Math 2263 Spring 2008 Midterm 3, WITH SOLUTIONS April 24, 2008

1. (10 points) Let B be the rectangular solid $0 \le x \le 4$, $1 \le y \le 2$, $0 \le z \le 2$. Find

$$\iiint_B \frac{xz^2}{y^2}\, dV.$$

SOLUTION: $\frac{xz^2}{y^2} = x\ y^{-2}\ z^2$, so $\iiint_B \frac{xz^2}{y^2}\, dV = \int_0^4 x\, dx\ \int_1^2 y^{-2}\, dy\ \int_0^2 z^2\, dz =$
$= \left[\frac{x^2}{2}\right]_{x=0}^4 \left[-\frac{1}{y}\right]_{y=1}^2 \left[\frac{z^3}{3}\right]_{x=0}^2 = \frac{32}{3}.$

2. (20 points) Let $\vec{F}$ be the vector field

$$\vec{F}(x, y) = (y^2 + e^x)\,\vec{\imath} + 2xy\vec{\jmath}.$$

(a) (10 points) Find a real-valued function $f(x, y)$ so that

$$\nabla f(x, y) = \vec{F}(x, y).$$

SOLUTION: We have $\frac{\partial f}{\partial x} = y^2 + e^x$ and $\frac{\partial f}{\partial y} = 2xy$. The formula for $\frac{\partial f}{\partial y}$ implies that $f(x, y) = xy^2 + g(x)$ for **some** function g of one variable. Differentiating this formula with respect to x yields $\frac{\partial f}{\partial x} = y^2 + g'(x)$, so we need $g'(x) = e^x$ and so $g(x) = e^x + C$. Therefore

$$f(x, y) = xy^2 + e^x + C.$$

(b) (10 points) Let C be the curve given by $x = \cos(t^2\pi)$ and $y = t$, $0 \le t \le 2$. Find the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

SOLUTION: The curve C starts at $x = 1$, $y = 0$ and ends at $x = \cos 4\pi = 1$, $y = 2$. So by the Fundamental Theorem of Calculus for Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(1, 2) - f(1, 0) = (4 + e + C) - (0 + e + C) = 4.$$

3. (15 points) Find the **surface area** of the portion of the paraboloid $z = x^2 + y^2$ which lies below the plane $z = 2$ (*Hint:* you might want to compute the double integral in polar coordinates.)

SOLUTION: $A(S) = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$, where R is the circle $x^2 + y^2 \le 2$ of radius $\sqrt{2}$. So

$$A(S) = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{2\pi}{8} \int_1^9 \sqrt{u}\, du =$$

$$\frac{\pi}{4}\left[\frac{u^{3/2}}{3/2}\right]_1^9 = \frac{\pi}{6}(27-1) = \frac{13\pi}{3}.$$

4. (20 points) Under the linear transformation $x = 4u+v$, $y = 5u+3v$ from the (u,v)-plane to the (x,y)-plane, the circular disk D given by the inequality $u^2+v^2 \le 4$ is transformed into the elliptical region E given by $34x^2 - 46xy + 17y^2 \le 156$. Compute the **area of** E **as an integral over** D.

   SOLUTION: D is a circular disk of radius 2, so $A(D) = 4\pi$. The Jacobian determinant

   $$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 4\cdot 3 - 1\cdot 5 = 7,$$

   a constant. So $A(E) = \iint_D 7\,dA = 7A(D) = 28\pi$.

5. (20 points) An oriented curve C in the (x,y)-plane consists of four pieces C_1, C_2, C_3, C_4: C_1 is the upper semicircle from $(2,0)$ to $(-2,0)$, given by $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le \pi$; C_2 is the segment of the x-axis from $(-2,0)$ to $(-1,0)$; C_3 is the upper semicircle from $(-1,0)$ to $(1,0)$, given by $x = -\cos t,\ y = \sin t,\ 0 \le t \le \pi$; and C_4 is the segment of the x-axis from $(1,0)$ to $(2,0)$. Let $\vec{F}$ be the vector field

   $$\vec{F}(x,y) = [e^{x^2} + \sin y]\ \vec{i} + [x\cos y + 3x + \ln(y+1)]\ \vec{j}.$$

   Find $\int_C \vec{F}\cdot d\vec{r}$. (*Hint:* Green's Theorem makes this much easier! You may use what you remember about areas inside circles.)

   SOLUTION: The curve C is the oriented boundary of a region D which can be described as the upper semi-circle of radius 2 centered at $(0,0)$ minus the semi-circle of radius 1 centered at $(0,0)$. In polar coordinates, D is described by $1 \le r \le 2$, $0 \le \theta \le \pi$. Write $\vec{F}(x,y) = P\vec{i} + Q\vec{j}$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos y + 3 - \cos y = 3$. By Green's Theorem,

   $$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \int_0^\pi \int_1^2 3r\,dr\,d\theta = 3\pi\left[\frac{r^2}{2}\right]_1^2 = \frac{9\pi}{2}.$$

6. (15 points) The portion of the ball of radius 2 with center at $(0,0,0)$, which lies above the cone

   $$z = \frac{1}{2}\sqrt{x^2+y^2+z^2},$$

   is described in spherical coordinates by $0 \le \rho \le 2$, $0 \le \phi \le \pi/3$, $0 \le \theta \le 2\pi$. Find the **volume** of this figure by computing an integral in spherical coordinates. (*Hint:* Recall that $\sin\pi/3 = \frac{1}{2}\sqrt{3}$ and $\cos\pi/3 = \frac{1}{2}$.)

   SOLUTION: The volume is $V =$

   $$\int_0^{2\pi}\int_0^{\pi/3}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\frac{\pi}{3}}\sin\phi\,d\phi\int_0^2\rho^2\,d\rho = 2\pi[-\cos\phi]_0^{\frac{\pi}{3}}\ i\left[\frac{\rho^3}{3}i\right]_0^2 = \frac{8\pi}{3}.$$