1. (10 points) Let $B$ be the box, or rectangular solid: $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 1$. Find
$$\iiint_B (xy - 2yz + x^2z^2) \, dV.$$  

**SOLUTION:**

$$\iiint_B (xy - 2yz + x^2z^2) \, dV = \int_0^1 \int_0^3 \int_0^1 \left( \frac{x^2y}{2} - 2yzx - \frac{x^3z^2}{3} \right) \, dx \, dz = \int_0^1 \int_0^3 \left( \frac{y^2}{2} - 2yz \frac{8z^2y^3}{3} \right) \, dz = \int_0^1 \int_0^3 (9 - 18z + 8z^2) \, dz = 9 - 9 + \frac{8}{3} = \frac{8}{3}.$$  

2. (15 points) Let $E$ be the solid region bounded below by the cone $z^2 = x^2 + y^2$ and above by the plane $z = 1$. Find $\iiint_E (x^2 + y^2) \, dV$.

**Hint:** Try cylindrical coordinates.

**SOLUTION:** Use cylindrical coordinates: $E$ is described by $0 \leq r \leq z \leq 1$, $0 \leq \theta \leq 2\pi$. So

$$\iiint_E (x^2 + y^2) \, dV = \int_0^1 \int_0^2 \int_0^z r^2 \, r \, dz \, dr \, d\theta = 2\pi \int_0^1 \frac{z^4}{4} \, dz = \frac{\pi}{10}.$$  

3. (25 points) $C_1$ and $C_2$ are oriented curves in the $(x, y)$-plane, each of which starts at $(0, 0)$ and ends at $(1, 1)$. $C_1$ is given by $y = x^2$, $0 \leq x \leq 1$; and $C_2$ is given by $x = y^2$, $0 \leq y \leq 1$. Let the vector field $\vec{F}$ be given by $\vec{F}(x, y) = 2xy \vec{i} + (x^2 - y^2) \vec{j}$.

**a** (10 points) Find $\int_{C_1} \vec{F} \cdot d\vec{r}$.

**SOLUTION:**

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \left( 2xy \, dx + (x^2 - y^2) \, dy \right) = \int_0^1 \left[ 2x(x^2) + (x^2 - x^4)2x \right] \, dx = \left[ \frac{x^4}{4} - \frac{2x^6}{6} \right]_0^1 = \frac{2}{3}.$$  

**b** (10 points) Find $\int_{C_2} \vec{F} \cdot d\vec{r}$.

**SOLUTION:**

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \left[ 2(y^2)(2y) + ((y^2)^2 - y^2) \right] \, dy = \int_0^1 \left[ 4y^5 + y^4 - y^2 \right] \, dy = \left[ \frac{4y^6}{6} + \frac{y^5}{5} - \frac{y^3}{3} \right]_{y=0}^1 = \frac{8}{15}.$$  

**c** (5 points) Is the vector field $\vec{F}(x, y)$ conservative? Why or why not?

**SOLUTION:** NO! $\frac{\partial}{\partial x} \left[ \frac{y^6}{6} + \frac{y^5}{5} - \frac{y^3}{3} \right]_{y=0}^1 = \frac{8}{15}$, so the integral from $(0, 0)$ to $(1, 1)$ depends on the path of integration connecting the two points.
4. (15 points) Under the transformation \( x = 4u + v, \ y = 5u + 2v \) from the \((u, v)\)-plane to the \((x, y)\)-plane, the circular disk \( D \) given by the inequality \( u^2 + v^2 \leq 9 \) is transformed into the elliptical region \( E \) given by

\[ 85x^2 - 76xy + 17y^2 \leq 9. \]

Compute the area of \( E \) as an integral over \( D \).

**SOLUTION:** First compute the Jacobian determinant:

\[
\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = (4)(2) - (1)(5) \equiv 3.
\]

Also, the area of a circular disk of radius 3 is \( A(D) = \pi(3)^2 = 9\pi \). So

\[
A(E) = \iint_D \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dA = \iint_D 3 \, dA = 27\pi.
\]

5. (20 points) Let \( C \) be the circle \( x^2 + y^2 = 9 \) in the \((x, y)\)-plane, oriented counterclockwise. Find

\[
\oint_C (2xye^{x^2} + 2y^2 - y) \, dx + (e^{x^2} + 4xy - 3x) \, dy.
\]

**Hint:** try Green’s theorem!

**SOLUTION:** Write \( P(x, y) = 2xye^{x^2} + 2y^2 - y \) and \( Q(x, y) = e^{x^2} + 4xy - 3x \). Following the hint, \( C \) is the oriented boundary of the circular disk \( D: \ x^2 + y^2 \leq 9 \), so by Green’s Theorem

\[
\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.
\]

But \( \frac{\partial Q}{\partial x} = 2xe^{x^2} + 4y - 3 \) and \( \frac{\partial P}{\partial y} = 2xe^{x^2} + 4y - 1 \). Subtracting, you get \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv -2 \) for all \((x, y)\), so

\[
\oint_C P \, dx + Q \, dy = \iint_D (-2) \, dA = -18\pi.
\]

6. (15 points) The portion \( E \) of the ball of radius 2 with center at \((0, 0, 0)\), which lies above the cone

\[ z = \sqrt{x^2 + y^2}, \]

is described in spherical coordinates by \( 0 \leq \rho \leq 2, \ 0 \leq \phi \leq \pi/4, \ 0 \leq \theta \leq 2\pi \). Find the volume of this figure by computing an integral in spherical coordinates. (**Hint:** Recall that \( \sin \pi/4 = \frac{1}{2}\sqrt{2} = \cos \pi/4. \)

**SOLUTION:** Using spherical coordinates, the volume of \( E \) is

\[
V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/4} \rho^2 \sin \phi \, d\phi =
\]

\[
2\pi \frac{8}{3} \left[ -\cos \phi \right]_0^{\pi/4} = \frac{8\pi}{3} (2 - \sqrt{2}).
\]