1. (7 points) Let \( R \) be the rectangle \( 0 \leq x \leq 3, 0 \leq y \leq 1 \). Find the double integral
\[
\int \int_R \frac{xy}{(y^2 + 1)^2} \, dA.
\]

**SOLUTION:** The iterated integral is \( \int_0^1 \int_0^3 \frac{xy}{(y^2 + 1)^2} \, dx \, dy \). The inside integral is \( \int_0^3 x \, dx = \frac{9}{2} \), so the double integral is
\[
\frac{9}{2} \int_0^1 \frac{y}{(y^2 + 1)^2} \, dy.
\]
Make the substitution \( u = y^2 + 1 \), so \( du = 2y \, dy \) and the answer is
\[
\frac{9}{2} \left( \frac{1}{2} \right)^{1/2} = \frac{9}{8}.
\]

2. (8 points) Let \( R \) be the rectangle \( 0 \leq x \leq 5, 0 \leq y \leq 2 \) in the \((x, y)\)-plane. If a continuous function \( f(x, y) \) satisfies
\[-1 \leq f(x, y) \leq xy^2,
\]
what does this tell you about the value of \( \int \int_R f(x, y) \, dA \)?

**SOLUTION:** Evaluate
\[
\int \int_R xy^2 \, dA = \int_0^2 \int_0^3 xy^2 \, dx \, dy = [x^2/2]_0^3 [y^3/3]_0^2 = \frac{27}{3} = 9.
\]
Also,
\[
\int \int_R (-1) \, dA = -A(R) = -6,
\]
so
\[-6 \leq \int \int_R f(x, y) \, dA \leq 12.
\]

3. (20 points) Suppose \( x = X \) and \( y = Y \) are random variables with joint density function
\[
f(x, y) = \frac{\alpha}{1 + x^2 + y^2} \quad \text{if} \quad x^2 + y^2 \leq 1,
\]
and
\[
f(x, y) = 0 \quad \text{if} \quad x^2 + y^2 > 1.
\]

(a) (10 points) What does the constant \( \alpha \) need to be? (Your answer will involve \( \ln 2 \). Do not evaluate \( \ln 2 \).)

**SOLUTION:** Compute in polar coordinates, and substitute \( u = 1 + r^2 \):
\[
\int \int f(x, y) \, dA = \int_0^{2\pi} \int_0^1 \frac{\alpha}{1 + r^2} r \, dr \, d\theta = 2\pi \int_0^1 \frac{\alpha}{2} \frac{1}{u} \, du = \alpha \pi \ln 2.
\]
The constant \( \alpha \) is determined by the requirement that \( \int \int f(x, y) \, dA = 1 = 100\% \), so \( \alpha = \frac{1}{\pi \ln 2} \).
(b) (10 points) Find the “median” radius \( R \), so that the probability that \( X^2 + Y^2 \leq R^2 \) is 50%. For full credit, your answer should not involve \( \ln 2 \).

**SOLUTION:** For each radius \( R \), the probability that \( X^2 + Y^2 \leq R^2 \) is

\[
\int_{r \leq R} f(x, y) \, dA = 2\pi \int_{0}^{R} \frac{\alpha}{1 + \alpha^2} r \, dr \, d\theta = \pi \int_{1}^{R^2} \frac{\alpha}{u} \, du = \frac{1}{\pi \ln 2} \pi \ln(1 + R^2).
\]

To make this equal to 50% = \( \frac{1}{2} \), we need \( 1 + R^2 = e^{\frac{\ln 2}{2}} \), so

\[ R = \sqrt{\sqrt{2} - 1}. \]

4. (15 points) A plate is in the shape of the triangle \( D \): \( 0 \leq y \leq 2 - |x| \), with corners \((-2, 0), (0, 2) \) and \((2, 0)\). The plate has mass density at the point \((x, y)\) equal to \( \rho(x, y) = y + |x| \) per unit area.

(a) (5 points) Find the total mass \( m \) of the plate.

**SOLUTION:** Let \( T \) denote the triangle. The total mass \( m = \iint_T \rho(x, y) \, dA = \int_{-2}^{2} \int_{0}^{2-|x|} y + |x| \, dy \, dx = \int_{-2}^{2} \int_{0}^{2-|x|} (y^2 + |x|y)_{y=0}^{2} \, dx = 2 \int_{0}^{2} (2-2x)^2+(2-2x)|x| \, dx = 2 \int_{0}^{2} [-x^2+2x]^2 \, dx = 2[-x^3 + 2x^2]_0^2 = -\frac{8}{3} + 8 = \frac{16}{3}. \)

(b) (10 points) Find the center of mass \((\bar{x}, \bar{y})\) of the plate.

