Math 2263 Fall 2008 Midterm 1, WITH SOLUTIONS October 2, 2008

1. (15 points) The lines given parametrically by

$$\langle x, y, z\rangle = \langle 5-t, 3+2t, 1+2t\rangle, \quad -\infty < t < \infty$$

and

$$\langle x, y, z\rangle = \langle 5+2s, 3+2s, 1-s\rangle, \quad -\infty < s < \infty$$

intersect at the point $\langle x, y, z\rangle = \langle 5, 3, 1\rangle$. Find an equation for the **plane** which contains both lines.

SOLUTION: A normal vector $\vec{v}$ is the cross product of the vectors $\langle -1, 2, 2\rangle$ and $\langle 2, 2, -1\rangle$ in the directions of the two given lines.

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{vmatrix} = -6\vec{i} + 3\vec{j} - 6\vec{k}.$$

We can divide $\vec{v}$ by 3, so an equation for the plane is $-2(x-5) + (y-3) - 2(z-1) = 0$, or simplifying:

$$-2x + y - 2z = -9.$$

2. (15 points) Find an equation for the **elliptical cylinder** in (x, y, z)-space containing infinitely many lines parallel to the z-axis, and containing the slanted circle $z = y$, $x^2 + y^2 + z^2 = 4$.

SOLUTION: Eliminate z from the equations $z = y$, $x^2+y^2+z^2 = 4$ to get $x^2+2y^2 = 4$. This is the equation of the ellipse which is the projection of the slanted circle into the (x, y)-plane. So the elliptical cylinder is given by the same equation:

$$x^2 + 2y^2 = 4,$$

as an equation for a surface in (x, y, z)-space.

3. (15 points) Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2 + xy + y^2},$$

or state that it does not exist, giving reasons.

SOLUTION: First try plugging in $x = 0$ and $y = 0$: the numerator and the denominator of the quotient both converge to 0. So it's not easy to tell what the limit is. Try approaching $(0, 0)$ along the line $y = mx$ of slope m: the function is constant along that line and equal to

$$\frac{x^2 - mx^2 + m^2x^2}{x^2 + mx^2 + m^2x^2} = \frac{1 - m + m^2}{1 + m + m^2},$$

which depends on m. So the limit **does not exist**, because you get different limits along lines of different slope m.

4. (15 points) For the function

$$f(x,y) = e^{3y}\cos 2x,$$

find the **second partial derivatives**

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$

SOLUTION: Compute $f_x = \frac{\partial f}{\partial x} = -2e^{3y}\sin 2x$ and $f_y = \frac{\partial f}{\partial y} = 3e^{3y}\cos 2x$. Then

$$f_{xx} = -4e^{3y}\cos 2x = -4f, \quad f_{xy} = -6e^{3y}\sin 2x \quad \text{and} \quad f_{yy} = 9e^{3y}\cos 2x = 9f.$$

5. (10 points) Suppose $z = f(x,y)$ is a function with partial derivatives $f_x(3,4) = 3$ and $f_y(3,4) = -2$. If x and y are both functions of t: $x = 4 - t^2$ and $y = 3t + t^2$, find

$$\frac{dz}{dt} = \frac{d}{dt} f\big(x(t), y(t)\big)$$

at $t = 1$.

SOLUTION: The **chain rule** says that

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

Compute $x(1) = 4 - 1 = 3$, $y(1) = 3 + 1 = 4$ so f_x and f_y are to be evaluated at $x = 3$, $y = 4$. Compute $\frac{dx}{dt} = -2t$ so $\frac{dx}{dt}(1) = -2$ and $\frac{dy}{dt} = 3 + 2t$ so $\frac{dy}{dt}(1) = 5$. Finally,

$$\frac{dz}{dt}(1) = (3)(-2) + (-2)(5) = -16.$$

6. (15 points) The point $\langle x,y,z\rangle = \langle -2,1,0\rangle$ lies on the surface S:

$$x^2 - y^2 + xz + xy - 4z^2 = 1.$$

Find the equation of the **tangent plane** to the surface S at $\langle -2,1,0\rangle$, in the form $ax + by + cz = d$.

SOLUTION: Compute the **gradient** of $g(x,y,z) = x^2 - y^2 + xz + xy - 4z^2$: $\vec{\nabla} g = (2x+z)\vec{i} + (-2y+x)\vec{j} + (x-8z)\vec{k}$. Then $\vec{\nabla} g(-2,1,0) = -4\vec{i} - 4\vec{j} - 2\vec{k}$ is a normal vector to the surface S given by $g(x,y,z) = 1$ at $\langle -2,1,0\rangle$. The equation of the **tangent plane** to S at $\langle -2,1,0\rangle$ is

$$-4(x+2) - 4(y-1) - 2(z-0) = 0, \quad \text{or} \quad 2x + 2y + z = -2.$$

7. **(15 points) (a)**Find the **gradient** of the function $f(x, y, z) = (x + z^2)\sin(xy)$ at the point $\langle x, y, z\rangle = \langle 1, \frac{\pi}{2}, 2\rangle$.

   (15 points) **SOLUTION:** $f_x = \sin(xy) + (x + z^2)y\cos(xy)$; $f_y = (x + z^2)x\cos(xy)$; and $f_z = 2z\sin(xy)$. So the partial derivatives of f at $\langle x, y, z\rangle = \langle 1, \frac{\pi}{2}, 2\rangle$ are $f_x = 1 + \frac{5\pi}{2}(0) = 1$, $f_y = (-2 + 1)(-2)(0) = 0$ and $f_z = 4(1) = 4$. Together, the gradient

   $$\vec{\nabla} f(1, \frac{\pi}{2}, 2) = \vec{i} + 4\vec{k}.$$

   (b) Find the **directional derivative** of f at the point $\langle 1, \frac{\pi}{2}, 2\rangle$ in the direction of the unit vector

   $$\vec{u} = \frac{1}{3}\left(2\vec{i} - \vec{j} - 2\vec{k}\right).$$

   SOLUTION: We know that $D_{\vec{u}} f(1, \frac{\pi}{2}, 2) = \vec{u} \cdot \vec{\nabla} f(1, \frac{\pi}{2}, 2) = \frac{1}{3}\left(2\vec{i} - \vec{j} - 2\vec{k}\right) \cdot \left(\vec{i} + 4\vec{k}\right) = \frac{2-8}{3} = -2$.