Brownian Motion and Geometric Brownian Motion

Graphical representations

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1 Standard Brownian Motion

Definition. A Wiener process \( W(t) \) (standard Brownian Motion) is a stochastic process with the following properties:

1. \( W(0) = 0 \).

2. Non-overlapping increments are independent: \( \forall 0 \leq t < T \leq s < S \), the increments \( W(T) - W(t) \) and \( W(S) - W(s) \) are independent random variables.

3. \( \forall 0 \leq t < s \) the increment \( W(s) - W(t) \) is a normal random variable, with zero mean and variance \( s - t \).

4. \( \forall \omega \in \Omega \), the path \( t \mapsto W(t)(\omega) \) is a continuous function.

For each \( t > 0 \) the random variable \( W(t) = W(t) - W(0) \) is the increment in \([0, t]\): it is normally distributed with zero mean, variance \( t \) and density

\[
f(t, x) = \frac{1}{\sqrt{2\pi}t} e^{-x^2/2t}.
\]

For \( p \in [0, 1] \) the \( p \)-th percentile of \( W(t) \) is \( N^{-1}(p) \sqrt{t} \), where \( N^{-1} \) is the inverse function of the std. normal distribution function

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du.
\]
Wiener process. Density functions for $t = 0.25, 0.5, 0.75, 1, \ldots, 5$.

Wiener process. Plot of $(t, x) \mapsto f(t, x)$.
Wiener process. The mean path (red), 20 sample paths for $t \in [0, 5]$ (green), 1%, 5%, 10%, 90%, 95% and 99% percentile paths (red dashed).
2 Brownian Motion (with drift)

**Definition.** A *Brownian Motion (with drift)* \(X(t)\) is the solution of an SDE with constant drift and diffusion coefficients 

\[
dX(t) = \mu \, dt + \sigma \, dW(t),
\]

with initial value \(X(0) = x_0\).

By direct integration

\[
X(t) = x_0 + \mu t + \sigma W(t)
\]

and hence \(X(t)\) is *normally distributed*, with *mean* \(x_0 + \mu t\) and *variance* \(\sigma^2 t\).

Its density function is

\[
f(t, x) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}}
\]

and for \(p \in [0, 1]\) the *\(p\)-th percentile* is \(x_0 + \mu t + N^{-1}(p)\sigma \sqrt{t}\). 

Brownian Motion \((\mu = 0.15, \sigma = 0.20\) and \(x_0 = 0\)). Density functions for \(t = 0.25, 0.5, 0.75, 1, \ldots, 5\).
Brownian Motion ($\mu = 0.15$, $\sigma = 0.20$ and $x_0 = 0$). Plot of $(t, x) \mapsto f(t, x)$.

Brownian Motion ($\mu = 0.15$, $\sigma = 0.20$ and $x_0 = 0$). The mean path (red), 20 sample paths for $t \in [0, 5]$ (green), 1%, 5%, 10%, 90%, 95% and 99% percentile paths (red dashed).
3 Geometric Brownian Motion

Definition. A Geometric Brownian Motion $X(t)$ is the solution of an SDE with linear drift and diffusion coefficients

$$dX(t) = \mu X(t) \, dt + \sigma X(t) \, dW(t),$$

with initial value $X(0) = x_0$.

A straightforward application of Itô’s lemma (to $F(X) = \log(X)$) yields the solution

$$X(t) = e^{\log x_0 + \tilde{\mu} t + \sigma W(t)} = x_0 e^{\tilde{\mu} t + \sigma W(t)},$$

where $\tilde{\mu} = \mu - \frac{1}{2} \sigma^2$ and hence $X(t)$ is lognormally distributed, with

- mean $E(X(t)) = x_0 e^{\tilde{\mu} t}$,
- variance $\text{var}(X(t)) = x_0^2 e^{2\tilde{\mu} t} \left(e^{\sigma^2 t} - 1\right)$,
- density $f(t, x) = \frac{1}{\sigma x \sqrt{2\pi t}} e^{-\left(\log x - \log x_0 - \tilde{\mu} t\right)^2 / 2\sigma^2 t}$.

For $p \in [0, 1]$ the $p$-th percentile is $x_0 e^{\tilde{\mu} t + N^{-1}(p) \sigma \sqrt{t}}$.

Geometric B.M. ($\mu = 0.15$, $\sigma = 0.20$, $x_0 = 1$, $\tilde{\mu} = 0.11$). Density functions for $t = 0.25, 0.5, 0.75, 1, \ldots, 5$. 

\[\text{Graph}\]
Geometric B.M. ($\mu = 0.15, \sigma = 0.20, x_0 = 1, \hat{\mu} = 0.11$). Plot of $(t, x) \mapsto f(t, x)$.

Geometric B.M. ($\mu = 0.15, \sigma = 0.20, x_0 = 1, \hat{\mu} = 0.11$). The mean path (red), 20 sample paths for $t \in [0, 5]$ (green), 1%, 5%, 10%, 90%, 95% and 99% percentile paths (red dashed).