Critical Attractive Spin Systems

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Abstract. We study a class of attractive spin systems. We prove that for these processes, the system dies out when the parameters are on the critical surface. We also prove that the supercritical process survives with positive probability in a sufficiently thick space-time 'slab'.

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1. Introduction

In Bezuidenhout and Grimmett (1990), it was shown that the critical contact process dies out. The proof presented in that paper is easily generalized to show that in any one-parameter family of additive processes with translation-invariant spatially symmetric rates, the critical process dies out starting from any finite initial state, as long as the minimal death rate is positive. However since the argument given there relies heavily on spatial symmetry, it implies nothing, for example, about the critical behaviour of the one-sided one-dimensional contact process. A corollary of the main result of this paper, Corollary 2.6, implies that *any* critical additive process with minimal positive death rate dies out with probability 1 from any finite initial state. In particular, the one-sided contact process dies out at its critical value.

In fact our argument has implications for a wider class of attractive finite-range translation-invariant spin systems: we show that if the birth and death parameters of such a system are critical for the process starting from a single occupied site and if the death rate is strictly positive, then the process dies out with probability 1. See Theorem 2.4.

As a by-product of the construction used to prove the main result, we show that no matter what the spatial dimension of the process, survival starting from a single occupied site is essentially a one-dimensional phenomenon. More precisely, we obtain that for any spatial dimension $d \geq 2$, if there is a positive survival probability for the process started from a single occupied site and the death rate is strictly positive, then there is also a positive probability that this process survives when it is restricted to a sufficiently thick two-dimensional space-time 'slab' (or equivalently, after linear change of space-time coordinates that leaves the time coordinate unchanged, the process survives with positive probability when it is restricted spatially to a sufficiently thick one-dimensional 'tube'). See Theorem 2.8. Note that the analogous result for the contact process followed easily from symmetry. In the absence of symmetry we have to work somewhat harder to obtain the result about survival in a space-time slab.

We use a refinement of the argument of Bezuidenhout and Grimmett (1990). That argument in turn was an adaptation to the oriented setting of an earlier result due to Barsky, Grimmett and Newman (1990) about (unoriented) percolation in a half-space. Related techniques have been used in Grimmett and Marstrand (1990) to study the supercritical phase for bond percolation in the full space.

2. Statement of results

Systems with attractive rates

We consider the class of spin systems in $\mathbf{Z}^d$ with translation-invariant finite-range attractive rates. A spin system in $\mathbf{Z}^d$ is a Markovian system of processes $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ with values in the set $2^{\mathbf{Z}^d}$, the collection of all subsets of $\mathbf{Z}^d$. We write ξ_t^A for the state at time t of the process with initial state A. The state ξ_t^A is a subset of $\mathbf{Z}^d$. It is sometimes called the set of 'occupied sites' in $\mathbf{Z}^d$ and its complement the set of 'vacant sites'. In a finite-range spin system, the occupancy of a site changes at a rate determined by the occupancy of nearby sites. Let $N_r = \{y \in \mathbf{Z}^d : \max_i |y_i| \le r\}$ and $N'_r = N_r \setminus \{0\}$. The class of translation-invariant finite-range spin systems with range $r < \infty$ is indexed by the set of parameters (rates) $(\boldsymbol{\beta}, \boldsymbol{\delta})$ where

$$\begin{aligned} \boldsymbol{\beta} &= (\beta(\eta) : \eta \subseteq N'_r) \\ \boldsymbol{\delta} &= (\delta(\eta) : \eta \subseteq N'_r) \end{aligned}$$

with $\beta(\eta) \ge 0$ and $\delta(\eta) \ge 0$. The quantity $\beta(\eta)$ is known as a 'birth rate'. It equals the rate at which a vacant site x in the state ξ becomes occupied when $(\xi - x) \cap N'_r = \eta$, where $\xi - x = \{y - x : y \in \xi\}$. Similarly, $\delta(\eta)$ is the rate at which an occupied site x in the state ξ becomes vacant when $(\xi - x) \cap N'_r = \eta$. A vector of rate parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ is called *attractive* if $\beta(\eta) \le \beta(\eta')$ and $\delta(\eta) \ge \delta(\eta')$ whenever $\eta \subseteq \eta'$. In the attractive case, the minimal birth and death rates are, respectively, $\beta(\emptyset)$ and $\delta(N'_r)$.

For a given set of rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$, it is possible to construct all the corresponding processes $\{\xi_t^A : t \ge 0, A \subseteq \mathbf{Z}^d\}$ jointly on the same probability space. We shall discuss the details of this construction in Section 3. The probability space resulting from this construction will be denoted by Ω, and the corresponding probability measure by $P_{\boldsymbol{\beta},\boldsymbol{\delta}}$.

Under certain conditions on the birth and death rates, the construction mentioned in the preceding paragraph can be carried out so that the following holds:

$$\xi_t^A \cup \xi_t^B = \xi_t^{A \cup B} \text{ for all } A, B \subseteq \mathbf{Z}^d \text{ and } t \ge 0 \ .$$

The rate parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ of such a system are called *additive*. Systems with additive rates are closely connected with percolation models, so it is not surprising that many results from percolation theory more easily generalize to such systems. See the book by Griffeath (1979) or the more recent and somewhat less formal book by Durrett (1988) for conditions on the rates that lead to additive systems.

Critical processes

There are several notions of 'criticality' for systems $\{\xi_t^A : t \ge 0, A \subseteq \mathbf{Z}^d\}$ with attractive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$. For $A \subseteq \mathbf{Z}^d$, we say that ξ^A *survives* for a given sample point $\omega \in \Omega$ if $\xi_t^A(\omega) \ne \emptyset$ for all $t \ge 0$, and we call $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^A \text{ survives})$ the *survival probability* of the process ξ^A. If the survival probability of ξ^A is positive, we say that ξ^A is *viable*. If $|A| < \infty$, we define

$$\mathcal{S}_A(r) = \{\text{attractive } (\boldsymbol{\beta}, \boldsymbol{\delta}) : \xi^A \text{ is viable}\} \ .$$

When A is a singleton, which by translation invariance we may take to be $\{0\}$, we write $\mathcal{S}_0(r)$ for $\mathcal{S}_A(r)$. We might call $\mathcal{S}_A(r)$ the 'supercritical region' in parameter space. Any parameter vector $(\boldsymbol{\beta}, \boldsymbol{\delta})$ on the boundary of $\mathcal{S}_A(r)$ is called *critical* for the process starting at A, and the corresponding process ξ^A is called a *critical process*.

Our main result (see Theorem 2.4 below) is that if $(\boldsymbol{\beta}, \boldsymbol{\delta})$ is critical for the process starting at $\{0\}$ and if $\delta(N'_r) > 0$, then the corresponding critical process $\xi^{\{0\}}$ is not viable. Equivalently, the theorem asserts that $\mathcal{S}_0(r) \cap \{(\boldsymbol{\beta}, \boldsymbol{\delta}) : \delta(N'_r) > 0\}$ is open in the set of attractive rates.

For any translation-invariant additive system it can be shown that for all finite $A \subset \mathbf{Z}^d$,

$$\xi^{\{0\}} \text{ is viable if and only if } \xi^A \text{ is viable.}$$

—see Durrett (1988). It follows from Theorem 2.4 that critical additive processes with strictly positive death rates are not viable (Corollary 2.6).

One-parameter families and critical values

Consider a one-parameter family of spin systems with range-r translation-invariant attractive rates, indexed by a curve

$$\{(\boldsymbol{\beta}_\lambda, \boldsymbol{\delta}_\lambda) : 0 \le \lambda < \infty\} \tag{2.1}$$

in parameter space. We assume that each component of $\boldsymbol{\beta}_\lambda$ is non-decreasing and each component of $\boldsymbol{\delta}_\lambda$ is non-increasing in λ, and that $\delta_0(N_r') > 0$ and $\beta_0(N_r') = 0$. We shall sometimes write P_λ for the corresponding probability measure. In the most common examples, $(\boldsymbol{\beta}_\lambda, \boldsymbol{\delta}_\lambda) = (\lambda\boldsymbol{\beta}, \boldsymbol{\delta})$ for some fixed set of attractive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$. For example the symmetric contact process is such a one-parameter family with the choices: $r = 1$, $\beta_\lambda(\eta) = \lambda|\eta|$, and $\delta_\lambda(\eta) = 1$ for every η.

It can be shown by using attractiveness and the hypothesis of monotonicity in λ that the survival probability of $\xi^{\{0\}}$ is non-decreasing in λ. Therefore there exists a (possibly infinite) critical value λ_c defined by

$$P_\lambda(\xi^{\{0\}} \text{ survives}) \begin{cases} = & 0 \qquad \lambda < \lambda_c \\ > & 0 \qquad \lambda > \lambda_c. \end{cases} \tag{2.2}$$

As a consequence of our main theorem, we have that for any such one-parameter family for which $\lambda_c < \infty$,

$$P_{\lambda_c}(\xi^{\{0\}} \text{ survives}) = 0 \ .$$

See Corollary 2.5.

Discrete-time systems

The spin systems $\{\xi_t^A : t \ge 0, A \subseteq \mathbf{Z}^d\}$ described above evolve in continuous time. One can define analogous discrete-time spin systems in which t is a non-negative integer and the parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ are probabilities instead of rates: thus if $\eta \in 2^{N_r'}$,

$$\begin{aligned} P_{\boldsymbol{\beta},\boldsymbol{\delta}}\left(x \notin \xi_{t+1} \mid (\xi_t - x) \cap N_r = \eta \cup \{x\}\right) &= \delta(\eta) \\ P_{\boldsymbol{\beta},\boldsymbol{\delta}}\left(x \in \xi_{t+1} \mid (\xi_t - x) \cap N_r = \eta\right) &= \beta(\eta). \end{aligned} \tag{2.3}$$

Note that translation invariance and the finite-range condition are built into (2.3). Attractiveness is defined as in continuous time, except that there is an additional condition, namely that

$$\beta(\eta) + \delta(\eta) \le 1,$$

for all $\eta \subseteq N_r'$. This extra condition is required in discrete time to preserve the monotonicity properties that are fundamental in attractive systems. In particular, it ensures that for any given configuration of occupied and vacant sites in $\mathbf{Z}^d \setminus \{x\}$, the probability that the site x is occupied at some time t is greater if x is occupied at time $t-1$ than if x is vacant at that time.

All of our main results are valid for discrete-time systems as well as continuous-time ones, and the proofs are, apart from technical details, identical. We need to discuss both types of systems, however, because Theorem 2.8 below is proved by induction on the dimension d, and even when the original system has a continuous time variable, after one step of the induction argument, we are forced to deal with a discrete-time system.

All the quantities discussed above have analogues in the discrete-time setting. We use the same notation as above to denote these discrete-time analogues.

Statement of results about critical processes

Our main result is:

(2.4) THEOREM. *For every $r < \infty$, in discrete or continuous time, the set*

$$\{(\boldsymbol{\beta}, \boldsymbol{\delta}) : \delta(N_r') > 0 \ , \ \xi^{\{0\}} \text{ is viable}\}$$

is open in the space of parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ corresponding to range-r attractive systems.

An immediate consequence of Theorem 2.4 for one-parameter families of attractive spin systems is given in Corollary 2.5. See the paragraphs containing (2.1) and (2.2) for an explanation of the notation.

(2.5) COROLLARY. *Consider a one-parameter family of spin systems with attractive rates $(\boldsymbol{\beta}_\lambda, \boldsymbol{\delta}_\lambda)$, $\lambda \in [0, \infty)$, in discrete or continuous time. Assume that the components of $\boldsymbol{\beta}_\lambda$ are continuous and non-decreasing and the components of $\boldsymbol{\delta}_\lambda$ are continuous and non-increasing in λ. Further assume that $\beta_0(\emptyset) = 0$, $\delta_{\lambda_c}(N_r') > 0$, and, in the discrete-time case, that $\delta_{\lambda_c}(\emptyset) < 1$. Then $\xi^{\{0\}}$ is not viable under P_{λ_c}.*

Theorem 2.4 has the following consequence for additive processes.

(2.6) COROLLARY. *Let A be a finite subset of $\mathbf{Z}^d$ and ξ^A a critical process in discrete or continuous time with finite-range translation-invariant additive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$. Assume that $\delta(N_r') > 0$. Then ξ^A is not viable.*

Survival of supercritical processes in restricted space-time

As we shall see, an immediate consequence of the proof of Theorem 2.4 is:

(2.7) COROLLARY. *Let ξ^A be a viable process with finite-range translation-invariant attractive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$. Assume that $\delta(N_r') > 0$, and in the discrete-time case, that $\delta(\emptyset) < 1$. Then, possibly after a linear change of space-time coordinates that leaves the time coordinate fixed has been made, there exists $w_{d-1} > 0$ so that with positive $P_{\boldsymbol{\beta},\boldsymbol{\delta}}$-probability, $\xi^{\{0\}}$ survives inside the d-dimensional space-time region*

$$\mathbf{Z}^{d-2} \times [-w_{d-1}, w_{d-1}] \times \mathbf{Z} \times \mathbf{R}^+.$$

By working somewhat harder, we can show:

(2.8) THEOREM. *Suppose* $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ *satisfies the hypotheses of Corollary 2.7. Then, possibly after making a linear change of space-time coordinates that leaves the time coordinate fixed, we can find a* $w > 0$ *so that with positive* $P_{\boldsymbol{\beta},\boldsymbol{\delta}}$*-probability,* $\xi^{\{0\}}$ *survives inside the 2-dimensional space-time region*

$$[-w, w]^{d-1} \times \mathbf{Z} \times \mathbf{R}^+.$$

This last result can be stated informally as: if $\xi^{\{0\}}$ survives with positive probability, then it does so when restricted to a sufficiently thick 2-dimensional space-time slab, or after a linear change of space-time coordinates coordinates that leaves time unchanged, when restricted spatially to a sufficiently thick 1-dimensional tube in $\mathbf{Z}^d$.

