1. (16 points) Let $D$ be the region $0 \leq x \leq 4$, $0 \leq y \leq \sqrt{x}$ in the $(x,y)$-plane. Find the double integral
\[ \int\int_D x^2 y \, dA. \]
ANSWER: 
\[ \int\int_D x^2 y \, dA = \int_0^4 \int_0^{\sqrt{x}} x^2 y \, dy \, dx = \int_0^4 x^2 \left[ \frac{y^2}{2} \right]_0^{\sqrt{x}} \, dx = \frac{1}{2} \int_0^4 x^3 \, dx = 32. \]

2. (18 points) Let $E$ be the triangular solid 
\[ E = \{ (x,y,z) : x \geq 0, \ y \geq 0, \ x + y \leq 1, \ 0 \leq z \leq 1 \}. \]
Find the triple integral 
\[ \int\int\int_E xyz \, dV. \]
ANSWER: 
\[ \int\int\int_E xyz \, dV = \int_0^1 \int_0^{1-y} \int_0^{1-x} xyz \, dz \, dy \, dx = \frac{1}{2} \int_0^1 \int_0^{1-y} z \, dy \, dx = \frac{1}{2} \int_0^1 \left( \frac{1}{2} - \frac{x}{3} + \frac{1}{4} \right) \, dx = \frac{1}{48}. \]

3. (16 points) Let $R$ be the rectangle $-3 \leq x \leq 4$, $0 \leq y \leq 3$ in the $(x,y)$-plane. If a continuous function $f(x,y)$ satisfies 
\[ -|x|y^2 \leq f(x,y) \leq 2, \]
for all $(x,y) \in R$, what does this tell you about the value of $\int\int_R f(x,y) \, dA$?
ANSWER: 
\[ A(R) = 21, \ \text{and} \ \int\int_R (-|x|y^2) \, dA = -\left[ \frac{x|x|}{2} \right]_{-3}^{4} \left[ \frac{y^3}{3} \right]_{0}^{3} = (-16/2 - 9/2)(27/3) = -\frac{225}{2}, \] 
so $-\frac{225}{2} \leq \int\int_R f(x,y) \, dA \leq 42$. 

4. (28 points) Suppose $x = X$ and $y = Y$ are random variables with joint density function 
\[ f(x,y) = \alpha(x^2 + y^2) \quad \text{if} \quad x^2 + y^2 \leq 1, \]
and 
\[ f(x,y) \equiv 0 \quad \text{if} \quad x^2 + y^2 > 1. \]

(a) (16 points) What does the constant $\alpha$ need to be?
ANSWER: We need $\int\int_{R^2} f(x,y) \, dA = 1$. So in polar coordinates: 
\[ 1 = \int_0^{2\pi} \int_0^1 \alpha(x^2+y^2) r \, dr \, d\theta = 2\pi \int_0^1 \alpha r^2 \, dr = \frac{\pi}{2} \alpha. \] 
This gives $\alpha = \frac{2}{\pi}$. 

(b) (12 points) Find the “median” radius $R$, so that the probability that $X^2 + Y^2 \leq R^2$ is 50%.
ANSWER: We want to find $R$ so that 
\[ \frac{1}{2} = \int_0^{2\pi} \int_0^R \alpha(x^2 + y^2) r \, dr \, d\theta = 2\pi \int_0^R \alpha r^3 \, dr = R^4. \] 
So 
\[ R = \left( \frac{1}{2\pi} \right)^{1/4}. \]
5. (28 points) A plate (or lamina) is in the shape of the triangle \( D \): \( 0 \leq x \leq 2, 0 \leq y \leq 1 - \frac{x}{2} \), with corners \((0,0), (0,1)\) and \((2,0)\). The plate has mass density at the point \((x,y)\) equal to \( \rho(x,y) = xy \) per unit area.

(a) (12 points) Find the total mass \( m \) of the plate.

**ANSWER:** \( m = \iint_{D} \rho(x,y) \, dA = \int_{0}^{2} \int_{0}^{1-x/2} xy \, dy \, dx = \int_{0}^{2} \left[ \frac{xy^2}{2} \right]_{0}^{1-x/2} \, dx = \frac{1}{3} \int_{0}^{2} x(1-x^2/2) \, dx = \frac{1}{3} \int_{0}^{2} \frac{1}{8} x(1-x^2/2) \, dx = \frac{1}{8} \int_{0}^{2} (4x - 4x^2 + x^3) \, dx = \frac{1}{8} \left[ \frac{4x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4} \right]_{0}^{2} = \frac{1}{8}(16/2 - 32/3 + 16/4) = \frac{6-8+3}{8} = \frac{1}{8}. \)

(b) (16 points) Find the center of mass \((\bar{x}, \bar{y})\) of the plate.

