Math 2263 Fall 2009 Midterm 2, WITH SOLUTIONS November 5, 2009

1. (7 points) Let R be the rectangle $0 \le x \le 3$, $0 \le y \le 1$. Find the double integral

$$\iint_R \frac{xy}{(y^2+1)^2}\, dA.$$

SOLUTION: The iterated integral is $\int_0^1 \int_0^3 \frac{xy}{(y^2+1)^2}\, dx\, dy$. The inside integral is $\int_0^3 x\, dx = \frac{9}{2}$, so the double integral is

$$\frac{9}{2}\int_0^1 \frac{y}{(y^2+1)^2}\, dy.$$

Make the substitution $u = y^2 + 1$, so $du = 2y\, dy$ and the answer is

$$\frac{9}{2}\frac{1}{2}\int_1^2 \frac{du}{u^2} = \frac{9}{4}[-u^{-1}]_1^2 = \frac{9}{8}.$$

2. (8 points) Let R be the rectangle $0 \le x \le 5$, $0 \le y \le 2$ in the (x, y)-plane. If a continuous function $f(x, y)$ satisfies

$$-1 \le f(x, y) \le xy^2,$$

what does this tell you about the value of $\iint_R f(x, y)\, dA$?

SOLUTION: Evaluate

$$\iint_R xy^2\, dA = \int_0^2 \int_0^3 xy^2\, dx\, dy = [x^2/2]_{x=0}^3 [y^3/3]_{y=0}^2 = \frac{9}{2}\frac{8}{3} = 12.$$

Also,

$$\iint_R (-1)\, dA = -A(R) = -6,$$

so

$$-6 \le \iint_R f(x, y)\, dA \le 12.$$

3. (20 points) Suppose $x = X$ and $y = Y$ are random variables with joint density function

$$f(x, y) = \frac{\alpha}{1 + x^2 + y^2} \quad \text{if} \quad x^2 + y^2 \le 1,$$

and

$$f(x, y) \equiv 0 \quad \text{if} \quad x^2 + y^2 > 1.$$

(a) (10 points) What does the constant α need to be? (Your answer will involve $\ln 2$. Do not evaluate $\ln 2$.)

SOLUTION: Compute in polar coordinates, and substitute $u = 1 + r^2$: $\iint f(x, y)\, dA = \int_0^{2\pi} \int_0^1 \frac{\alpha}{1+r^2} r\, dr\, d\theta = 2\pi \int_1^2 \frac{\alpha}{u}\frac{1}{2} du = \alpha\pi \ln 2$. The constant α is determined by the requirement that $\iint f(x, y)\, dA = 1 = 100\%$, so $\alpha = \frac{1}{\pi \ln 2}$.

(b) (10 points) Find the "median" radius R, so that the probability that $X^2 + Y^2 \le R^2$ is 50%. For full credit, your answer should **not** involve $\ln 2$.

SOLUTION: For each radius R, the probability that $X^2 + Y^2 \le R^2$ is

$$\iint_{r \le R} f(x,y)\, dA = 2\pi \int_0^R \frac{\alpha}{1+r^2} r\, dr\, d\theta = \pi \int_1^{1+R^2} \frac{\alpha}{u}\, du = \frac{1}{\pi \ln 2} \pi \ln(1+R^2).$$

To make this equal to $50\% = \frac{1}{2}$, we need $1 + R^2 = e^{\frac{\ln 2}{2}}$, so

$$R = \sqrt{\sqrt{2} - 1}.$$

4. (15 points) A plate is in the shape of the triangle D: $0 \le y \le 2 - |x|$, with corners $(-2, 0)$, $(0, 2)$ and $(2, 0)$. The plate has mass density at the point (x, y) equal to $\rho(x, y) = y + |x|$ per unit area.
(a) (5 points) Find the **total mass** m of the plate.

SOLUTION: Let T denote the triangle. The total mass $m = \iint_T \rho(x,y)\, dA = \int_{-2}^{2} \int_0^{2-|x|} y + |x|\, dy\, dx = \int_{-2}^{2} [\frac{y^2}{2} + |x|y]_{y=0}^{2-|x|}\, dx = \int_{-2}^{2} [\frac{(2-2|x|)^2}{2} + (2-2|x|)|x|]\, dx = 2\int_0^2 [\frac{(2-2x)^2}{2} + (2-2x)x]\, dx = 2\int_0^2 [-fracx^2 2 + 2]\, dx = 2[-\frac{x^3}{6} + 2x]_0^2 = -\frac{8}{3} + 8 = \frac{16}{3}$.