3. Preliminaries

Graphical construction of spin systems

Our goal in this section is to define a single 'universal' probability space on which all of the processes used in the arguments throughout the rest of the paper are jointly defined. We use the so-called 'graphical construction' of spin systems. It enables us to make convenient comparisons among several processes. The reader who is willing to accept that such a construction exists may wish to read only about the construction in the discrete time case, skip the details for the continuous-time case, and then read about the auxiliary processes that we define in terms of the graphical construction.

We start with the discrete-time case. Let $U_{x,t}, V_{x,t}, (x,t) \in \mathbf{Z}^d \times \mathbf{Z}^+$ be independent random variables, uniformly distributed on $[0,1]$. Thus, to each point (x,t) in space-time, we assign two uniformly distributed random variables. The corresponding probability space is called $(\Omega, \mathcal{F}, P)$.

Given a set of parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ with range r and an initial state A, we construct the corresponding discrete-time process on Ω inductively as follows. Let $\xi_0^A = A$, and having defined ξ_t^A for a nonnegative integer t, define ξ_{t+1}^A by the condition

$$\text{(3.1)} \qquad \begin{aligned} x \in \xi_{t+1}^A \quad & \text{iff either} \\ & \qquad x \notin \xi_t^A \text{ and } \beta((\xi_t^A - x) \cap N_r') \geq U_{x,t+1} \\ & \text{or} \\ & \qquad x \in \xi_t^A \text{ and } \delta((\xi_t^A - x) \cap N_r') < V_{x,t+1}. \end{aligned}$$

It is easily checked that (3.1) implies (2.3), so this construction does indeed give us the desired discrete-time spin system. Note that in the discrete-time case, one probability space $(\Omega, \mathcal{F}, P)$ serves for all possible parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ and all initial states A. Thus, $P_{\boldsymbol{\beta},\boldsymbol{\delta}}$ equals P for all $(\boldsymbol{\beta}, \boldsymbol{\delta})$. In the discrete-time setting, we let

$$\mathcal{F}_t = \sigma\{U_{x,s}, V_{x,s} : s \leq t, x \in \mathbf{Z}^d\}.$$

Now we turn to the continuous-time setting. Choose parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$ with range r. Let $\mathcal{B}_x, \mathcal{D}_x, x \in \mathbf{Z}^d$ be independent Poisson point processes in $[0, \infty)$. We let the Poisson point processes $\mathcal{B}_x$ have intensity parameter equal to the maximal birth rate $\beta(N_r')$, and the Poisson point processes $\mathcal{D}_x$ have intensity parameter equal to the maximal death rate $\delta(\emptyset)$. In other words, each $\mathcal{B}_x$ is a random, discrete subset of the time line $[0,\infty)$ with the property that the distances between points in $\mathcal{B}_x \cup \{0\}$ are independent, identically distributed exponential random variables with mean $1/\beta(N_r')$, and similarly for the random sets $\mathcal{D}_x$. (When the mean distance between points is infinite, we understand the random set to be empty.) To each point $t \in \mathcal{B}_x$ we assign a random variable $U_{x,t}$ and to each point $t \in \mathcal{D}_x$ we assign a random variable $V_{x,t}$. These random variables are all uniformly distributed on $[0,1]$, independent of one another, and independent of the entire collection of Poisson point processes $\mathcal{B}_x, \mathcal{D}_x, x \in \mathbf{Z}^d$. Note the similarity to the discrete-time case. Let $(\Omega, \mathcal{F}, P_{\boldsymbol{\beta},\boldsymbol{\delta}})$ be the underlying probability space. By removing a $P_{\boldsymbol{\beta},\boldsymbol{\delta}}$-null set, we may assume that the random sets $\mathcal{B}_x, \mathcal{D}_x, x \in \mathbf{Z}^d$ are pairwise disjoint. For this continuous-time setting, we let

$$\mathcal{F}_t = \sigma\{\mathcal{B}_x \cap [0,t], \mathcal{D}_x \cap [0,t], U_{x,r}, V_{x,s} : 0 \le r, s \le t, x \in \mathbf{Z}^d\}.$$

Given an initial state A, it can be shown (see Gray and Griffeath (1982; Section 2)) that there is a unique process ξ_t^A defined on Ω satisfying the following four conditions. The first condition is that $\xi_0^A = A$. To state the remaining conditions, fix a site x and a time $t > 0$. Let $s_1 = 0 \vee \max\{s < t : s \in \mathcal{B}_x \cup \mathcal{D}_x\}$ and $s_2 = \min\{s \ge t : s \in \mathcal{B}_x \cup \mathcal{D}_x\}$. Because of the nature of Poisson point processes, both s_1 and s_2 are members of $\mathcal{B}_x \cup \mathcal{D}_x \cup \{0\}$, and $s_1 < t \le s_2$, with probability 1. The second condition is that

$$\text{if } t < s_2, \text{ then } x \in \xi_t^A \text{ iff } x \in \xi_{s_1}^A.$$

The final two conditions are analogous to (3.1):

$$\begin{aligned} &\text{if } t = s_2 \in \mathcal{B}_x, \text{ then } x \in \xi_t^A \text{ iff either} \\ &\quad x \in \xi_{s_1}^A \quad \text{or} \quad \beta((\xi_t^A - x) \cap N_r') \ge \beta(N_r')U_{x,t} \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\text{if } t = s_2 \in \mathcal{D}_x, \text{ then } x \notin \xi_t^A \text{ iff either} \\ &\quad x \notin \xi_{s_1}^A \quad \text{or} \quad \delta((\xi_t^A - x) \cap N_r') \ge \delta(\emptyset)V_{x,t}. \end{aligned} \tag{3.3}$$

The collection of processes $\{\xi_t^A : t \ge 0, A \subseteq \mathbf{Z}^d\}$ is Markov with respect to the σ-fields $\mathcal{F}_t$.

Let us give an informal description of the content of the preceding paragraph. The first condition says that the initial state is A. The second condition says that the occupancy of a site x does not change during any time interval that does not contain a point in either $\mathcal{B}_x$ or $\mathcal{D}_x$. The third condition tells us when a vacant site x becomes occupied. This can occur only at a time $t \in \mathcal{B}_x$ such that the ratio $\beta((\xi_t^A - x) \cap N_r')/\beta(N_r')$ is sufficiently large in comparison with the corresponding uniform random variable. Thus, the points in $\mathcal{B}_x$ are the only times at which x may become occupied if it is vacant, and the random variables $U_{x,n}$ determine conditions under which occupation actually takes place. The third condition also implies that an occupied site x does not become vacant at times $t \in \mathcal{B}_x$. The fourth condition says that an occupied site x may be vacated only at a time $t \in \mathcal{D}_x$, and then only if the ratio $\delta((\xi_t^A - x)\cap N_r')/\delta(\emptyset)$ is sufficiently large in comparison with the corresponding uniform random variable. Vacant sites x do not become occupied at times $t \in \mathcal{D}_x$.

In the proofs of our main results, we shall use several different auxiliary processes, all of which can be defined on the probability space of the above graphical construction. First, it will be convenient to be able to start processes at times τ other than 0. It should not be surprising that if τ is a finite stopping time with respect to the σ-fields $\mathcal{F}_t$, then the graphical construction described above can be used to construct processes $\xi_t^{A,\tau}, t \geq \tau$ with initial state (at time τ) A and birth and death parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$, all on the probability sample space Ω. These processes will 'fit together in a Markovian way' in the sense that

$$\xi_t^{A,\sigma} = \xi_t^{\xi_\tau^{A,\sigma},\tau} \tag{3.4}$$

if $\sigma \leq \tau \leq t$.

Second, we shall want to work with *restricted processes.* A restricted process is one in which we allow a site x to be occupied at time t only if (x,t) lies in some given space-time set B. We shall use notation like $\tilde{\xi}_t^A(B)$ and $\tilde{\xi}_t^{A,\tau}(B)$ for such restricted processes. It should be clear that restricted processes can be constructed on the probability sample space Ω. One simply modifies conditions (3.1), (3.2), and (3.3) appropriately, adding the restriction that $x \notin \tilde{\xi}_t^{A,\tau}$ if $(x,t) \notin B$. In the next several paragraphs, we shall introduce the various types of space-time boxes B that will be of interest to us. It is a consequence of the properties of Poisson point processes and our graphical construction that if B and B' are disjoint space-time boxes, then for any two initial states A and A' and initial times s and s',

$$\tilde{\xi}_t^{A,s}(B) \quad \text{and} \quad \tilde{\xi}_t^{A',s'}(B') \quad \text{are independent.} \tag{3.5}$$

Notation

We use the following notation. Suppose that w_i and h are non-negative and $\alpha_i \in \mathbf{R}$. Let $\mathbf{w} = (w_1, \ldots, w_d)$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. Then $B(\mathbf{w}, h; \boldsymbol{\alpha})$ will denote a space-time box of width $2w_i$ in the i^{th} spatial coordinate direction, height h (in the time direction), and angles of inclination from the vertical α_i in the d spatial coordinate directions. Thus

$$B(\mathbf{w}, h; \boldsymbol{\alpha}) \;=\; \{(\mathbf{x}, t) \in \mathbf{R}^d \times \mathbf{R}^+ : 0 \leq t \leq h, -w_i \leq x_i - t\tan\alpha_i \leq w_i \text{ for } i = 1, \ldots, d\}. \tag{3.6}$$

When $\boldsymbol{\alpha} = \mathbf{0} = (0, \ldots, 0) \in \mathbf{R}^d$, we write $B(\mathbf{w}, h)$ for $B(\mathbf{w}, h; \boldsymbol{\alpha})$. See Figure 1. We also define six related boxes

$$\begin{aligned} R(\mathbf{w}, h) &= \{(\mathbf{x}, t) \in B(\mathbf{w}, h) : (x_d, t) \in [w_d/3, w_d] \times [h/3, h]\} \\ R^{\pm}(\mathbf{w}, h) &= \{(\mathbf{x}, t) \in R(\mathbf{w}, h) : \text{ sign } x_{d-1} = \pm\} \\ L(\mathbf{w}, h) &= \{(\mathbf{x}, t) \in B(\mathbf{w}, h) : (x_d, t) \in [-w_d, -w_d/3] \times [h/3, h]\} \\ L^{\pm}(\mathbf{w}, h) &= \{(\mathbf{x}, t) \in L(\mathbf{w}, h) : \text{ sign } x_{d-1} = \pm\} \end{aligned} \tag{3.7}$$

One may think of these boxes as 'upper corner' portions of $B(\mathbf{w},h)$. For the case of non-zero $\boldsymbol{\alpha}$, we also define the upper corner boxes $R(\mathbf{w},h;\boldsymbol{\alpha})$ and $L(\mathbf{w},h;\boldsymbol{\alpha})$ much as $B(\mathbf{w},h;\boldsymbol{\alpha})$ was defined. Figure 5 shows a view of $L(\mathbf{w},h)$ and $R(\mathbf{w},h)$ for a particular choice of $\mathbf{w}$ and h. We sometimes replace w_i and/or h by ∞: we interpret $B(\infty,w_2,\dots,w_d,h;\boldsymbol{\alpha})$, for example, as the union of $B(w_1,w_2,\dots,w_d,h;\boldsymbol{\alpha})$ over finite w_1.

Suppose that the range of the rates is r, and let $B=B(\mathbf{w},h;\boldsymbol{\alpha})$ be as in (3.6), with $2r\le w_i\le\infty$, $\alpha_i\ge 0$ and $h\ge 0$. In the discrete time case, assume that h is an integer. Let $S_i^{\pm}(B)$ stand for strips of width $2r$ along the right $(+)$ and left $(-)$ sides of B in the i^{th} coordinate direction, and let $T(B)$ be the top of B. Thus

$$\begin{aligned}T(B) &= \{(\mathbf{x},t)\in\mathbf{R}^d\times\mathbf{R}^+ : t=h, -w_i\le x_i-t\tan\alpha_i\le w_i \text{ for } i=1,\dots,d\}\\ S_i^+(B) &= \{(\mathbf{x},t)\in\mathbf{R}^d\times\mathbf{R}^+ : 0\le t\le h, w_i-2r\le x_i-t\tan\alpha_i\le w_i\}\cap B \\ S_i^-(B) &= \{(\mathbf{x},t)\in\mathbf{R}^d\times\mathbf{R}^+ : 0\le t\le h, -w_i\le x_i-t\tan\alpha_i\le -w_i+2r\}\cap B.\end{aligned}\tag{3.8}$$

See Figure 1 again.

Finally, we introduce notation to count the number of occupied sites in $T(B)$ and $S_i^{\pm}(B)$ for a process restricted to B. During the course of the argument, we shall fix an initial state D. Let $N_T(B)$ denote the cardinality of the set

$$T(B)\cap\{(\mathbf{x},h):\mathbf{x}\in\tilde{\xi}_h^D(B)\}.$$

Similarly, in discrete time, we let $N_{S_i^{\pm}}(B)$ be the cardinality of the set

$$S_i^{\pm}(B)\cap\{(\mathbf{x},t):\mathbf{x}\in\tilde{\xi}_t^D(B)\}.$$

In continuous time, we let $N_{S_i^{\pm}}(B)$ be the measure of this set (using the product of counting measure and Lebesgue measure). Also let

$$\begin{aligned}N_S(B) &= \sum_{i=1}^d\big(N_{S_i^+}(B)+N_{S_i^-}(B)\big),\\ N(B) &= N_T(B)+N_S(B).\end{aligned}$$

Preliminary lemmas

In this section, we state several known results as lemmas for future reference. For all but the first of them, we also give short proofs for completeness.