**SOLUTION:** \( \bar{x} = \iint_T x\rho(x, y) \, dA = 0 \), because \( \rho(x, y) = \rho(-x, y) \), and \( T \) is symmetric, so the contribution from \(-2 \leq x \leq 0\) equals minus the contribution from \(0 \leq x \leq 3\). And \( m\bar{y} = \iint_T y\rho(x, y) \, dA = \int_{-2}^{2} \int_{0}^{2-|x|} y^2 + |x|y \, dy \, dx = \int_{-2}^{2} \int_{0}^{2} \frac{y^3}{3} + |x|y^2 \frac{2}{5} \, dy \, dx = 2 \frac{y^3}{3} + x \frac{y^2}{2} |y=0 \, dx = 2 \int_{0}^{2} \frac{1}{5}[16 - 12x + x^3] \, dx = \frac{1}{3}(32 - 24 + \frac{8}{7}) = \frac{10}{3} \). So

\[ \bar{y} = \frac{\frac{10}{3}}{\frac{16}{3}} = \frac{5}{8}. \]

5. (10 points) Let \( D \) be the circular disk of radius \( R \) and center \((0, 0)\) in the \((x, y)\)-plane. Find

\[
\iint_D e^{x^2+y^2} \, dA.
\]

(*Hint*: polar coordinates.)

**SOLUTION:** This integral is not possible as an \((x, y)\) iterated integral. But in polars, substituting \( u = r^2 \), \( \iint_D e^{x^2+y^2} \, dA = \int_{0}^{2\pi} \int_{0}^{R} e^{r^2} r \, dr \, d\theta = 2\pi \int_{0}^{R^2} \frac{1}{2} e^u \, du = \pi(e^{R^2} - 1) \).

6. (15 points) Find the maximum and minimum values of

\[ f(x, y) = xy - y \]
subject to the side condition
\[ g(x, y) = 4x^2 + y^2 = 4. \]

(Hint: Lagrange multipliers.)

**SOLUTION:** By Lagrange multipliers, a maximum or minimum point \((x, y)\) must satisfy

\[ \nabla f = \lambda \nabla g \]

for some scalar \(\lambda\). But this means

\[ \nabla f = y\vec{i} + (x - 1)\vec{j} = \lambda(8x\vec{i} + 2y\vec{j}), \]

so \(y = 8\lambda x\) and \(x - 1 = 2\lambda y\). Then \(y^2 = 8\lambda xy = 4x(x - 1)\). Using \(g(x, y) = 4\) gives \(8x^2 - 4x = 4\), so \(x = 1\) or \(x = -\frac{1}{2}\). If \(x = 1\), then \(y = 0\). If \(x = -\frac{1}{2}\), then \(y^2 = 3\). Check \(f(x, y)\) at these three points:

- \(f(1, 0) = 0\); \(f(-\frac{1}{2}, \sqrt{3}) = -\frac{3}{2}\sqrt{3}\) which is the minimum; and \(f(-\frac{1}{2}, -\sqrt{3}) = +\frac{3}{2}\sqrt{3}\) which is the maximum.

7. (25 points) Let \(f(x, y) = 3x^3 - xy^2 + yx^2 + \frac{7}{2}x^2\).
   (a) (5 points) Compute the first and second partial derivatives of \(f(x, y)\).

   **SOLUTION:**
   
   \[
   \begin{align*}
   f_x &= 9x^2 - y^2 + 2xy + 7x; \\
   f_y &= -2xy + x^2; \\
   f_{xx} &= 18x + 2y + 7; \\
   f_{xy} &= -2y + 2x; \\
   f_{yy} &= -2x.
   \end{align*}
   \]

   (b) (10 points) Find all the critical points of \(f(x, y)\).

   **SOLUTION:** \(f_y = 0\) requires either \(x = 0\) or \(x = 2y\).
   
   - If \(x = 0\) then \(f_x = -y^2 = 0\) only for \(y = 0\): \((0, 0)\) is a critical point.
   - If \(x = 2y\), then \(f_x = 36y^2 - y^2 + 4y^2 + 14y = 0\) requires \(y = 0\) or \(y = -\frac{14}{39}\).

   So there are only two critical points: \((x, y) = (0, 0)\) or \((-\frac{28}{39}, -\frac{14}{39})\).

   (c) (10 points) For each critical point, state whether it is a local minimum point, a local maximum point or a saddle point.

   **SOLUTION:** At \((x, y) = (0, 0)\), the second partial derivatives are \(f_{xx} = 7, f_{xy} = 0,\) and \(f_{yy} = 0\). So the origin \((0, 0)\) is a degenerate critical point.
   
   At \((x, y) = (-\frac{28}{39}, -\frac{14}{39})\), the second partial derivatives are \(f_{xx} = 7, f_{xy} = 0,\) and \(f_{yy} = 0\).