1. (15 points) The lines given parametrically by
\[
\langle x, y, z \rangle = \langle 5 - t, 3 + 2t, 1 + 2t \rangle, \quad -\infty < t < \infty
\]
and
\[
\langle x, y, z \rangle = \langle 5 + 2s, 3 + 2s, 1 - s \rangle, \quad -\infty < s < \infty
\]
intersect at the point \( \langle x, y, z \rangle = \langle 5, 3, 1 \rangle \). Find an equation for the plane which contains both lines.

**SOLUTION:** A normal vector \( \vec{v} \) is the cross product of the vectors \( \langle -1, 2, 2 \rangle \) and \( \langle 2, 2, -1 \rangle \) in the directions of the two given lines.

\[
\vec{v} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
-1 & 2 & 2 \\
2 & 2 & -1
\end{vmatrix} = -6\vec{i} + 3\vec{j} - 6\vec{k}.
\]

We can divide \( \vec{v} \) by 3, so an equation for the plane is \(-2(x - 5) + (y - 3) - 2(z - 1) = 0\), or simplifying:

\[-2x + y - 2z = -9.\]

2. (15 points) Find an equation for the **elliptical cylinder** in \((x, y, z)\)-space containing infinitely many lines parallel to the \(z\)-axis, and containing the slanted circle \( z = y, x^2 + y^2 + z^2 = 4 \).

**SOLUTION:** Eliminate \( z \) from the equations \( z = y, x^2 + y^2 + z^2 = 4 \) to get \( x^2 + 2y^2 = 4 \). This is the equation of the ellipse which is the projection of the slanted circle into the \((x, y)\)-plane. So the elliptical cylinder is given by the same equation:

\[x^2 + 2y^2 = 4,\]

as an equation for a surface in \((x, y, z)\)-space.

3. (15 points) Evaluate the limit
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2 + xy + y^2},
\]
or state that it does not exist, giving reasons.

**SOLUTION:** First try plugging in \( x = 0 \) and \( y = 0 \): the numerator and the denominator of the quotient both converge to 0. So it’s not easy to tell what the limit is. Try approaching \((0, 0)\) along the line \( y = mx \) of slope \( m \): the function is constant along that line and equal to

\[
\frac{x^2 - mx^2 + m^2x^2}{x^2 + mx^2 + m^2x^2} = \frac{1 - m + m^2}{1 + m + m^2},
\]

which depends on \( m \). So the limit **does not exist**, because you get different limits along lines of different slope \( m \).
4. (15 points) For the function 
\[ f(x, y) = e^{3y} \cos 2x, \]
find the **second partial derivatives**
\[ f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}. \]

**SOLUTION:** Compute 
\[ f_x = \frac{\partial f}{\partial x} = -2e^{3y} \sin 2x \quad \text{and} \quad f_y = \frac{\partial f}{\partial y} = 3e^{3y} \cos 2x. \]
Then 
\[ f_{xx} = -4e^{3y} \cos 2x = -4f, \quad f_{xy} = -6e^{3y} \sin 2x \quad \text{and} \quad f_{yy} = 9e^{3y} \cos 2x = 9f. \]

5. (10 points) Suppose \( z = f(x, y) \) is a function with partial derivatives \( f_x(3, 4) = 3 \) and \( f_y(3, 4) = -2 \). If \( x \) and \( y \) are both functions of \( t \): \( x = 4 - t^2 \) and \( y = 3t + t^2 \), find 
\[ \frac{dz}{dt} = \frac{d}{dt} f(x(t), y(t)) \]
at \( t = 1 \).

**SOLUTION:** The **chain rule** says that
\[ \frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}. \]
Compute \( x(1) = 4 - 1 = 3 \), \( y(1) = 3 + 1 = 4 \) so \( f_x \) and \( f_y \) are to be evaluated at \( x = 3, \ y = 4 \). Compute \( \frac{dx}{dt} = -2t \) so \( \frac{dx}{dt}(1) = -2 \) and \( \frac{dy}{dt} = 3 + 2t \) so \( \frac{dy}{dt}(1) = 5 \). Finally,
\[ \frac{dz}{dt}(1) = (3)(-2) + (-2)(5) = -16. \]

6. (15 points) The point \( \langle x, y, z \rangle = \langle -2, 1, 0 \rangle \) lies on the surface \( S \):
\[ x^2 - y^2 + xz + xy - 4z^2 = 1. \]
Find the equation of the **tangent plane** to the surface \( S \) at \( \langle -2, 1, 0 \rangle \), in the form 
\[ ax + by + cz = d. \]

**SOLUTION:** Compute the gradient of \( g(x, y, z) = x^2 - y^2 + xz + xy - 4z^2 \): \( \nabla g = (2x + z)\mathbf{i} + (-2y + x)\mathbf{j} + (x - 8z)\mathbf{k} \). Then \( \nabla g(-2, 1, 0) = -4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k} \) is a normal vector to the surface \( S \) given by \( g(x, y, z) = 1 \) at \( \langle -2, 1, 0 \rangle \). The equation of the tangent plane to \( S \) at \( \langle -2, 1, 0 \rangle \) is 
\[-4(x + 2) - 4(y - 1) - 2(z - 0) = 0, \quad \text{or} \quad 2x + 2y + z = -2. \]
7. (15 points) (a) Find the gradient of the function \( f(x, y, z) = (x + z^2) \sin(xy) \) at the point \( \langle x, y, z \rangle = \langle 1, \frac{\pi}{2}, 2 \rangle \).

(15 points) SOLUTION: \( f_x = \sin(xy) + (x + z^2)y \cos(xy); \) \( f_y = (x + z^2)x \cos(xy); \) and \( f_z = 2z \sin(xy) \). So the partial derivatives of \( f \) at \( \langle x, y, z \rangle = \langle 1, \frac{\pi}{2}, 2 \rangle \) are \( f_x = 1 + \frac{5\pi}{2}(0) = 1, \) \( f_y = (-2 + 1)(-2)(0) = 0 \) and \( f_z = 4(1) = 4. \) Together, the gradient \( \nabla f(1, \frac{\pi}{2}, 2) = \vec{i} + 4\vec{k}. \)

(b) Find the directional derivative of \( f \) at the point \( \langle 1, \frac{\pi}{2}, 2 \rangle \) in the direction of the unit vector \( \vec{u} = \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}). \)

SOLUTION: We know that \( D_{\vec{u}} f(1, \frac{\pi}{2}, 2) = \vec{u} \cdot \nabla f(1, \frac{\pi}{2}, 2) = \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}) \cdot (\vec{i} + 4\vec{k}) = \frac{2 \cdot 1}{3} = \frac{2 - \pi}{3} = -2. \)