The first result is the famous Harris-FKG inequality (see Harris (1960), Fortuin, Kasteleyn and Ginibre (1971)). In order to state this inequality, we need some definitions. Recall the probability space $(\Omega,\mathcal{F},P_{\boldsymbol{\beta},\boldsymbol{\delta}})$ defined earlier in terms of the graphical construction. We put a partial ordering '$<$' on Ω. We say that $\omega<\omega'$ if for all $x\in\mathbf{Z}^d$, (i) $B_x(\omega)\subseteq B_x(\omega')$, (ii) $D_x(\omega)\supseteq D_x(\omega')$, (iii) $U_{x,t}(\omega)\ge U_{x,t}(\omega')$ for $t\in B_x(\omega)$, and (iv) $V_{x,t}(\omega)\le V_{x,t}(\omega')$ for $t\in D_x(\omega')$. (For the discrete-time case, to make sense of these conditions we let B_x and D_x equal $\mathbf{Z}^+$ for all x.) This partial ordering goes back to Harris (1978), at least in the additive case. Informally, the conditions that define the partial ordering say that there are fewer births and/or more deaths for the sample point ω than for ω' if $\omega<\omega'$. We call an event $E\in\mathcal{F}$ a *positive event* if $\omega\in E$ implies $\omega'\in E$ for all ω' such that $\omega<\omega'$. Note that intersections of positive events are themselves positive. The following is a very useful tool in the analysis of systems with attractive rates.

(3.9) LEMMA. *Positive events are positively correlated.*

The following result was first used by Russo (1978) and Seymour and Welsh (1978).

(3.10) LEMMA. *Suppose A and B are positive events, and that*

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A \cup B) \ \geq 1 - \varepsilon_1 \quad \text{and} \quad P_{\boldsymbol{\beta},\boldsymbol{\delta}}(B) \ < \ 1 - \varepsilon_2.$$

Then $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A) > 1 - \varepsilon_1/\varepsilon_2$.

PROOF: By the Harris-FKG inequality, A and B (and therefore A^c and B^c) are positively correlated. Therefore:

$$\begin{aligned} \varepsilon_1 \ &\geq \ P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A^c \cap B^c) \ \geq P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A^c) P_{\boldsymbol{\beta},\boldsymbol{\delta}}(B^c) \\ &> \ \varepsilon_2 P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A^c). \end{aligned}$$

The result follows after some rearrangement. □

Our next two lemmas rely (in part) on the Harris-FKG inequality. They concern the behavior of viable processes.

(3.11) LEMMA. *Let $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ be a spin system with finite-range translation-invariant attractive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$ such that $\delta(N_r') > 0$. Suppose $D \subseteq \mathbf{Z}^d$ is such that $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^D$ survives$) > 1 - \varepsilon$ with $\varepsilon > 0$. Choose positive integers N_0 and $N_i^{\pm}$, and let $N = N_0 + \sum_{i=1}^d [N_i^+ + N_i^-]$. Then there exists a space-time box $B = B(\mathbf{w}, h)$ depending on N and ε such that the following statements are true. Let $B'(\mathbf{w}', h'; \boldsymbol{\alpha}')$ be a space-time box containing B. In the discrete-time case, assume that the components $\boldsymbol{\alpha}_i'$ of the vector $\boldsymbol{\alpha}'$ are bounded above by* arctan r *in absolute value. Then*

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N(B') \geq N) \ > \ 1 - \varepsilon,$$

and

(3.12)

$$(1 - P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B') \geq N_0)) \prod_{i=1}^d \left[(1 - P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_i^+}(B') \geq N_i^+))(1 - P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_i^-}(B') \geq N_i^-))\right] \ < \ \varepsilon.$$

PROOF: To prove the first statement assume to get a contradiction that no such B exists. Then there exists a nested sequence $B_k \uparrow \mathbf{Z}^d$ of space-time boxes (each of which is tilted no more than arctan r in any of the d spatial directions in the discrete-time case), so that if E_k is the event $\{N(B_k) < N\}$ then $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_k) \geq \varepsilon$. But then

$$P(E_k \text{ i.o.}) \ \geq \ \limsup_{n \to \infty} P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_n) \ \geq \ \varepsilon.$$

We may assume without loss of generality that for each k, the boundaries of the boxes B_k and B_{k+1} are at least one unit apart (in all directions). Then it can be shown that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^D \text{ dies out in } B_{k+1} | E_k) \geq c > 0 \tag{3.13}$$

where c is a constant independent of k. The key facts here are the Markov property and the fact that the sets $T(B)$ and $S_i^{\pm}$ have been chosen so that all occupied space-time points outside of B are necessarily "descendants" of occupied space-time points in the sets $T(B)$ and/or $S_i^{\pm}$. If the time coordinate is discrete, a proof of (3.13) based on these facts is straightforward. In the continuous-time case, the proof is slightly more technical because we count particles in terms of the measure of the set of occupied space-time points. See Bezuidenhout and Grimmett (1990, (17) in the proof of Lemma (7)) or Durrett (1989) for details. It is easy to show using (3.13) that $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^D \text{ dies out } |E_k \text{ i.o.}) = 1$, and hence that $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^D \text{ dies out}) \geq \varepsilon$. This contradiction establishes the first part of the lemma. The second part, (3.12), follows from this and the FKG inequalities because the events in (3.12) are all positive. □

Note that the essential ingredients in the proof of this lemma are (i) the FKG inequality, and (ii) the lower bound (3.13).

(3.14) LEMMA. *Let $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ be a spin system with finite-range translation-invariant attractive rates $(\boldsymbol{\beta},\boldsymbol{\delta})$ such that $\delta(N_r') > 0$ and $\xi^{\{0\}}$ is viable. Fix $\varepsilon > 0$. Then there exists a finite set D and an integer J so that*

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(D \subseteq \tilde{\xi}_J^{\{0\}}(D \times [0, J])) > 0 \tag{3.15}$$

and

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^D \textit{ survives}) \;>\; 1 - \varepsilon. \tag{3.16}$$

PROOF: First we shall show that there exists a finite set D and a positive integer J such that (3.16) holds and

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(D \subseteq \xi_J^{\{0\}}) > 0. \tag{3.17}$$

Suppose not. Then for every finite set D satisfying (3.17) for some J, (3.16) fails. But then for each such set D there exists a finite time τ_D so that $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi_{\tau_D}^D = \emptyset) \geq c > 0$. By a standard stopping-time argument, one obtains from this that $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^{\{0\}} \text{ survives}) = 0$.

A standard limiting argument also shows that (3.17) implies the existence of a minimal finite set $C \subseteq \mathbf{Z}^d$ such that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(D \subseteq \tilde{\xi}_J^{\{0\}}(C \times [0, J])) > 0$$

(just take limits as $C \uparrow \mathbf{Z}^d$). Since C is minimal, the probability that every site in C is occupied at least once during $[0, J]$ must be positive. Since there is a positive probability that any site that is occupied at some time before time J is still occupied by time J (recall that in the discrete-time case, the death probabilities are bounded away from 1), it follows from the Harris-FKG inequality that there is a positive probability that every site in C is occupied at time J by the process starting at $\{0\}$ and restricted to C. After enlarging D

if necessary to equal C, (3.15) follows. □

Our final lemma states that, for a spin system with translation-invariant finite-range rates and with minimal birth rate equal to 0 started from a finite set, the set of occupied sites is contained with probability close to 1 inside a space-time cone whose width grows linearly with time.

(3.18) LEMMA. *Let $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ be a spin system with finite-range translation-invariant rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$. Assume that $\beta(\emptyset) = 0$. Then for each $\varepsilon > 0$, there exists an angle $\alpha_0 \in [0, \pi/2)$ such that for all finite initial states A,*

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi_t^A \subseteq [-\rho - t\tan\alpha_0, \rho + t\tan\alpha_0]^d \text{ for all } t \geq 0) \geq 1 - \varepsilon, \tag{3.19}$$

where $\rho = \max\{\max_i |x_i| : \mathbf{x} \in A\}$. The quantity α_0 depends only on ε, the range r, and the maximal birthrate $\beta(N_r')$.

PROOF: For the discrete-time case, this result is trivial, since growth occurs only at integer times, and since a vacant site cannot become occupied unless it lies within distance r of one or more occupied sites (let $\alpha_0 = \arctan r$, independently of ε). For the continuous time case, we first note that the result is obviously true if we restrict t to a bounded interval (that is, replace the expression '$t \geq 0$' in (3.19) by '$t \in [0, T]$' for T finite). For large t, we compare the process ξ_t^A to a range-r process with death rate equal to 0 and birth rate equal to $\beta(N_r')$. It is known that for such a process, $\rho(t)/t \to c$ a.s. for some constant c independent of A, where $\rho(t)$ is the maximum diameter of the set ξ_t^A. This convergence of the quantity $(\rho(t) - \rho(0))/t$ to the constant c happens uniformly in the size of A, in an appropriate sense. See the discussion of Richardson's growth model in Durrett (1988). Using on these observations, it is easy to finish the proof. □

4. The fundamental lemma

The following lemma is the heart of the proof and we go into some detail in proving it. Unfortunately, the details in the discrete-time case are somewhat messier than in continuous time, although there is no essential difference. We give the proof for continuous time first and indicate the necessary changes for discrete time afterward. In the statement of the lemma, the set D and the integer J are provided by Lemma 3.14, and the quantity α_0 comes from Lemma 3.18. The quantities $N_T(B)$ and $N_{S_d^\pm}(B)$ were defined earlier, in terms of D and J. See the "Notation" section above.

(4.1) LEMMA. *Let $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ be a spin system with finite-range translation-invariant attractive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$ such that $\delta(N_r') > 0$ and $\xi^{\{0\}}$ is viable. Assume that $\delta(\emptyset) < 1$ in the discrete-time case. Fix $\delta \in (0, 1)$. Suppose that D is a finite subset of $\mathbf{Z}^d$ and J is a positive integer satisfying (3.15) and (3.16), with $\varepsilon = \delta^3/2$ (in the discrete-time case, we need to take $\varepsilon = \delta^4/4$). Fix N_0 and $N^\pm \geq 1$. Let α_0 be given by Lemma 3.18, with $\varepsilon = \delta$. Then, possibly after a relabeling of coordinates, there exists a box $B = B(\mathbf{w}, h; \boldsymbol{\alpha})$*

with $\boldsymbol{\alpha} = (0,\dots,0,\alpha_d)$ *and* $|\alpha_d| \le \alpha_0$, *so that*

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B) \ge N_0) \ \ge\ 1-\delta \tag{4.2}$$

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^{\pm}}(B) \ge N^{\pm}) \ \ge\ 1-2\delta. \tag{4.3}$$

Furthermore, the coordinates of $\mathbf{w}$ *and the quantity* h *can be chosen as large as we wish.*

Proof in the continuous-time case: Because $\delta(N_r') > 0$, the process ξ_t^D is sure to die out if the size of ξ_t^D falls below N_0 infinitely often. Consequently, by (3.16) with $\varepsilon = \delta^3/2$, there exists a positive number t_0 so that if $t \ge t_0$ then

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(|\xi_t^D| \ge N_0) \ \ge\ 1-\delta^2/2. \tag{4.4}$$

Noting that the proof of Lemma 3.11 is still valid even if we require in the statement of that lemma that the first $d-1$ coordinates of $\mathbf{w}$ equal ∞, we see that there exist $\mathbf{w}_0 = (\infty,\dots,\infty,w_0^d)$ and $h_0 \ge t_0$ so that if B' is any box containing $B(\mathbf{w}_0,h_0)$, then

$$\begin{aligned}(1-P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B') \ge N_0))(1-P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^+}(B') \ge N^+))(1-P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^-}(B') \ge N^-))\\ <\ \delta^3/2.\end{aligned} \tag{4.5}$$

By (4.4), there exist $\mathbf{w}_1 = (\infty,\dots,\infty,w_1^d)$ and h_1, with $w_1^d \ge w_0^d$ and $h_1 \ge h_0$, such that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B(\mathbf{w}_1,h_1)) \ge N_0) \ \ge\ 1-\delta/2. \tag{4.6}$$

By our choice of α_0,

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi_t^D \not\subset \mathbf{R}^{d-1}\times[-\rho - t\tan\alpha_0, \rho + t\tan\alpha_0] \text{ for some } t) \ <\ 1-\delta, \tag{4.7}$$

where

$$\rho = \max\{\max_i |x_i| : \mathbf{x} \in D\}. \tag{4.8}$$

Fix a width $w_2^d \notin \mathbf{Z}$ so that

$$w_2^d - w_1^d > 2r \quad \text{and} \quad \arctan[(w_2^d - w_1^d - 2r)/h_1] > \alpha_0, \tag{4.9}$$

and let $\mathbf{w}_2 = (\infty,\dots,\infty,w_2^d)$. We have chosen w_2^d so that any space-time box $B = B(\mathbf{w}_2,h;\boldsymbol{\alpha})$ with $h \ge h_1$ that does not contain $B(\mathbf{w}_1,h_1)$ has either its right hand side or its left hand side in the d^{th} co-ordinate direction (depending on whether α_d is positive or negative) outside the 'light cone' $\{(x,t) \in \mathbf{R}^d\times\mathbf{R}^+ : -\rho - t\tan\alpha_0 \le x_d \le \rho + t\tan\alpha_0\}$ and consequently (by (4.7)) either $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^+} \ge N^+) \ <\ 1-\delta$ or $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^-} \ge N^-) \ <\ 1-\delta$. See Figure 2. In the proofs of the two claims below, all of the boxes that we define will be of the form $B(\mathbf{w}_2,h;\boldsymbol{\alpha})$, for $h \ge h_1$ and $\boldsymbol{\alpha} = (0,\dots,0,\alpha)$. It will follow from (4.7) and the way in which these boxes will be defined that we shall never have to consider angles α larger than α_0 in absolute value. Therefore, any such box will contain $B(\mathbf{w}_1,h_1)$ (and hence $B(\mathbf{w}_0,h_0)$), so we shall be able to use (4.5) to obtain information about the numbers of particles on the sides and top of such a box.

