometric, inverse trigonometric.

(iii) Learn how to use scaling and shifting to transform the graphs of the functions $x, x^2, e^x$, and $\log x$ into all of the other functions of the same type.

(iv) What happens if you graph a power function on log-log graph paper? In other words, if you start with the function $y = ax^p$ with $a > 0$ and for each pair $(x, y)$ that satisfies this relationship, you plot the point $(\log x, \log y)$, what is the result?

(v) What happens if you start with an exponential function $y = ae^{bx}$, and for each point $(x, y)$ that satisfies the equation, you plot the point $(x, \log y)$, what is the result?

(vi) Review how to find the derivatives of all the functions that I listed.

(vii) Review how to find the integrals of the polynomials, power functions, exponential functions, and basic trigonometric functions that I listed.

(viii) Review how to find the integral of a function of the form $xf(x)$ or $x^2f(x)$ if $f(x)$ is an exponential function or $\sin x$ or $\cos x$ (integration by parts).

(ix) Review how to find the integral of a function of the form $f(ax + b)$ if $f(x)$ is a function that you know how to integrate (substitution).

(x) Make sure you have answers to the following questions: What good are affine linear transformations? What good is the derivative of a function? What good is the integral of a function? Why do I need to know about quadratic, power, logarithmic, exponential and trigonometric functions?

(xii) Review matrix multiplication and finding the inverse of a matrix.