1. (10 points) Let $B$ be the rectangular solid $0 \leq x \leq 4, 1 \leq y \leq 2, 0 \leq z \leq 2$. Find
$$\iiint_B xz^2 \, y^2 \, dV.$$  

**SOLUTION:** $xz^2 = x \cdot y \cdot -2z^2$, so 
$$\iiint_B xz^2 \, y^2 \, dV = \int_0^4 x \, dx \int_1^2 y \cdot -2 \, dy \int_0^2 z^2 \, dz = \left[ \frac{x^2}{2} \right]_{x=0}^{x=4} \left[ -\frac{1}{3} \right]_{y=1}^{y=2} \left[ \frac{z^3}{3} \right]_{z=0}^{z=2} = \frac{32}{3}.$$  

2. (20 points) Let $\vec{F}$ be the vector field
$$\vec{F}(x, y) = (y^2 + e^x) \vec{i} + 2xy \vec{j}.$$  

(a) (10 points) Find a real-valued function $f(x, y)$ so that 
$$\nabla f(x, y) = \vec{F}(x, y).$$  

**SOLUTION:** We have $\frac{\partial f}{\partial x} = y^2 + e^x$ and $\frac{\partial f}{\partial y} = 2xy$. The formula for $\frac{\partial f}{\partial x}$ implies that $f(x, y) = xy^2 + g(x)$ for some function $g$ of one variable. Differentiating this formula with respect to $x$ yields $\frac{\partial f}{\partial x} = y^2 + g'(x)$, so we need $g'(x) = e^x$ and so $g(x) = e^x + C$. Therefore 
$$f(x, y) = xy^2 + e^x + C.$$  

(b) (10 points) Let $C$ be the curve given by $x = \cos(t^2 \pi)$ and $y = t, \ 0 \leq t \leq 2$. Find the line integral 
$$\int_C \vec{F} \cdot d\vec{r}.$$  

**SOLUTION:** The curve $C$ starts at $x = 1, y = 0$ and ends at $x = \cos 4\pi = 1, y = 2$. So by the Fundamental Theorem of Calculus for Line Integrals, 
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1, 2) - f(1, 0) = (4 + e + C) - (0 + e + C) = 4.$$  

3. (15 points) Find the **surface area** of the portion of the paraboloid $z = x^2 + y^2$ which lies below the plane $z = 2$ (Hint: you might want to compute the double integral in polar coordinates.)  

**SOLUTION:** $A(S) = \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dA$, where $R$ is the circle $x^2 + y^2 \leq 2$ of radius $\sqrt{2}$. So 
$$A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2} \, dA = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \frac{2\pi}{8} \int_1^{\sqrt{2}} \sqrt{u} \, du =$$
6. (15 points) The portion of the ball of radius 2 with center at $(0, 0, 4)$ is the upper semicircle from $(2, 0, 0)$ to $(-2, 0, 0)$, given by $x = 2 \cos t$, $y = 2 \sin t$, $0 \leq t \leq \pi$; $C_2$ is the segment of the $x$-axis from $(-2, 0, 0)$ to $(-1, 0, 0)$; $C_3$ is the upper semicircle from $(-1, 0, 0)$ to $(1, 0, 0)$, given by $x = -\cos t$, $y = \sin t$, $0 \leq t \leq \pi$; and $C_4$ is the segment of the $x$-axis from $(1, 0, 0)$ to $(2, 0, 0)$. Let $\vec{F}$ be the vector field

$$
\vec{F}(x, y) = [e^{x^2} + \sin y] \vec{i} + [x \cos y + 3x + \ln(y + 1)] \vec{j}.
$$

Find $\int_C \vec{F} \cdot d\vec{r}$. (Hint: Green’s Theorem makes this much easier! You may use what you remember about areas inside circles.)

**SOLUTION:** The curve $C$ is the oriented boundary of a region $D$ which can be described as the upper semi-circle of radius 2 centered at $(0, 0, 0)$ minus the semi-circle of radius 1 centered at $(0, 0, 0)$. In polar coordinates, $D$ is described by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$. Write $\vec{F}(x, y) = P\vec{i} + Q\vec{j}$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos y + 3 - \cos y = 3$. By Green’s Theorem,

$$
\int_C \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_0^\pi \int_1^2 3r \, dr \, d\theta = 3\pi \left[ \frac{r^2}{2} \right]_1^2 = \frac{9\pi}{2}.
$$

5. (20 points) Under the linear transformation $x = 4u + v$, $y = 5u + 3v$ from the $(u, v)$-plane to the $(x, y)$-plane, the circular disk $D$ given by the inequality $u^2 + v^2 \leq 4$ is transformed into the elliptical region $E$ given by $34x^2 - 46xy + 17y^2 \leq 156$. Compute the area of $E$ as an integral over $D$.

**SOLUTION:** $D$ is a circular disk of radius 2, so $A(D) = 4\pi$. The Jacobian determinant

$$
\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = 4 \cdot 3 - 1 \cdot 5 = 7,
$$

a constant. So $A(E) = \iint_E 7 \, dA = 7A(D) = 28\pi$.

6. (15 points) The portion of the ball of radius 2 with center at $(0, 0, 0)$, which lies above the cone

$$
z = \frac{1}{2} \sqrt{x^2 + y^2 + z^2},
$$

is described in spherical coordinates by $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/3$, $0 \leq \theta \leq 2\pi$. Find the volume of this figure by computing an integral in spherical coordinates. (Hint: Recall that $\sin \pi/3 = \frac{1}{2} \sqrt{3}$ and $\cos \pi/3 = \frac{1}{2}$.)

**SOLUTION:** The volume is $V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/3} \sin \phi \, d\phi \int_0^2 \rho^2 \, d\rho = 2\pi \left[ -\cos \phi \right]_0^{\pi/3} \left[ \frac{\rho^3}{3} \right]_0^2 = \frac{8\pi}{3}$. 

$$
\frac{\pi}{4} \left[ \frac{u^{3/2}}{3/2} \right]_1^9 = \frac{\pi}{6} (27 - 1) = \frac{13\pi}{3}.
$$