We show first (in Claim 4.10 below) that it is possible to choose a space-time box (unbounded in the first $d-1$ spatial co-ordinate directions and with vertical sides in the d^{th}) for which (4.2) is satisfied with δ replaced by $\delta/2$, and for which at least one of the two inequalities in (4.3) is satisfied.

(4.10) CLAIM. *There exists $h_2 \ge h_1$ so that*

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B(\mathbf{w}_2, h_2)) \ge N_0) \ \ge \ 1 - \delta/2 \tag{4.11}$$

and either

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^+}(B(\mathbf{w}_2, h_2)) \ge N^+) \ \ge \ 1 - \delta \tag{4.12}$$

or

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^-}(B(\mathbf{w}_2, h_2)) \ge N^-) \ \ge \ 1 - \delta. \tag{4.13}$$

PROOF OF CLAIM 4.10: Let us introduce some temporary notation, to be used throughout the remainder of the proof of Lemma 4.1. Let

$$\tilde{B}(h, \alpha) = B(\infty, \dots, \infty, w_2^d, h; 0, \dots, 0, \alpha).$$

(As mentioned earlier, if $h \ge h_1$ and $\alpha \in [-\alpha_0, \alpha_0]$, the box $\tilde{B}(h,\alpha)$ contains $B(\mathbf{w}_1, h_1)$, by the definition of w_2^d.) Since $T(\tilde{B}(h_1, 0)) \supseteq T(B(\mathbf{w}_1, h_1))$, it follows from (4.6) that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(\tilde{B}(h, 0)) \ge N_0) \ \ge \ 1 - \delta/2 \tag{4.14}$$

for $h = h_1$. Suppose there exists $h \ge h_1$ for which (4.14) fails. Let

$$h_2 \ = \ \inf\{h > h_1 : (4.14) \text{ fails}\}. \tag{4.15}$$

In the continuous-time case, the left hand side of (4.14) is continuous in h and therefore there is equality in (4.14) for $h = h_2$. So since $\tilde{B}(h_2, 0) \supseteq B(\mathbf{w}_0, h_0)$, we have by (4.5) that either (4.12) or (4.13) holds with $\tilde{B} = \tilde{B}(h_2, 0)$, and Claim 4.10 is verified in this case.

If the set in (4.15) is empty, then (4.14) holds for every $h \ge h_1$, and hence the probability that the system survives inside the d-dimensional space-time slab $\tilde{B}(\infty, 0)$ is at least $1 - \delta/2$. Assume that the birth rate is such that when only the origin is occupied, there is at least one point with non-zero d^{th} coordinate at which the birth rate is positive. Since we are assuming that the process survives when only the origin is occupied in the initial state, this assumption is true, at least after a possible relabeling of coordinates. We shall assume that this point has positive d^{th} coordinate (it is obvious how to modify the argument if its d^{th} coordinate is negative). Then it is easy to see that, conditioned on survival in the slab, $N_{S_d^+}(\tilde{B}(\infty, 0))$ is infinite with probability 1. Therefore there exists an $h_2 \ge h_1$ so that, conditioned on survival in the slab, the probability that $N_{S_d^+}(\tilde{B}(h_2, 0)) \ge N^+$ exceeds $1 - \delta/2$. Combined with the fact that the survival probability is at least $1 - \delta/2$, this yields (4.12) with $\tilde{B} = \tilde{B}(h_2, 0)$. This proves Claim 4.10. □

We now show that there exists a space-time box, infinite in the first $d - 1$ spatial directions and possibly tilted in the d^{th}, so that (4.11), (4.12) and (4.13) all hold.

(4.16) Claim. *There exist* h_3 *and* α_1^d *so that if* $\tilde{B} = B(\infty, \ldots, \infty, w_2^d, h_3; 0, \ldots, 0, \alpha_1^d)$ *then the analogues of (4.11), (4.12) and (4.13) with* $B(\mathbf{w}_2, h_2)$ *replaced by* $\tilde{B}$ *all hold.*

Proof of Claim 4.16: In this proof, we use the same temporary notation $\tilde{B}(h, \alpha)$ as in the proof of Claim 4.10. Let us assume that h_2 has been chosen so that (4.11) and (4.12) hold. By Claim 4.10, if we cannot find such an h_2, then we can instead find an h_2 so that (4.11) and (4.13) hold, and we modify the argument accordingly.

Consider the set of non-negative angles α for which there exists an $h \geq h_1$ so that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(\tilde{B}(h,\alpha)) \geq N_0) \ \geq \ 1 - \delta/2 \tag{4.17}$$

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^+}(\tilde{B}(h,\alpha)) \geq N^+) \ \geq \ 1 - \delta. \tag{4.18}$$

This set is not empty since by assumption it contains $\alpha = 0$ (at $h = h_2$). By the definition of α_0 (see (4.7)), the set is contained in $[0, \alpha_0)$. Let

$$\alpha_1^d \ = \ \sup\{\alpha > 0 : (4.17) \text{ and } (4.18) \text{ hold for some } h \geq h_1\}. \tag{4.19}$$

Since we have chosen the width w_2^d to be a non-integer, for fixed h the quantities on the left sides of (4.18) and (4.19) are constant for α near 0. It follows that $0 < \alpha_1^d \leq \alpha_0$.

By definition of α_1^d, there exists a sequence of positive angles $\alpha(n) \uparrow \alpha_1^d$ and a sequence of heights $h(n) \geq h_1$ so that both (4.17) and (4.18) hold when $(h, \alpha) = (h(n), \alpha(n))$. First consider the case in which it is possible to choose the sequence of heights $h(n)$ so that $h(n) \to \infty$ as $n \to \infty$. We may assume that $\alpha(n) \geq A$ where A is some positive constant. For any box $\tilde{B}(h, \alpha)$ with $\alpha \geq A$, a particle in the box $\tilde{B}(h, \alpha)$ that survives at least $2w_2^d \cot(A)$ time units must hit the left side of the box. Since the maximal death rate is finite, there is a (possibly small) quantity $p > 0$ such that the probability is at least p that any given particle in $\tilde{B}(h, \alpha)$ will survive long enough to hit the left side of the $\tilde{B}(h, \alpha)$. It is easy to prove from this fact and the fact that survival events are positive events, that for all sufficiently large h, conditioned on the event $E = \{\tilde{\xi}_h^D(\tilde{B}(h, \alpha)) \neq \emptyset\}$, the probability that $N_{S_d^-}(\tilde{B}(h, \alpha)) \geq N^-$ is at least $1 - \delta/2$. The event E is implied by the event in (4.17), so we conclude that for all sufficiently large n, $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^-}(\tilde{B}(h(n), \alpha(n))) \geq N^-)$ is at least $1 - \delta$. The statement of Claim 4.16 thus holds in this case.

We now turn to the case in which we cannot choose the sequence $h(n)$ to tend to infinity. Let H be the (bounded) set of heights $h \geq h_1$ such that (4.17) and (4.18) both hold when $\alpha = \alpha_1^d$. This set is nonempty, since it contains any limit point of the sequence $h(n)$ (the left sides of (4.17) and (4.18) are jointly continuous in h and α if time is continuous). If there exists an $h_3 \in H$ so that for $(h, \alpha) = (h_3, \alpha_1^d)$,

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^-}(\tilde{B}(h,\alpha)) \geq N^-) \ \geq \ 1 - \delta, \tag{4.20}$$

then the assertion of Claim 4.16 follows with $\tilde{B} = \tilde{B}(h_3, \alpha_1^d)$.

It remains to deal with the possibility that no height h_3 in the bounded set H exists such that (4.20) holds with $h = h_3$ and $\alpha = \alpha_1^d$. We show that this leads to a contradiction, completing the proof of Claim 4.16. This part of the argument is illustated in Figure 3.

Define

$$(4.21) \qquad h_4 \;=\; \sup H = \; \sup\{h \geq h_1 : \; (4.17) \text{ holds when } \alpha = \alpha_1^d\}.$$

Since the left hand side of (4.18) is increasing in h for fixed α, the fact that H is bounded implies that there exists $h \in (h_1, \infty)$ so that (4.17) fails for $\alpha = \alpha_1^d$. So $h_4 < \infty$. By the continuity in h of the left hand side of (4.17) we have that there is equality in (4.17) when $\alpha = \alpha_1^d$ and $h = h_4$. Recall that we are assuming that h_3 does not exist – in other words that whenever (4.17) and (4.18) hold simultaneously for $\alpha = \alpha_1^d$ and $h \geq h_1$, (4.20) fails. Therefore since $\tilde{B}(h_4, \alpha_1^d) \supseteq B(\mathbf{w}_0, h_0)$, it follows from (4.5) that there must be strict inequality in (4.18) for $h = h_4$ and $\alpha = \alpha_1^d$. See Figure 3(a). Define:

$$(4.22) \qquad \alpha_2^d \;=\; \inf\{\alpha > \alpha_1^d : (4.18) \text{ fails when } h = h_4\}.$$

By definition of α_0 (see (4.7)), (4.18) fails for every h when $\alpha = \alpha_0$. We also have that (4.18) holds with strict inequality when $h = h_4$ and $\alpha = \alpha_1^d$. By the continuity in α of its left hand side, (4.18) holds with equality when $\alpha = \alpha_2^d$ and $h = h_4$. So $\alpha_1^d < \alpha_2^d < \alpha_0$. Therefore by definition of α_1^d, (4.17) fails for $h = h_4$ and $\alpha = \alpha_2^d$. Therefore since $\alpha_2^d \in [0, \alpha_0)$ and hence $\tilde{B}(h_4, \alpha_2^d) \supseteq B(\mathbf{w}_0, h_0)$, (4.5) implies that (4.20) holds with strict inequality for $h = h_4$ and $\alpha = \alpha_2^d$. See Figure 3(b). Let

$$(4.23) \qquad \alpha_3^d \;=\; \sup\{\alpha < \alpha_2^d : (4.20) \text{ fails when } h = h_4\}.$$

When $h = h_4$, we have that (4.20) fails for $\alpha = \alpha_1^d$ and that it holds with strict inequality for $\alpha = \alpha_2^d$. By continuity, (4.20) holds with equality when $h = h_4$ and $\alpha = \alpha_3^d$. Therefore $\alpha_1^d < \alpha_3^d < \alpha_2^d$ and, by definition of α_2^d and α_1^d, (4.18) holds with strict inequality and (4.17) fails when $h = h_4$ and $\alpha = \alpha_3^d$. See Figure 3(c). Let

$$h_5 \;=\; \sup\{h < h_4 : (4.17) \text{ holds when } \alpha = \alpha_3^d\}.$$

By definition of w_2^d and because $\alpha_3^d \leq \alpha_0$, (4.17) holds when $h = h_1$ and $\alpha = \alpha_3^d$. We also have that (4.17) fails when $h = h_4$ and $\alpha = \alpha_3^d$. By the continuity in h of its left hand side, (4.17) holds with equality when $h = h_5$. Therefore $h_5 \in [h_1, h_4)$ and, since $\alpha_3^d > \alpha_1^d$, (4.18) fails for $h = h_5$ and $\alpha = \alpha_3^d$. Hence since $\tilde{B}(h_5, \alpha_3^d) \supseteq B(\mathbf{w}_0, h_0)$, (4.5) implies that (4.20) holds with strict inequality when $h = h_5$ and $\alpha = \alpha_3^d$. See Figure 3(d). Since $h_5 < h_4$ this contradicts our earlier observation that (4.20) holds with equality when $(h, \alpha) = (h_4, \alpha_3^d)$.

This completes the proof of Claim 4.16. □

It follows from Claim 4.16 that if we choose $w_1, \ldots, w_{d-1}$ sufficiently large and let $B = B(w_1, \ldots, w_{d-1}, w_2^d, h_3; 0, \ldots, 0, \alpha_d^1)$, then (4.2) and (4.3) hold. It is clear that the coordinates of the vector $(w_1, \ldots, w_d)$ can be chosen as large as we wish. Just before (4.6) in the proof of Lemma 4.1, the height h_1 can be chosen as large as we wish, so h_3 (which is at least as large as h_1) can also be chosen as large as we wish. Thus we have established Lemma 4.1 in the continuous-time case. □

PROOF OF LEMMA 4.1 IN THE DISCRETE-TIME CASE: In the continuous-time case, we relied several times on the continuity in h and α of certain probabilities. In discrete time, h is restricted to integer values, and for fixed h, these probabilities are not continuous as functions of α. However, it is easily checked from the definitions that for fixed h, the quantities on the left sides of (4.17), (4.18), and (4.20) are upper semi-continuous as functions of α. That is

$$\limsup_{\alpha \to \alpha'} f(h, \alpha) \leq f(h, \alpha')$$

for f equal to each such quantity. This upper semi-continuity will be useful in the proof that now follows.

