Math 2263
Spring 2016
Midterm 1, WITH SOLUTIONS
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Name (Print): ______________________
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Section Number: __________
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1. (20 points) Find an equation for the **plane** passing through the two points $(x, y, z) = (-2, 0, 1)$ and $(3, 3, 2)$ so that the vector $\vec{i} + \vec{j}$ is tangent to the plane.
ANSWER: The vector between the points is $(5, 3, 1)$; its cross product with $\vec{i} + \vec{j}$ is $(-1, 1, 2)$; so an equation of the plane is $-(x+2) + (y-0) + 2(z-1) = 0$ or $-x + y + 2z = 4$.

2. (15 points) Suppose $z = f(x, y)$ is a function with first partial derivatives $f_x(3, -1) = 5$ and $f_y(3, -1) = 3$. If x and y are both functions of t: $x = g(t) = 1 + 2t$ and $y = h(t) = 3 - 4t$, find the **derivative of z with respect to t at $t = 1$** :
$$\frac{dz}{dt}(1) = \frac{d}{dt} f\Big(g(t), h(t)\Big).$$
ANSWER: By the chain rule, $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (5)(2) + (3)(-4) = -2$.

3. (20 points) The lines given parametrically by
$$(x, y, z) = (2t, 2 - 3t, 2 + 3t), \quad -\infty < t < \infty$$
and
$$(x, y, z) = (s, 3 - 2s, 9 - 2s), \quad -\infty < s < \infty$$
intersect at the point $(x, y, z) = (2, -1, 5)$. Find an equation for the **plane** which contains both lines.
ANSWER: The first line is in the direction of the vector $\langle 2, -3, +3\rangle$ and the second is in direction $\langle 1, -2, -2\rangle$. Their cross product is $12\vec{i} + 7\vec{j} - \vec{k}$, so one equation of the plane is $12(x-2) + 7(y+1) - (z-5) = 0$, or simplifying: $12x + 7y - z = 12$.

4. (20 points) For the function $f(x, y) = e^{2x-y^2} \sin y$, find the **second partial derivatives**
$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$
Write each answer as a polynomial times $f(x, y)$ plus another polynomial times $g(x, y) := e^{2x-y^2} \cos y$.
ANSWER: $f_x = 2e^{2x-y^2} \sin y$, $f_y = -2ye^{2x-y^2} \sin y + e^{2x-y^2} \cos y$. So $f_{xx} = 4e^{2x-y^2} \sin y = 4f$, $f_{xy} = -4ye^{2x-y^2} \sin y + 2e^{2x-y^2} \cos y = -4yf + 2g$, and $f_{yy} = -2e^{2x-y^2} \sin y + (-2y)^2 e^{2x-y^2} \sin y - 2ye^{2x-y^2} \cos y + (-2y)e^{2x-y^2} \cos y - e^{2x-y^2} \sin y = (-3 + 4y^2)f + (-4y)g$.

5. (25 points) The point $(x, y, z) = (3, 1, -3)$ lies on the surface S:
$$2x^2 + z^2 - 3xz - 5y^2 = 49.$$
Find the equation of the **tangent plane** to the surface S at $(3, 1, -3)$. Write it in the form $ax + by + cz = d$.
ANSWER: The gradient of $2x^2 + z^2 - 3xz - 5y^2$ is $(4x - 3z)\vec{i} - 10y\vec{j} + (2z - 3x)\vec{k} = 17\vec{i} - 10\vec{j} - 15\vec{k}$, which is a normal vector to the tangent plane, so the tangent plane is $17(x-3) - 10(y-1) - 15(z+3) = 0$ or $17x - 10y - 15z = 86$.

6. **(50 points) (a)** (10 points) Compute the first and second partial derivatives of $f(x, y) = x^3 + xy^2 - 3x^2 - y^2 - 6x$.
ANSWER: $f_x = 3x^2 + y^2 - 6x - 6$; $f_y = 2xy - 2y$; $f_{xx} = 6x - 6$; $f_{xy} = 2y$; $f_{yy} = 2x - 2$.

(50 points) (b) (15 points) Find all the **critical points** of $f(x, y)$.
ANSWER: Since $f_y(x, y) = 2xy - 2y = 0$, we have either $y = 0$ or $x = 1$. If $y = 0$, $f_x(x, y) = 3x^2 - 6x - 6 = 0$ so $x = 1 \pm \sqrt{3}$. If $x = 1$, then $y^2 = 9$, so $y = \pm 3$. Thus there are four critical points: $(1 + \sqrt{3}, 0)$, $(1 - \sqrt{3}, 0)$, $(1, 3)$ and $(1, -3)$.

(50 points) (c) (25 points) For each critical point, determine whether it is a **local maximum point,** a **local minimum point,** or a **saddle point.**
ANSWER: The matrix of second partial derivatives at $(1, \pm 3)$ has determinant $f_{xx}f_{yy} - f_{xy}^2 = -36$: these are both **saddle points**. At $(1 \pm \sqrt{3}, 0)$, the determinant is $f_{xx}f_{yy} - f_{xy}^2 = (\pm 6\sqrt{3})(\pm 2\sqrt{3}) - 0^2 = +36$: since $f_{xx} > 0$ at $(1 + \sqrt{3}, 0)$, this is a **local minimum point**; but $f_{xx} < 0$ at $(1 - \sqrt{3}, 0)$, so this critical point is a **local maximum point**.