**ANSWER:** \( \bar{x} = \frac{1}{m} \int_{D} x \rho(x,y) \, dA = \frac{6}{8} \int_{0}^{2} \int_{0}^{1-x/2} xy^2 \, dy \, dx = \frac{3}{4} \int_{0}^{2} (4x^2 - 4x^3 + x^4) \, dx = \frac{3}{4} \left[ \frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right]_{0}^{2} = 3 \left( \frac{8}{3} - 4 + \frac{8}{5} \right) = \frac{4}{5}. \)

The other component of the center of mass is \( \bar{y} = 6 \int_{0}^{2} \int_{0}^{1-x/2} yx^2 \, dy \, dx = \frac{1}{2} \int_{0}^{2} (8x - 12x^2 + 6x^3 - x^4) \, dx = \frac{1}{2} \left[ 4x^2 - 4x^3 + 3x^4/2 - x^5/5 \right]_{0}^{2} = 4 - 8 + 6 - \frac{8}{5} = \frac{2}{5}. \)

An alternative computation is \( \bar{y} = 6 \int_{0}^{1} \int_{0}^{2(1-y)} xy^2 \, dx \, dy = 6 \int_{0}^{1} y^2 \left[ \frac{xy^3}{3} \right]_{0}^{2(1-y)} \, dy = 3 \int_{0}^{1} y^2[4(1-y)^2] \, dy = 12 \left( \frac{1}{3} - \frac{2}{3} + \frac{1}{5} \right) = \frac{-10+12}{5} = \frac{2}{5}. \) So the center of mass of the plate is \((\bar{x}, \bar{y}) = (\frac{4}{5}, \frac{2}{5})\). 

6. (22 points) Use the method of Lagrange multipliers to find the maximum and minimum values of

\[ f(x,y) = xy \]

among points \((x,y)\) which lie on the ellipse

\[ g(x,y) = x^2 + xy + y^2 = 3. \]

**ANSWER:** At any extreme point, we have \( \nabla f = \lambda \nabla g \) so \( yj + x\bar{i} = \lambda [(2x + y)\bar{i} + (x + 2y)\bar{j}] \), so \( \frac{y}{x+2y} = \lambda = \frac{x}{x+2y} \), which implies that \( y = \pm x \). If \( y = +x \), then \( g(x,y) = g(x,x) = x^2 + x^2 + x^2 = 3 \), so \( x = \pm 1 \) and \( f(x,y) = f(x,x) = x^2 = 1 \). On the other hand, if \( y = -x \), then \( g(x,y) = g(x,-x) = x^2 - x^2 + x^2 = 3 \), so \( x = \pm \sqrt{3} \) and \( f(x,y) = f(x,-x) = -x^2 = -3 \). That is, the maximum of \( f \) along the ellipse is \( f(1,1) = f(-1,-1) = +1 \), and the minimum is \( f(\sqrt{3}, -\sqrt{3}) = f(-\sqrt{3}, \sqrt{3}) = -3. \)

7. (22 points) Compute the surface area \( A(S) \) of the parabolic hyperboloid \( S = \{(x,y,z) : x^2 + y^2 \leq 1, z = y^2 - x^2 \} \). (Hint: polar coordinates.)

**ANSWER:** Write \( D = \{(x,y) : x^2 + y^2 \leq 1 \} \) and \( f(x,y) = y^2 - x^2 \). Then \( A(S) = \iint_{D} \sqrt{1 + (fx)^2 + (fy)^2} \, dA \).

Compute \( fx = -2x \) and \( fy = 2y \), so \( A(S) = \iint_{D} \sqrt{1 + 4x^2 + 4y^2} \, dx \, dy \). In polar coordinates, this integral becomes computable: \( A(S) = \int_{0}^{2\pi} \int_{0}^{1} \sqrt{1 + 4r^2} \, r \, dr \, d\theta = 2\pi \frac{1}{8} \int_{0}^{1} \sqrt{u} \, du \), where \( u = 1 + 4r^2 \). This yields \( A(S) = \frac{\pi}{5\sqrt{5} - 1}. \)