Starting from the beginning of the continuous-time proof, we now indicate the places where modifications are needed. The first change is that we assume (without loss of generality) that $N_0 \geq \max\{N^+, N^-\}$. We then choose $\mathbf{w}_0 = (\infty, \ldots, \infty, w_0^d)$ and h_0 so that for all boxes B' containing $B(\mathbf{w}_0, h_0)$,

(4.24)
$$(1 - P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B') \geq 2CN_0))(1 - P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^+}(B') \geq N^+))(1 - P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_{S_d^-}(B') \geq N^-)) < \delta^4/4,$$

where $C > 1$ is chosen large enough so that the following is true:

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(|G \cap \xi_t^D| > CN) \leq P_{\boldsymbol{\beta},\boldsymbol{\delta}}(|G \cap \xi_{t+1}^D| > N) + \delta/4 \tag{4.25}$$

for all integer times $t \geq 0$, finite sets G, and positive integers N. We can choose such a constant C because of the assumption that the death probability is bounded away from 1. Now we choose $\mathbf{w}_1 = (\infty, \ldots, \infty, w_1^d)$ with $w_1^d \geq w_0^d$, and $h_1 \geq h_0$, so that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(B(\mathbf{w}_1, h_1)) \geq CN_0) \geq 1 - \delta/4. \tag{4.26}$$

Throughout the rest of the proof, we shall use (4.24) and (4.26) in the place of (4.5) and (4.6).

The proof now follows the continuous-time case until the first place that continuity is used, namely in the paragraph containing (4.14) and (4.15). Replace (4.14) by

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(\tilde{B}(h, 0) \geq CN_0)) \geq 1 - \delta/4, \tag{4.27}$$

which holds for $h = h_1$ by (4.26). Define h_2 accordingly. Then (4.27) fails at $h = h_2$, so by (4.24), either (4.12) or (4.13) holds. Since (4.27) holds at $h = h_2 - 1$, it follows from (4.25) that (4.11) also holds at $h = h_2$. The rest of the proof of Claim 4.10 is unchanged.

We now move on to the proof of Claim 4.16 in discrete time. If the analogue of the sequence $\{h(n)\}$ introduced in the first paragraph after (4.19) is unbounded, we proceed as in the continuous-time case. If this sequence is bounded, we may assume that is is constant since time is now discrete. Now upper semi-continuity of the left sides of (4.17) and (4.18) in α for fixed h is sufficient to prove that the set H is non-empty.

This brings us to the last part of the proof of Claim 4.16, in which we assume that the bounded set H does not contain a height h_3 such that (4.20) holds with $h = h_3$ and $\alpha = \alpha_1^d$. We define h_4 as before (see (4.21)). Lacking continuity, we know only that (4.17) and (4.18) both hold and (4.20) fails at $h = h_4$ and $\alpha = \alpha_1^d$. Now consider the situation at $h = h_4$ and angles α slightly larger than α_1^d. By upper-semicontinuity, (4.20) still fails. There are two possibilities concerning (4.17) and (4.18): either (4.17) fails and (4.18) holds, or (4.18) fails and (4.17) holds (they cannot both fail because of (4.24), and they cannot both hold because of the definition of α_1^d). In the second case, (4.24) and upper semi-continuity imply that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(N_T(\tilde{B}(h,\alpha)) \geq 2CN_0) \geq 1 - \delta^2/4 \tag{4.28}$$

at $h = h_4$ and $\alpha = \alpha_1^d$. At the end of this proof, we shall show that (4.28) implies that either (4.20) holds, or (4.17) holds with h replaced by $h+1$. Since (4.20) fails at $h = h_4$ and $\alpha = \alpha_1^d$, and (4.17) fails at $h = h_4 + 1$ and $\alpha = \alpha_1^d$ (by the definition of h_4), we see that the second case leads to a contradiction.

For the moment, then, let us assume that the second case mentioned in the preceding paragraph does not happen. That is, we assume that (4.17) fails and (4.18) holds at $h = h_4$ and angles α slightly larger than α_1^d, and we continue the argument as in the continuous-time case. Namely, we define α_2^d as before (see (4.22)), and because of our assumption, we are assured that $\alpha_1^d < \alpha_2^d$. We also define α_3^d as in the continuous-time case (see (4.23)). Since (4.20) fails at $h = h_4$ and angles α slightly larger than α_1^d, we have that $\alpha_3^d > \alpha_1^d$. However we cannot prove, as we did in the continuous-time case, that $\alpha_3^d < \alpha_2^d$. Instead, we merely have the trivial inequality $\alpha_3^d \leq \alpha_2^d$.

Now consider the situation at height $h = h_4$ and angles α slightly less than α_3^d. By the definitions of the various angles α_i^d, we can find an angle $\alpha_4^d \in (\alpha_1^d, \alpha_3^d)$ such (4.17) fails, (4.18) holds, and (4.20) fails at $h = h_4$ and $\alpha = \alpha_4^d$. Define

$$h_5 = \sup\{h \leq h_4 : \text{ (4.17) holds at } \alpha = \alpha_4^d\}.$$

As in the continuous-time case, $h_5 \geq h_1$, and since $\alpha_4^d > \alpha_1^d$, (4.18) fails at $h = h_5$ and $\alpha = \alpha_4^d$. Also as in the continuous-time case, we have by monotonicity that (4.20) fails at this location. By definition of h_5, (4.17) fails at $h = h_5 + 1$ and $\alpha = \alpha_4^d$. By (4.24), (4.28) holds at $h = h_5$ and $\alpha = \alpha_4^d$. Thus we have the same contradiction as in the paragraph containing (4.28), once we show that if (4.28) holds, then either (4.20) holds, or else (4.17) holds with h replaced by $h+1$.

To complete the proof, assume that (4.28) holds. Of course, it has been implicit in our discussion that we are interested only in heights $h \geq h_1$ and angles $\alpha \in [0, \alpha_0]$. Because we are in discrete time, we may take $\alpha_0 = \arctan r$, where r is the range of the birth and death rates. Let B denote the box $\tilde{B}(h,\alpha)$, and define two sets:

$$E = T(B) \cap S_d^-(B)$$
$$F = T(B) \setminus E.$$

Let N_E be the cardinality of $(\tilde{\xi}_h^D(B) \times \{h\}) \cap E$, and let N_F be the cardinality of $(\tilde{\xi}_h^D(B) \times \{h\}) \cap F$. In other words, N_E counts the particles on the top of B that are also within $2r$

units of the side of B in the negative d^{th} coordinate direction, while N_F counts particles on the top of B that are more than $2r$ units from that same side, so $N_E + N_F = N_T(B)$. It is obvious that $N_{S_d^-}(B) \geq N_E$. It follows that if the event in (4.28) occurs, then at least one of the following two events also occurs:

$$A_1 = \{N_{S_d^-}(\tilde{B}(h, \alpha)) \geq N^-\}$$
$$A_2 = \{N_F \geq CN_0\}.$$

(We have used here the assumption that $N_0 \geq N^-$.) Both A_1 and A_2 are positive events. By (4.28), their union has probability at least $1 - \delta^2/4$. It follows from Lemma 3.10 that either $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A_1) \geq 1-\delta$, or $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A_2) \geq 1-\delta/4$. The first inequality is (4.20). We claim that the second inequality implies (4.17) with h replaced by $h+1$. To see this, first note that since $\alpha_0 = \arctan r$, then if we let $G = \{\mathbf{x} : (\mathbf{x}, h) \in F\}$, then $G \times \{h+1\} \subseteq T(\tilde{B}(h+1, \alpha))$. Now apply (4.25). This completes the proof of Claim 4.16 in the discrete-time case. The rest of the proof of Lemma 4.1 is the same as in the continuous-time case. □

5. The basic ingredients of our main construction

We state three propositions. Each concerns the ability of a viable process to reproduce a given finite set D within a given target area. These will be the basis of the construction we use in proving our main results. The notation is as in Section 3. See especially (3.7).

(5.1) PROPOSITION. *Suppose $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ is a spin system with translation-invariant attractive rates $(\boldsymbol{\beta}, \boldsymbol{\delta})$ with range r such that $\delta(N_r') > 0$, and in the discrete-time case, $\delta(\emptyset) < 1$. Further suppose that $\xi^{\{0\}}$ is viable. Fix $\varepsilon > 0$. Then, possibly after making a linear change of space-time coordinates that leaves the time coordinate fixed, one can find a finite set $D \subseteq \mathbf{Z}^d$ and numbers $h \geq 0$ and $w_i \geq 0$ so that with probability at least $1 - \varepsilon$, $\{(\mathbf{x}, t) : \mathbf{x} \in \tilde{\xi}_t^D(B(\mathbf{w}, h))\} \cap R(\mathbf{w}, h)$ contains a translate of $D \times \{0\}$, and similarly for $L(\mathbf{w}, h)$.*

This proposition suffices for the proof of Theorem 2.4. In order to prove Corollary 2.7 and Theorem 2.8, we need the following refinement of Proposition 5.1:

(5.2) PROPOSITION. *Let $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ be as in the statement of Proposition 5.1. Fix $\varepsilon > 0$. Then after making a linear change of space-time coordinates that leaves the time coordinate fixed, one can find a finite set $D \subseteq \mathbf{Z}^d$ and numbers $h \geq 0$ and $w_i \geq 0$ so that with probability at least $1 - \varepsilon$, $\{(\mathbf{x}, t) : \mathbf{x} \in \tilde{\xi}_t^D(B(\mathbf{w}, h))\} \cap R^+(\mathbf{w}, h)$ contains a translate of $D \times \{0\}$, and similarly for $R^-(\mathbf{w}, h), L^+(\mathbf{w}, h)$, and $L^-(\mathbf{w}, h)$.*

Proposition 5.2 is used to prove the following proposition, which constitutes the basic building block in the construction. Translates of the event described in this proposition will be used to construct a supercritical discrete-time process.

(5.3) PROPOSITION. *Suppose $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ satisfies the hypotheses of Proposition*

5.1. Fix $k \geq 1$ and for $w_i \geq 0$ and $h \geq 0$, define

$$(5.4) \qquad \begin{aligned} B^0 &= B(0,\ldots,0,w_{d-1},w_d,h) \\ B_k^{\pm} &= B(3kw_1,\ldots,3kw_{d-2},2w_{d-1},2w_d,(k+1)h;0,\ldots,0,\arctan(\frac{\pm w_d}{3h})). \end{aligned}$$

Choose $\varepsilon > 0$. Then there exists a finite set $D \subseteq \mathbf{Z}^d$, numbers w, w_i and h, and a neighbourhood U of $(\boldsymbol{\beta},\boldsymbol{\delta})$ in parameter space, such that for every $(\boldsymbol{\beta}',\boldsymbol{\delta}') \in U$ and every $(\mathbf{x},s) \in B^0$, with $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$-probability at least $1-\varepsilon$, after a suitable change of variables that leaves the time co-ordinate fixed, $\{(\mathbf{y},t) : \mathbf{y} \in \tilde{\xi}_t^{D+\mathbf{x},s}(B_k^{\pm})\}$ contains a translate of $D \times \{0\}$ lying in the set

$$(0,\ldots,0,\pm k\frac{w_d}{3},kh) + B^0 + [-3kw,3kw]^{d-2} \times 0 \times 0 \times 0.$$

Let us briefly describe in words the content of the last proposition. The sets $B_k^{\pm}$ should be visualized as slanted slabs that are quite narrow in the $(d-1)^{\text{st}}$ and d^{th} coordinate directions. The quantity k is a scaling factor which will later be chosen to be ≥ 20. The set B^0 is a 'source' located at the center of the bottom of the slabs $B_k^{\pm}$ and containing an occupied translate of $D \times \{0\}$. The proposition asserts that with high probability, the process restricted to the slabs and starting at the source copy of $D \times \{0\}$ will reproduce occupied translated copies of $D \times \{0\}$ in some target sets located at the tops of the slabs. These target sets are like the source set B^0, except that they are allowed to be thicker in the first $d-2$ coordinate directions.

Proof of Proposition 5.1.

We shall restrict our attention to the part of the proposition concerned with finding an occupied translate of the set D in the box $R(\mathbf{w},h)$. The proof for $L(\mathbf{w},h)$ is completely analogous.

Let $B = B(\mathbf{w}',h';\boldsymbol{\alpha}), D, J$, and α_0 be as in Lemma 4.1, with h' and $\mathbf{w}'$ playing the roles of h and $\mathbf{w}$ in that lemma. The quantity δ of that lemma is assumed to have been chosen suitably small, as described later in this proof. Define $\bar{h} = h' + J$, $\bar{\mathbf{w}} = \mathbf{w}' + (\rho,\rho,\ldots,\rho,\rho + J\tan\alpha_0)$, where $\rho = \max\{\max_i |x_i| : \mathbf{x} \in D\}$, and $\bar{B} = B(\bar{\mathbf{w}},\bar{h};\boldsymbol{\alpha})$. We may think of the dimensions of B as large in comparison with ρ and $J\tan\alpha_0$, so $\bar{B}$ is a slightly thickened version of B. Let $\mathbf{w} = 4\bar{\mathbf{w}}$, $h = 3\bar{h}$ and $\hat{B} = B(\mathbf{w},h;\boldsymbol{\alpha})$. We are going to consider processes restricted to $\bar{B}$ and $\hat{B}$.