(b) (10 points) Find the **center of mass** $(\overline{x}, \overline{y})$ of the plate.

SOLUTION: $\overline{x} = \iint_T x\rho(x,y)\, dA = 0$, because $\rho(x,y) = \rho(-x,y)$, and T is symmetric, so the contribution from $-2 \le x \le 0$ equals minus the contribution from $0 \le x \le 3$. And $m\overline{y} = \iint_T y\rho(x,y)\, dA = \int_{-2}^{2} \int_0^{2-|x|} y^2 + |x|y\, dy\, dx = \int_{-2}^{2} [\frac{y^3}{3} + |x|\frac{y^2}{2}]_{y=0}^{2-|x|}\, dx = 2\int_0^2 [\frac{y^3}{3} + x\frac{y^2}{2}]_{y=0}^{2-x}\, dx = 2\int_0^2 \frac{1}{6}[16 - 12x + x^3]\, dx = \frac{1}{3}(32 - 24 + \frac{8}{2}) = \frac{10}{3}$. So

$$\overline{y} = \frac{\frac{10}{3}}{\frac{16}{3}} = \frac{5}{8}.$$

5. (10 points) Let D be the circular disk of radius R and center $(0, 0)$ in the (x, y)-plane. Find

$$\iint_D e^{x^2+y^2}\, dA.$$

(*Hint:* polar coordinates.)

SOLUTION: This integral is not possible as an (x, y) iterated integral. But in polars, substituting $u = r^2$: $\iint_D e^{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^R e^{r^2} r\, dr\, d\theta = 2\pi \int_0^{R^2} \frac{1}{2} e^u\, du = \pi(e^{R^2} - 1)$.

6. (15 points) Find the **maximum** and **minimum** values of

$$f(x, y) = xy - y$$

subject to the **side condition**

$$g(x, y) = 4x^2 + y^2 = 4.$$

(*Hint:* Lagrange multipliers.)

SOLUTION: By Lagrange multipliers, a maximum or minimum point (x, y) must satisfy

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

fo some scalar λ. But this means $\vec{\nabla} f = y\vec{i} + (x - 1)\vec{j} = \lambda(8x\vec{i} + 2y\vec{j})$, so $y = 8\lambda x$ and $x - 1 = 2\lambda y$. Then $y^2 = 8\lambda xy = 4x(x - 1)$. Using $g(x, y) = 4$ gives $8x^2 - 4x = 4$, so $x = 1$ or $x = -\frac{1}{2}$. If $x = 1$, then $y = 0$. If $x = -\frac{1}{2}$, then $y^2 = 3$. Check $f(x, y)$ at these three points: $f(1, 0) = 0$; $f(-\frac{1}{2}, \sqrt{3}) = -\frac{3}{2}\sqrt{3}$ which is the **minimum**; and $f(-\frac{1}{2}, -\sqrt{3}) = +\frac{3}{2}\sqrt{3}$ which is the **maximum**.

7. (25 points) Let $f(x, y) = 3x^3 - xy^2 + yx^2 + \frac{7}{2}x^2$.
(a) (5 points) Compute the **first** and **second** partial derivatives of $f(x, y)$.

SOLUTION: $\frac{\partial f}{\partial x} = f_x = 9x^2 - y^2 + 2xy + 7x$;
$f_y = -2xy + x^2$;
$f_{xx} = 18x + 2y + 7$;
$f_{xy} = -2y + 2x$;
$f_{yy} = -2x$.

(b) (10 points) Find all the **critical points** of $f(x, y)$.

SOLUTION: $f_y = 0$ requires either $x = 0$ or $x = 2y$.

If $x = 0$ then $f_x = -y^2 = 0$ only for $y = 0$: $(0, 0)$ is a critical point.

If $x = 2y$, then $f_x = 36y^2 - y^2 + 4y^2 + 14y = 0$ requires $y = 0$ or $y = -\frac{14}{39}$.

So there are only two critical points: $(x, y) = (0, 0)$ or $(-\frac{28}{39}, -\frac{14}{39})$.

(c) (10 points) For each critical point, state whether it is a **local minimum point**, a **local maximum point** or a **saddle point**.

SOLUTION: At $(x, y) = (0, 0)$, the second partial derivatives are $f_{xx} = 7$, $f_{xy} = 0$, and $f_{yy} = 0$. So the origin $(0, 0)$ is a degenerate critical point.

At $(x, y) = (-\frac{28}{39}, -\frac{14}{39})$, the second partial derivatives are $f_{xx} = 7$, $f_{xy} = 0$, and $f_{yy} = 0$.