Let E^T be the event that there exists an $\mathbf{x} \in \mathbf{Z}^d$ such that $D + \mathbf{x} \subseteq \tilde{\xi}_{\bar{h}}^D(\bar{B})$. One way in which this event can occur is for a site $\mathbf{x}$ in $\tilde{\xi}_{h'}^D(B)$ to 'produce' the occupied set $D + \mathbf{x}$ after J time units. More precisely, the event E^T occurs if there exists an $\mathbf{x} \in \tilde{\xi}_{h'}^D(B)$ such that $D + \mathbf{x} \subseteq \tilde{\xi}_{\bar{h}}^{\{\mathbf{x}\},h'}((D+\mathbf{x}) \times [h',\bar{h}])$. (Note that if $\mathbf{x} \in \tilde{\xi}_{h'}^D(B)$, then $\tilde{\xi}_{\bar{h}}^{\{\mathbf{x}\},h'}((D+\mathbf{x}) \times [h',\bar{h}]) \subseteq \tilde{\xi}_{\bar{h}}^D(\bar{B})$, because of (3.4) and the way in which we have thickened up B to form $\bar{B}$.) We shall see in a moment how to use this fact and (3.15) to show that for any $\gamma > 0$, there exists an N such that

$$(5.5) \qquad P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E^T | N_T(B) \geq N) > 1 - \gamma$$

for all choices of B such that the event that $N_T(B) \geq N$ has positive probability. The value of N can be chosen in a way that depends only on γ, D, J, and the probability on the left side of (3.15). Similarly, we shall indicate how to show that if we let E^+ be the event that there exists a space-time point $(\mathbf{x}, s)$ such that $D + \mathbf{x} \subseteq \tilde{\xi}_s^D(\bar{B})$ and $(\mathbf{x}, s - J) \in S_d^+(B)$, then for all $\gamma > 0$, then there exists an N such that

$$(5.6) \qquad P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E^+ | N_{S_d^+}(B) \geq N) > 1 - \gamma$$

for all choices of B such that the event that $N_{S_d^+}(B) \geq N$ has positive probability. Intuitively, the idea behind both (5.5) and (5.6) is that if a space-time set contains enough occupied points, then it is likely that at least one of those occupied points will produce an occupied translate of the set D in time J. A similar fact can be found in Bezuidenhout and Grimmett (1990; see the proof of Lemma 18). But the argument here is technically slightly more complicated than the one needed in Bezuidenhout and Grimmett (1990), so we shall indicate briefly how it goes.

We focus on the proof of (5.6). The argument for (5.5) is very similar (and slightly easier). Let $X = \{(\mathbf{x}, t) \in S_d^+(B) : \mathbf{x} \in \tilde{\xi}_t^D(B)\}$. In order to be systematic when we refer to various points in X, we lexicographically order the points in space-time, with the time dimension being given precedence over the spatial dimensions, and with the precedence among the spatial dimensions being fixed arbitrarily. Since $S_d^+(B)$ is a closed set and since the processes under consideration have right-continuous paths, it is clear that, with respect to this ordering, there is a minimal space-time point $(\mathbf{x}_0, t_0)$ in X, provided of course that X is non-empty. Now we proceed inductively to define $(\mathbf{x}_k, t_k)$ to be the minimal space-time point such that $(\mathbf{x}_k + D) \times (t_k, t_k + J]$ does not intersect $(\mathbf{x}_j + D) \times [t_j, t_j + J]$ for any $j = 0, \dots, k - 1$. Of course, from some k on, there will not exist a point $(\mathbf{x}_k, t_k)$ that satisfies this definition. Let N' equal the largest integer k such that $(\mathbf{x}_k, t_k)$ exists. A routine argument can be used to show that $N' \geq a N_{S_d^+}(B)$ for some appropriate positive constant a. For each k, let

$$A_k = \{N' \geq k\} \cap \{D + \mathbf{x}_k \subseteq \tilde{\xi}_{t_k+J}^{\{\mathbf{x}_k\}, t_k}((D + \mathbf{x}_k) \times [t_k, t_k + J])\}.$$

Thus, if $N' \geq k$, A_k is the event that the occupied site $(\mathbf{x}_k, t_k)$ produces the set $D + \mathbf{x}_k$ in time J. Note that E^+ contains the union of the events A_k. Thus, because of the relationship between $N_{S_d^+}(B)$ and N', in order to prove (5.6), it is enough to show

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\cup A_k | N' \geq N) > 1 - \gamma$$

for sufficiently large N. This last inequality is an immediate consequence of the following:

$$(5.7) \qquad P_{\boldsymbol{\beta},\boldsymbol{\delta}}(A_{k+1} | N' > k, A_0^c, \dots, A_k^c) > b$$

for all k and some positive constant b independent of k. And (5.7) follows in a straightforward manner from the strong Markov property together with (3.5) and (3.15). This completes our justification of (5.5) and (5.6).

If we choose $\gamma > 0$ sufficiently small and then choose a sufficiently large N so that (5.5) and (5.6) both hold, then by applying Lemma 4.1 with $N_0 = N^{\pm} = N$ and a sufficiently small $\delta > 0$, we see that B, D, J, and α_0 can be chosen so that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E^T)(P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E^+))^3 > 1 - \varepsilon. \tag{5.8}$$

Lemma 4.1 also allows us to choose the width w'_d as large as we wish, so we may assume that

$$w_d > 48(r + \rho + J \tan \alpha_0) \tag{5.9}$$

where, as above, $w_d = 4(w'_d + \rho + J \tan \alpha_0)$. Let us assume that the event E^+ occurs. Then there exists a space-time point $(\mathbf{x}, s)$ near the right edge of the box $\bar{B}$ such that $D + \mathbf{x} \subseteq \tilde{\xi}^D_s(\bar{B})$. We can choose the time s to be as small as possible, and choose the site $\mathbf{x}$ in some systematic way that does not depend on what happens after time s. We describe the situation informally by saying that 'the process has moved the set D to the right from $(0, 0)$ to $(\mathbf{x}, s)$'. We employ similar informal terminology when the event E^T occurs: we say that 'the process has moved the set D up from $(0, 0)$ to $(\mathbf{x}, \bar{h})$ if $D + \mathbf{x} \subseteq \tilde{\xi}^D_{\bar{h}}(\bar{B})$'. Again, we can choose the site $\mathbf{x}$ in a systematic way that does not depend on what happens after time $\bar{h}$.

Assume that the process has moved the set D to the right from $(0, 0)$ to (x_1, s_1). This happens with high probability. Now think of restarting the process at the space-time point $(\mathbf{x}_1, s_1)$. Using the strong Markov property, we see that with high probability, the restarted, restricted process $\tilde{\xi}^{D+\mathbf{x}_1,s_1}(\bar{B} + (\mathbf{x}_1, s_1))$ moves the set D to the right from $(\mathbf{x}_1, s_1)$ to a point $(\mathbf{x}_2, s_2)$ near the right edge of the box $\bar{B} + (\mathbf{x}_1, s_1)$. Next, instead of moving again to the right, we move up. That is, the process

$$\tilde{\xi}^{D+\mathbf{x}_2,s_2}(\bar{B} + (\mathbf{x}_2, s_2))$$

moves the set D up from $(\mathbf{x}_2, s_2)$ to a point $(\mathbf{x}_3, s_3)$ at the top of the box $\bar{B} + (\mathbf{x}_2, s_2)$. And finally, we move the set D once more to the right, this time from the point $(\mathbf{x}_3, s_3)$ to a point $(\mathbf{x}_4, s_4)$ near the right edge of $\bar{B} + (\mathbf{x}_3, s_3)$. All of this happens with high probability. In fact, the strong Markov property and (5.8) imply that this sequence of four movements of the set D (two to the right, followed by one up, followed by another to the right) happens with probability greater than $1 - \varepsilon$.

Now we observe that the bound in (5.9) and the boxes $\bar{B}$ and $\hat{B}$ have been chosen in such a way that at least one of the sets $D \times \{0\} + (\mathbf{x}_i, s_i)$ $(i = 2, 3, 4)$ lies inside $\hat{R} = R(\mathbf{w}, h; \boldsymbol{\alpha})$, where, as above, $\mathbf{w} = 4\bar{\mathbf{w}}$ and $h = 3\bar{h}$. Note that $\hat{R}$ is the upper right corner box of $\hat{B}$. Choose j to be the smallest value of i such that $(D \times \{0\}) + (\mathbf{x}_i, s_i)$ lies inside $\hat{R}$. We have chosen the dimensions of the boxes so that it is necessarily the case that the first j moves described in the preceding paragraph all take place with processes that are restricted to subsets of the box $\hat{B}$.

Let us summarize. We have described a procedure whereby, with probability at least $1 - \varepsilon$, an occupied translated copy of the set D lying in the set $R(\mathbf{w}, h; \boldsymbol{\alpha})$ is produced by the restricted process $\tilde{\xi}^D(\hat{B})$. To complete the proof, we simply make the change of coordinates

$$(x_1, \ldots, x_d, t) \to (x_1, \ldots, x_{d-1}, x_d - t \tan \alpha_d, t). \tag{5.10}$$

Since $\alpha_1, \ldots, \alpha_{d-1}$ are all equal to 0, the result now follows. □

Proof of Proposition 5.2

We now complete the proof of Proposition 5.2, making use of Proposition 5.1. It follows from Proposition 5.1 that after a change of co-ordinates (as in (5.10)), there exists a box $B' = B(\mathbf{w}', h')$ such that if E_L is the event that the set $\{(\mathbf{x}, t) : \mathbf{x} \in \tilde{\xi}_t^D(B(\mathbf{w}', h'))\} \cap L(\mathbf{w}', h')$ contains a translate of $D \times \{0\}$ and if E_R is the analogous event involving $R(\mathbf{w}', h')$, then for the ε in the statement of Proposition 5.2,

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_L \cap E_R) \geq 1 - \varepsilon^2. \tag{5.11}$$

We shall work in the transformed co-ordinates just described.

The strategy of the proof is to find a hyperlane that splits the two corner boxes $R(\mathbf{w}', h')$ and $L(\mathbf{w}', h')$ into four boxes, each of which is likely to contain an occupied translate of the set $D \times \{0\}$. The procedure for finding such a hyperplane is very similar to that used in the proof of the Fundamental Lemma 4.1. Once this procedure is successfully carried out, a change of co-ordinates completes the proof.

For each real number a and angle θ, let $P(a, \theta)$ be the hyperplane in $\mathbf{R}^d \times \mathbf{R}^+$ that passes through the origin $(\mathbf{0}, 0)$ of space-time and contains the set:

$$\ell(a, \theta) \;=\; \{(\mathbf{y}, t) \in \mathbf{R}^d \times \mathbf{R}^+ : y_{d-1} = a + y_d \tan \theta, t = h'\}$$

in $\mathbf{R}^d \times \mathbf{R}^+$. Since $P(a, \theta) = P(a, \theta + \pi)$, we may restrict our attention to $\theta \in [-\pi/2, \pi/2]$. Each $P(a, \theta)$ divides $\mathbf{R}^d \times \mathbf{R}^+$ into two half-spaces $H^{\pm}(a, \theta)$, where $H^{\pm}(a, \theta)$ is the half-space containing $(0, \ldots, 0, a \pm 1, \mp \tan \theta, h')$. Let $L^{\pm}(a, \theta)$ and $R^{\pm}(a, \theta)$ be the respective closures of the intersections of $L(\mathbf{w}', h')$ and $R(\mathbf{w}', h')$ with $H^{\pm}(a, \theta)$. See Figures 4 and 5. Since at least one of these four sets is empty for $|a| > 3w_{d-1}$, we shall restrict our attention from here on to $a \in [-3w_{d-1}, 3w_{d-1}]$. Let $E_L^{\pm}(a, \theta)$ and $E_R^{\pm}(a, \theta)$ be the analogues of the events E_R and E_L with $L(\mathbf{w}', h')$ and $R(\mathbf{w}', h')$ replaced by $L^{\pm}(a, \theta)$ and $R^{\pm}(a, \theta)$. We show that there exists a hyperplane $P(a, \theta)$ so that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_i^{\pm}(a, \theta)) \;\geq\; 1 - \varepsilon \quad \text{for} \quad i = L, R. \tag{5.12}$$

The assertion of Proposition 5.2 will follow from this after an appropriate change of variables in space-time.

For $i = L, R$, define

$$f_i^{\pm}(a, \theta) \;=\; P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_i^{\pm}(a, \theta)).$$

Note that each $f_i^{\pm}(a, \theta)$ is jointly upper semi-continuous in (a, θ). By (5.11) we have that when $a = -3w_{d-1}$, $f_i^+(a, 0) \geq 1 - \varepsilon^2 > 1 - \varepsilon$ for $i = L, R$. Let

$$a_1 \;=\; \sup\{a \geq -3w_{d-1} : \exists\, \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \text{ with } f_i^+(a, \theta) \geq 1 - \varepsilon, i = L, R\}. \tag{5.13}$$

The set in (5.13) is contained in $[-3w_{d-1}, 3w_{d-1}]$ so a_1 is finite. Moreover since $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is compact and by the joint upper semi-continuity in (a, θ) of f_i^+, we have that a_1 is in the set of which it is the supremum.

Suppose there exists $\theta \geq 0$ for which $f_L^+(a_1, \theta)$ and $f_R^+(a_1, \theta)$ are both greater than or equal to $1 - \varepsilon$. If this does not occur then there exists $\theta < 0$ satisfying this condition and we modify the argument given below accordingly. Define:

$$\theta_1 \;=\; \sup\{\theta \in [0, \pi/2] : f_i^+(a_1, \theta) \geq 1 - \varepsilon, i = L, R\}.$$

Note that $\theta \leq \pi/2$ since $f_R^+(a, \pi/2) = 0$ for every a. By the upper semi-continuity in θ of $f_i^+(a_1, \theta)$, we have $f_i^+(a_1, \theta_1) \geq 1 - \varepsilon$ for $i = L, R$. By definition of θ_1 and since $f_R^+(a_1, \theta)$ is non-increasing and $f_L^+(a_1, \theta)$ is non-decreasing as θ increases, we must have that $f_R^+(a_1, \theta) < 1 - \varepsilon$ if $\theta_1 < \theta \leq \pi/2$. By Lemma 3.10 and because $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_R) \geq 1 - \varepsilon^2$ and $E_R = E_R^+(a, \theta) \cup E_R^-(a, \theta)$, $f_R^-(a_1, \theta) > 1 - \varepsilon$ for such θ. Therefore by upper semi-continuity in θ, $f_R^-(a_1, \theta_1) \geq 1 - \varepsilon$. If $f_L^-(a_1, \theta_1) \geq 1 - \varepsilon$, (5.12) holds with $(a, \theta) = (a_1, \theta_1)$.

So we may suppose that $f_L^-(a_1, \theta_1) < 1 - \varepsilon$. As θ decreases, $f_L^-(a_1, \theta)$ increases. Define

$$\theta_2 \;=\; \sup\{\theta \leq \theta_1 : f_L^-(a_1, \theta) \geq 1 - \varepsilon\}. \tag{5.14}$$

The set in (5.14) is non-empty since $f_L^-(a_1, -\pi/2) \geq 1 - \varepsilon^2$. By the upper semi-continuity in θ of $f_L^-(a_1, \theta)$, we have that $f_L^-(a_1, \theta_2) \geq 1 - \varepsilon$ and so $\theta_2 < \theta_1$. Since $f_L^-(a_1, -\frac{\pi}{2}) \geq 1 - \varepsilon$, we also have $\theta_2 \geq -\frac{\pi}{2}$. By definition of θ_2, $f_L^-(a_1, \theta) < 1 - \varepsilon$ if $\theta_2 < \theta \leq \theta_1$. Therefore for such θ, Lemma 3.10 implies as above that $f_L^+(a_1, \theta_2) > 1 - \varepsilon$. So by upper semi-continuity, $f_L^+(a_1, \theta_2) \geq 1 - \varepsilon$. If we had that for some $\tilde{\theta} \in (\theta_2, \theta_1)$, $f_R^-(a_1, \tilde{\theta}) < 1 - \varepsilon$, then by the upper semi-continuity in a of $f_L^-(a, \tilde{\theta})$ and $f_R^-(a, \tilde{\theta})$, there would exist $\tilde{a} > a_1$ so that $f_i^-(\tilde{a}, \tilde{\theta}) < 1 - \varepsilon$ for $i = L, R$. Since $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_L \cap E_R) \geq 1 - \varepsilon^2$, using Lemma 3.10 as above, we would have that $f_i^+(\tilde{a}, \tilde{\theta}) > 1 - \varepsilon$, violating the definition of a_1. So if $\theta \in (\theta_2, \theta_1)$, $f_R^-(a_1, \theta) \geq 1 - \varepsilon$. Since $f_R^+(a_1, \theta)$ increases as θ decreases, we have that $f_R^+(a_1, \theta) \geq f_R^+(a_1, \theta_1) \geq 1 - \varepsilon$ for such θ. So by upper semi-continuity, $f_R^\pm(a_1, \theta_2) \geq 1 - \varepsilon$. So (5.12) holds with $(a, \theta) = (a_1, \theta_2)$.

Assume that (5.12) holds. Make the change of variables

$$\begin{aligned}
y_i &\;\to\; y_i \quad \text{for} \quad i = 1, \ldots, d-2 \\
y_{d-1} &\;\to\; -\frac{at}{h'} \cos\theta + y_{d-1} \cos\theta - y_d \sin\theta \\
y_d &\;\to\; -\frac{at}{h'} \sin\theta + y_{d-1} \sin\theta + y_d \cos\theta \\
t &\;\to\; t
\end{aligned}$$

Let $w_i = w_i'$ for $1 \leq i \leq d-2$, $w_{d-1} = a\cos\theta + w_{d-1}' \cos\theta + w_d' |\sin\theta|$, $w_d = a|\sin\theta| + w_{d-1}' |\sin\theta| + w_d' \cos\theta$ and $h = h'$. The proof is now complete. □

Proof of Proposition 5.3

This result is the analogue of Lemma (19) in Bezuidenhout and Grimmett and, details aside, its proof is essentially the same as the proof of that result. The statement of Proposition 5.3 involves two parts involving regions $B_k^{\pm}$. We prove the one involving B_k^{+}. The proof of the other is entirely analogous.

Fix $\varepsilon_1 > 0$. Its value, depending on ε, will be determined later. We use Proposition 5.2 to choose a set $D \subseteq \mathbf{Z}^d$ satisfying (3.15) and numbers w_i and h so that (after a suitable change of co-ordinates if necessary) the following is true. Let $B^0 = B(\mathbf{w}, h)$, and let $R^{\pm} = R^{\pm}(\mathbf{w}, h)$ and $L^{\pm} = L^{\pm}(\mathbf{w}, h)$ be as in (3.7). Let $E_R^{\pm}$ be the event that $\{(\mathbf{y}, s) : \mathbf{y} \in \tilde{\xi}_s^D(B^0)\}$ contains a translate of $D \times \{0\}$ contained in $R^{\pm}$ and define $E_L^{\pm}$ similarly. Then by Proposition 5.2, we can make appropriate choices so that $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(E_i^{\pm}) > 1 - \varepsilon_1$ for $i = R, L$. Note that we may assume that $w_i = w$, say, for $i = 1, \dots, d-2$, so that

$$B^0 \;=\; [-w, w]^{d-2} \times [-w_{d-1}, w_{d-1}] \times [-w_d, w_d] \times [0, h],$$

and, for example,

$$R^- \;=\; [-w, w]^{d-2} \times [-w_{d-1}, 0] \times [\frac{w_d}{3}, w_d] \times [\frac{h}{3}, h].$$

Each event $E_i^{\pm}$ depends only on the configuration inside the space-time region B^0 and so its probability depends continuously on the parameters $(\boldsymbol{\beta}, \boldsymbol{\delta})$. Therefore there exists a neighbourhood U of $(\boldsymbol{\beta}, \boldsymbol{\delta})$ in the space of parameters so that if $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U$ then $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}(E_i^{\pm}) > 1 - \varepsilon_1$ for $i = R, L$. As a first step in establishing the part of Proposition 5.3 that concerns B_k^{+}, we show:

(5.15) Claim. *Let $B^1 = B(3w, \dots, 3w, 2w_{d-1}, 4w_d/3, 2h)$. Then for every $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U$, and every $(\mathbf{x}, t) \in B^0$, with $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$-probabilty at least $(1 - \varepsilon_1)^3$, $\{(\mathbf{y}, s) : \mathbf{y} \in \tilde{\xi}_s^{D+\mathbf{x},t}(B^1)\}$ contains a translate of $D \times \{0\}$ contained in*

$$B^2 = [-3w, 3w]^{d-2} \times [-w_{d-1}, w_{d-1}] \times [-2w_d/3, 4w_d/3] \times [h, 2h]. \tag{5.16}$$

Proof of Claim 5.15: We consider a number of space-time regions; the space-time region B^0 is contained in their union and four are auxiliary regions. For $i = -2, \dots, 3$ and $j = 1, 2, 3$, let

$$T^+(i,j) = [-(j-1)w, (j-1)w]^{d-2} \times [0, w_{d-1}] \times ((i-1)\frac{w_d}{3} + [0, \frac{w_d}{3}]) \times ((j-1)\frac{h}{3} + [0, \frac{h}{3}])$$
$$T^-(i,j) = [-(j-1)w, (j-1)w]^{d-2} \times [-w_{d-1}, 0] \times ((i-1)\frac{w_d}{3} + [0, \frac{w_d}{3}]) \times ((j-1)\frac{h}{3} + [0, \frac{h}{3}]).$$

We consider also the auxiliary regions $T^{\pm}(4,2)$ and $T^{\pm}(4,3)$. If $(\mathbf{x}, t) \in T^{\pm}(i,j)$ with $-2 \le i \le 1$, we use translation invariance together with the fact that $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}(E_R^{\mp}) > 1 - \varepsilon_1$ to conclude that with $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$-probability at least $1 - \varepsilon_1$, $\{(\mathbf{y}, s) : \mathbf{y} \in \tilde{\xi}_s^{D+\mathbf{x},t}(B^0 + (\mathbf{x}, t))\}$

contains a translate of $D \times \{0\}$ contained in $R^{\mp} + (\mathbf{x}, t)$. In particular if $(\mathbf{x}, t) \in T^{+}(i, 3)$ with $-2 \leq i \leq 1$, then since for such $(\mathbf{x}, t)$,

$$B^0 + (\mathbf{x}, t) \subseteq [-3w, 3w]^{d-2} \times [-w_{d-1}, 2w_{d-1}] \times [(i-4)\frac{w_d}{3}, (i+3)\frac{w_d}{3}] \times [\frac{2h}{3}, 2h],$$

and

$$R^{-} + (\mathbf{x}, t) \subseteq [-3w, 3w]^{d-2} \times [-w_{d-1}, w_{d-1}] \times [i\frac{w_d}{3}, (i+3)\frac{w_d}{3}] \times [h, 2h],$$

we have that the event in the statement of Claim 5.15 occurs with $P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}$-probability at least $1 - \varepsilon_1$ in this case. If $(\mathbf{x}, t) \in T^{-}(i, 3)$ with $-2 \leq i \leq 1$ we use a translate of E_R^{+} to draw a similar conclusion. Finally, for $(x, t) \in T^{\pm}(i, 3)$ with $2 \leq i \leq 4$ we use a translate of $E_L^{\mp}$. Thus we have that if $(\mathbf{x}, t) \in T^{\pm}(i, 3)$ with $-1 \leq i \leq 4$ then the event in the statement of Claim 5.15 occurs with $P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}$-probability at least $1 - \varepsilon_1$.

If $j = 2$ and $(\mathbf{x}, t) \in T^{\pm}(i, j)$ and one uses a translate of the event $E_R^{\mp}$ or of $E_L^{\mp}$, depending on whether $-2 \leq i \leq 1$ or $2 \leq i \leq 4$, one finds that the region into which the translate of $D \times \{0\}$ falls with large probability may not be entirely contained in the region B^2 of (5.16). However, the region obtained is contained in

$$B^2 \cup \bigcup_{i=-1}^{4} [T^{+}(i, 3) \cup T^{-}(i, 3)].$$

If the lowest translate of D in $\{(\mathbf{y}, s) : \mathbf{y} \in \xi_s^{D+\mathbf{x}, t}(B^0 + (\mathbf{x}, t))\}$ is in $T^{\pm}(i, 3)$, then we use the strong Markov property to restart and then use the previous step. We conclude that if $-2 \leq i \leq 4$ and $(\mathbf{x}, t) \in T^{\pm}(i, 2)$ then with $P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}$-probability at least $(1 - \varepsilon_1)^2$, the event in the statement of Claim 5.15 occurs.

Similarly we find that if $-2 \leq i \leq 3$ and $(\mathbf{x}, t) \in T^{\pm}(i, 1)$ then the event in the statement of Claim 5.15 occurs with $P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}$-probability at least $(1 - \varepsilon_1)^3$. This completes the proof of Claim 5.15. □

Iterating this step k times gives the result of Proposition 5.3 as long as ε_1 is chosen small enough to ensure that $(1 - \varepsilon_1)^{3k} \geq 1 - \varepsilon$. This completes the proof of Proposition 5.3. □

6. Proof of the main results

Construction of a process

Fix $\varepsilon_0 > 0$ and $k \geq 20$. Assume $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$ satisfies the hypotheses of Proposition 5.1. Choose $\mathbf{w}$, h, D, and a neighbourhood U of $(\boldsymbol{\beta}, \boldsymbol{\delta})$ so that the statement of Proposition 5.3 holds.

We shall define a discrete-time Markov process $\Xi_n = (I_n, P_n)$ taking values in

$$\left[\{0, 1\} \times (\mathbf{Z}^d \times \mathbf{R}^{+})\right]^{\mathbf{Z}^{d-1}}.$$

For $\mathbf{x} \in \mathbf{Z}^{d-1}$, $\Xi_n(\mathbf{x})$ will be defined only if $x_{d-1} + n$ is even.

Before giving precise definitions, we shall give an informal description of the various objects that are involved in the process we are about to construct. For $\mathbf{x} \in \mathbf{Z}^{d-1}$ and $n \geq 0$ subject to the above parity restriction, the quantity $I_n(\mathbf{x})$ is a random variable which can take only the values 0 and 1 and indicates whether or not the site $\mathbf{x}$ is to be considered occupied at time n. The space-time point $(\mathbf{x}, n)$ is identified with a space-time box in our original process $\{\xi_t^A : t \geq 0, A \subseteq \mathbf{Z}^d\}$, and occupation in the new process signifies that a certain event has occurred involving that space-time box in the original process. When $I_n(\mathbf{x}) = 1$, the quantity $P_n(\mathbf{x})$ gives us more precise information about about where in space-time the corresponding event occurred in the original process. The kinds of events that we shall be concerned with in the definitions of I_n and P_n will be of the type described in Proposition 5.3. Note that the spatial dimension of the new process is one less than that of the original process. In trying to match up the coordinates, the reader should think of the first $d-2$ coordinates of the two processes as corresponding to each other, and the $(d-1)^{\text{th}}$ coordinate of the new process as corresponding to the d^{th} coordinate of the old process. In our argument, the $(d-1)^{\text{th}}$ coordinate of the old process will essentially disappear, because we shall restrict everything that happens in that coordinate direction to the interval $[-w_{d-1}, w_{d-1}]$.

For $(i, n) \in \mathbf{Z} \times \mathbf{Z}^+$ with $i + n$ even, let

$$V(i,n) \;=\; (0, ik\frac{w_d}{3}, nkh) + [-w_{d-1}, w_{d-1}] \times [-w_d, w_d] \times [0, h].$$

If $\mathbf{x} = (x_1, \ldots, x_{d-1})$, $x_{d-1} + n$ is even, and $I_n(\mathbf{x}) = 1$, then $P_n(\mathbf{x})$ will lie in

$$x_1 \times \cdots \times x_{d-2} \times V(x_{d-1}, n). \tag{6.1}$$

For $(\mathbf{u}, t) \in \mathbf{Z}^d \times \mathbf{R}^+$, and $(u_{d-1}, u_d, t) \in V(i, n)$ for some i and n, define

$$\chi^{\pm}(\mathbf{u}, t) \;=\; \{(\mathbf{v}, s) \in \mathbf{Z}^d \times \mathbf{R}^+ : \mathbf{v} \in \tilde{\xi}_s^{D+\mathbf{x},t}((u_1, \ldots, u_{d-2}, 0, ik\frac{w_d}{3}, nkh) + B_k^{\pm})\}.$$

(See (5.4) for the definition of $B_k^{\pm}$ and also for the definition of B^0 in the next display.) For such $(\mathbf{u}, t)$, let $G^{\pm}(\mathbf{u}, t)$ be the event that $\chi^{\pm}(\mathbf{u}, t)$ contains a translate of $D \times \{0\}$ lying inside

$$(u_1, \ldots, u_{d-2}, 0, (i \pm 1)k\frac{w_d}{3}, (n+1)kh) + B^0 + [-3kw, 3kw]^{d-2} \times 0 \times 0 \times 0.$$

Then by translation invariance, Proposition 5.3, and the choice of $\mathbf{w}$, h, D, and U, for every $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U$ and every $(\mathbf{u}, t)$ in $\mathbf{Z}^d \times \mathbf{R}^+$ with

$$(u_{d-1}, u_d, t) \in \bigcup_{i,n: n+i \text{ even}} V(i, n),$$

we have that

$$P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}(G^{\pm}(\mathbf{u}, t)) \geq 1 - \varepsilon_0. \tag{6.2}$$

We define Ξ_n recursively, beginning with Ξ_0. For $n = 0$ and $\mathbf{x} \in \mathbf{Z}^{d-1}$, define $I_n(\mathbf{x})$ to be the indicator function of $\mathbf{x} = \mathbf{0}$ and define $P_0(\mathbf{x}) = (\mathbf{0}, 0) \in \mathbf{Z}^d \times \mathbf{R}^+$ for every $\mathbf{x} \in \mathbf{Z}^{d-1}$. Suppose that we have defined the random variables $\Xi_k(\mathbf{x})$ for $k = 0, \dots, n$ and $\mathbf{x} \in \mathbf{Z}^{d-1}$. Consider $\mathbf{x} \in \mathbf{Z}^{d-1}$ with $x_{d-1} + n + 1$ even. Unless there exists a $\mathbf{y} = (y_1, \dots, y_{d-1})$ in $\mathbf{Z}^{d-1}$ with

$$\begin{aligned} |x_\ell - y_\ell| &\le 3w_\ell \quad \text{for } \ell = 1, \dots, d-2, \\ |x_{d-1} - y_{d-1}| &= 1 \end{aligned} \tag{6.3}$$

so that $I_n(\mathbf{y}) = 1$, we define $I_{n+1}(\mathbf{x}) = 0$ and $P_{n+1}(\mathbf{x}) = (\mathbf{0}, 0)$. Suppose there exists a $\mathbf{y}$ satisfying (6.3) for which $I_n(\mathbf{y}) = 1$. Suppose in addition that for some such $\mathbf{y}$,

(i) $y_{d-1} = x_{d-1} \pm 1$,
(ii) the event $G^{\mp}(P_n(\mathbf{y}))$ occurs,
(iii) the lowest translate of $D \times \{0\}$ lying inside

$$\chi^{\mp}(P_n(\mathbf{y})) \cap \big[(y_1, \dots, y_{d-2}, 0, x_{d-1}kw_d, (n+1)kh) + B^0 + [-3kw, 3kw]^{d-2} \times 0 \times 0 \times 0\big]$$

is $(D + (x_1, \dots, x_{d-2}, \tilde{x}_{d-1}, \tilde{x}_d)) \times \{\tilde{t}\}$ for some $(\tilde{x}_{d-1}, \tilde{x}_d, \tilde{t}) \in V(x_{d-1}, n+1)$. (In discrete time there may not be a unique 'lowest translate', but in that case we can choose among the alternatives in some prescribed deterministic way that does not depend on the future.)

Then we define $I_{n+1}(\mathbf{x}) = 1$ and we choose $P_{n+1}(\mathbf{x}) = (\mathbf{z}, t)$ so that the lowest translate of $D \times \{0\}$ contained in

$$x_1 \times \dots \times x_{d-2} \times V(x_{d-1}, n) \cap \big[\chi^+(P_n(\mathbf{y})) \cup \chi^-(P_n(\mathbf{y}))\big]$$

for some $\mathbf{y}$ satisfying (6.3) is at $(D + \mathbf{z}) \times \{t\}$.

Let $\mathcal{F}_n$ be the σ-field generated by $\{\Xi_m : 0 \le m \le n\}$. Then for every $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U$, $(\Xi_n, \mathcal{F}_n, P_{\boldsymbol{\beta}', \boldsymbol{\delta}'})$ is a Markov process and in fact, the conditional distribution of $\Xi_{n+1}(\mathbf{x})$ given $\mathcal{F}_n$ depends only on the quantities $\Xi_n(\mathbf{y})$ for $\mathbf{y}$ such that (6.3) holds.

Proof of Theorem 2.4 and Corollary 2.7

With $\{\Xi_n(\mathbf{x}) : \mathbf{x} \in \mathbf{Z}^{d-1}, n \ge 0\}$ as above, define random variables $Z_n(x)$ for $x \in \mathbf{Z}$ and $n \ge 0$ with $x + n$ even as follows: $Z_n(x) = 1$ if $I_n(x_1, \dots, x_{d-2}, x) = 1$ for some $(x_1, \dots, x_{d-2}) \in \mathbf{Z}^{d-2}$ and $Z_n(x) = 0$ otherwise. Then it follows from the definitions that for $i = 0, 1$,

$$P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}\left(Z_{n+1}(x) = i | \mathcal{F}_n\right) = P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}\left(Z_{n+1}(x) = i | \{\Xi_n(y) : |y_{d-1} - x| = 1\}\right).$$

Using (6.2), we have that

$$P_{\boldsymbol{\beta}', \boldsymbol{\delta}'}\left(Z_{n+1}(x) = 1 | \mathcal{F}_n\right) \ge p(Z_n(x-1), Z_n(x+1))$$

where $p(0,0) = 0$ and $p(1,0) = p(0,1) = p(1,1) = 1 - \varepsilon_0$. For fixed n, conditioned on $\mathcal{F}_n$, the random variables $Z_{n+1}(x)$ and $Z_{n+1}(y)$ are correlated only if $x = y \pm 2$, as long as we choose k in the statement of Proposition 5.3 sufficiently large ($k \geq 20$ will do). Therefore one can modify the usual contour argument for 1-dependent oriented site percolation – see Durrett (1988, p. 85) – to show that if ε_0 is sufficiently small, then with positive $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$-probability, there exist infinitely many values of n for which

$$\{x : Z_n(x) = 1\} \neq \emptyset. \tag{6.4}$$

For any n for which (6.4) holds, we have that $\tilde{\xi}^D_{nkh}(B(\infty, \dots, \infty, 2w_{d-1}, \infty, \infty)) \neq \emptyset$. So we have that with positive $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$-probability, ξ^D survives inside

$$B \;=\; B(\infty, \dots, \infty, 2w_{d-1}, \infty, \infty). \tag{6.5}$$

Thus we have that if $P_{\boldsymbol{\beta},\boldsymbol{\delta}}(\xi^0 \text{ survives }) > 0$ and $\delta(N'_r) > 0$, and, in the discrete-time case, $\delta(\emptyset) < 1$, then there exists w_{d-1} and a neighbourhood U of $(\boldsymbol{\beta}, \boldsymbol{\delta})$ in the space of parameters so that for every $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U$, ξ^D survives inside B (see (6.5)) with positive $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$-probability.

We have just proved that there is a positive survival probability for a certain open set U of parameter values, with the initial state being a certain finite set D. It remains to prove an analogous result for the process starting at the singleton $\{0\}$. Since the event in (3.15) depends only on the configuration inside the finite space-time region $D \times [0, h]$, its probability is a continuous function of the parameters. Therefore there exists a neighbourhood U' of $(\boldsymbol{\beta}, \boldsymbol{\delta})$ in the space of parameters so that if $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U'$, then (3.15) holds with $P_{\boldsymbol{\beta},\boldsymbol{\delta}}$ replaced by $P_{\boldsymbol{\beta}',\boldsymbol{\delta}'}$. It follows from this and the Markov property that if $(\boldsymbol{\beta}', \boldsymbol{\delta}') \in U \cap U'$, ξ^0 survives inside $B(\infty, \dots, \infty, 2w_{d-1}, \infty, \infty)$ with positive probability. Theorem 2.4 and, in fact, Corollary 2.7 now follow. □

Proof of Theorem 2.8

The proof is by induction on the dimension d. We note that we have already established the result when $d = 2$ and that for general $d > 2$ we have constructed a discrete-time Markov process $\Xi_n = (I_n, P_n)$ where I_n takes values in $\{0, 1\}^{\mathbf{Z}^{d-1}}$ and P_n is an auxiliary process. If $\mathbf{x} \in \mathbf{Z}^{d-1}$ and $I_n(\mathbf{x}) = 1$, then $P_n(\mathbf{x})$ gives the precise location of the translate of $D \times \{0\}$ in the underlying process whose presence makes the 'cell' in (6.1) occupied.

In order to decrease the spatial dimension once more, we would like to apply the result of Corollary 2.7 to the process $\{I_n : n \geq 0\}$. However, since $\{I_n\}$ does not satisfy the hypotheses of that result (it is not even a Markov process), we have to check that the proof can be modified to accomodate a wider class of processes.

There is one respect in which the process $\{I_n : n \geq 0\}$ is easier to handle than the original one, and that is that we can assume that the probability of 'survival' (i.e. the probability that the set of n for which

$$\{x : I_n(x) = 1\} \neq \emptyset$$

is unbounded) starting with $\{x : I_0(x) = 1\} = \{0\}$ is as close to 1 as desired. The reason we can do this is that we can choose ε in the statement of Proposition 5.3 as small as we like. In particular, for the new process, we need not repeatedly generate translates of a large set ($D \times \{0\}$ for the original process); the process can be restarted from a single 'occupied site'.

The basic ingredients needed to make the proof of the technical results leading up to the proof of Corollary 2.7 work are:

(i) the FKG inequality for events like those whose probabilities appear in (3.12), and

(ii) a result which states that

$$P_{\boldsymbol{\beta},\boldsymbol{\delta}}\left(\Xi \text{ dies out } \mid |\{(\mathbf{x},n) \in \mathcal{R} : \Xi_n(\mathbf{x}) = 1\}| < N\right) \geq \gamma > 0.$$

for certain space-time regions $\mathcal{R}$.

In both cases, these results are available because the events can be interpreted in terms of events in the underlying process, in which the analogues of (*i*) and (*ii*) hold, and the result for the rescaled process follows. The only other potential difficulty is that, in the proofs of Propositions 5.1 and 5.3, we use the strong Markov property to restart the process. The process $\{I_n : n \geq 0\}$ is not Markov. However the process $\Xi_n = (I_n, P_n)$ does satisfy the Markov property, and this fact suffices. Keeping these observations in mind, one can check that the proof of Corollary 2.7 carries over to the present setting.

We obtain a process

$$\{\Xi_n^{(2)}(\mathbf{x}) = (I_n^{(2)}(\mathbf{x}), P_n^{(2)}(\mathbf{x})) : n \geq 0, \mathbf{x} \in \mathbf{Z}^{d-2}\}$$

where, if $I_n^{(2)}(\mathbf{x}) = 1$, $P_n^{(2)}(\mathbf{x})$ gives the precise location of the lowest occupied (rescaled) site in $\mathbf{Z}^{d-1} \times \mathbf{R}^+$ that causes $I_n^{(2)}(\mathbf{x})$ to take the value 1. As long as the probability that $\{I_n : n \geq 0\}$ survives is sufficiently close to 1, we have that $\{I_n^{(2)} : n \geq 0\}$ survives with probability as close to 1 as we like and hence that, for some $D \subseteq \mathbf{Z}^d$, and w_{d-2} and w_{d-1}, the original process $\{\xi_t^D : t \geq 0\}$ survives inside

$$\mathbf{Z}^{d-3} \times [-w_{d-2}, w_{d-2}] \times [-w_{d-1}, w_{d-1}] \times \mathbf{Z} \times \mathbf{R}^+ \tag{6.6}$$

with positive probability. As before, the fact that D satisfies (3.15) ensures that $\{\xi_t^0 : t \geq 0\}$ survives inside the region (6.6) with positive probability. Since $\{I_n^{(2)} : n \geq 0\}$ is the same sort of process as $\{I_n : n \geq 0\}$, the result follows by induction. □

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Figure Captions

FIG. 1. The projection of $B(\mathbf{w}, h; \alpha)$ onto the (x_i, t) plane. The subsets $T(B)$ and $S_i^{\pm}$ of B are defined in (3.8).

FIG. 2. The choice of w_2^d in the proof of Lemma 4.1. For $i = 0, 1$, B_i is the projection of $B(\mathbf{w}_i, h_i)$ onto the (x_d, t) plane. The heavily shaded region is $S_d^{+}(B)$ where B is a box not containing B_1. Note that it lies completely outside the lightly shaded 'light cone'. The angle labelled $\tilde{\alpha}_0$ is $\arctan[(w_2^d - w_1^d - 2r)/h_1]$.

FIG. 3. A schematic representation of the contradiction reached at the end of the proof of Lemma 4.1. In each case, the shaded region is the projection of $B(\mathbf{w}_0, h)$ onto the (x_d, t) plane.

FIG. 4. The projection onto the (y_d, y_{d-1}) plane at height h' of various regions defined in the proof of Proposition 5.2.

FIG. 5. A three-dimensional view. The regions R and L are projections of $R(\mathbf{w}', h')$ and $L(\mathbf{w}', h')$. The dot in the center is the origin of space-time.