Math 2263 **Spring 2016** ____________________

Midterm 2WITH SOLUTIONS **March 24, 2016** __________

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1. (16 points) Let D be the region $0 \le x \le 4$, $0 \le y \le \sqrt{x}$ in the (x, y)-plane. Find the double integral
$$\iint_D x^2 y \, dA.$$
ANSWER:$\iint_D x^2 y \, dA = \int_0^4 \int_0^{\sqrt{x}} x^2 y \, dy \, dx = \int_0^4 x^2 \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx = \frac{1}{2}\int_0^4 x^3 \, dx = 32.$

2. (18 points) Let E be the triangular solid
$$E = \{(x, y, z) : x \ge 0, \ y \ge 0, \ x + y \le 1, \ 0 \le z \le 1\}.$$
Find the triple integral
$$\iiint_E xyz \, dV.$$
ANSWER: $\iiint_E xyz \, dV = \int_0^1 \int_0^1 \int_0^{1-y} xyz \, dx \, dy \, dz = \int_0^1 \int_0^1 \frac{1}{2}\left[x^2 yz\right]_0^{1-y} dy \, dz =$
$\frac{1}{2}\int_0^1 \int_0^1 y(1-y)^2 z \, dy \, dz = \frac{1}{2}\int_0^1 z \, dz \int_0^1 (y - 2y^2 + y^3) \, dy = \frac{1}{4}(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}) = \frac{6-8+3}{4 \cdot 12} = \frac{1}{48}.$

3. (16 points) Let R be the rectangle $-3 \le x \le 4$, $0 \le y \le 3$ in the (x, y)-plane. If a continuous function $f(x, y)$ satisfies
$$-|x|y^2 \le f(x, y) \le 2,$$
for all $(x, y) \in R$, what does this tell you about the value of $\iint_R f(x, y) \, dA$?

ANSWER: $A(R) = 21$, and $\iint_R (-|x| \, y^2) \, dA = -\left[x|x|/2\right]_{-3}^4 \left[y^3/3\right]_0^3 = (-16/2 - 9/2)(27/3) =$ $-\frac{225}{2}$, so $-\frac{225}{2} \le \iint_R f(x, y) \, dA \le 42.$

4. (28 points) Suppose $x = X$ and $y = Y$ are random variables with joint density function
$$f(x, y) = \alpha(x^2 + y^2) \quad \text{if} \quad x^2 + y^2 \le 1,$$
and
$$f(x, y) \equiv 0 \quad \text{if} \quad x^2 + y^2 > 1.$$

(a) (16 points) What does the constant α need to be?

ANSWER: We need $\iint_{\mathbb{R}^2} f(x, y) \, dA = 1$. So in polar coordinates: $1 = \int_0^{2\pi} \int_0^1 \alpha(x^2+y^2) r \, dr \, d\theta =$ $2\pi \int_0^1 \alpha r^2 r \, dr \, d\theta = \frac{\pi}{2}\alpha$. This gives $\alpha = \frac{2}{\pi}$.

(b) (12 points) Find the "median" radius R, so that the probability that $X^2 + Y^2 \le R^2$ is 50%.

ANSWER: We want to find R so that $\frac{1}{2} = \int_0^{2\pi} \int_0^R \alpha(x^2 + y^2) r \, dr \, d\theta = 2\pi \int_0^R \alpha r^3 \, dr = R^4$. So $R = \frac{1}{2^{1/4}}$.

5. (28 points) A plate (or lamina) is in the shape of the triangle D: $0 \le x \le 2$, $0 \le y \le 1 - \frac{x}{2}$, with corners $(0,0)$, $(0,1)$ and $(2,0)$. The plate has mass density at the point (x,y) equal to $\rho(x,y) = xy$ per unit area.

(a) (12 points) Find the **total mass** m of the plate.

ANSWER: $m = \iint_D \rho(x,y)\,dA = \int_0^2 \int_0^{1-x/2} xy\,dy\,dx = \int_0^2 \left[xy^2/2\right]_0^{1-x/2} dx = \frac{1}{2}\int_0^2 x(1-x/2)^2\,dx = \frac{1}{8}\int_0^2 (4x - 4x^2 + x^3)\,dx = \frac{1}{8}\left[\frac{4x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2 = \frac{1}{8}(16/2 - 32/3 + 16/4) = \frac{6-8+3}{6} = \frac{1}{6}$.

(b) (16 points) Find the **center of mass** $(\overline{x}, \overline{y})$ of the plate.

ANSWER: $\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = 6\int_0^2\int_0^{1-x/2} x^2 y\,dy\,dx = \frac{6}{4}\int_0^2 (4x^2 - 4x^3 + x^4)\,dx = \frac{3}{4}\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2 = 3\left(\frac{8}{3} - 4 + \frac{8}{5}\right) = \frac{4}{5}$.

The other component of the center of mass is $\overline{y} = 6\int_0^2\int_0^{1-x/2} xy^2\,dy\,dx = \frac{1}{4}\int_0^2 (8x - 12x^2 + 6x^3 - x^4)\,dx = \frac{1}{4}\left[4x^2 - 4x^3 + 3x^4/2 - x^5/5\right]_0^2 = 4 - 8 + 6 - \frac{8}{5} = \frac{2}{5}$.

An alternative computation is $\overline{y} = 6\int_0^1\int_0^{2(1-y)} xy^2\,dx\,dy = 6\int_0^1 y^2\left[\frac{x^2}{2}\right]_0^{2(1-y)} dy = 3\int_0^1 y^2[4(1-y)^2]\,dy = 12\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{-10+12}{5} = \frac{2}{5}$. So the center of mass of the plate is $(\overline{x}, \overline{y}) = (\frac{4}{5}, \frac{2}{5})$.

6. (22 points) Use the method of **Lagrange multipliers** to find the **maximum** and **minimum** values of

$$f(x,y) = xy$$

among points (x,y) which lie on the ellipse

$$g(x,y) = x^2 + xy + y^2 = 3.$$

ANSWER: At any extreme point, we have $\vec{\nabla} f = \lambda \vec{\nabla} g$ so $y\vec{i} + x\vec{j} = \lambda[(2x+y)\vec{i} + (x+2y)\vec{j}]$, so $\frac{y}{2x+y} = \lambda = \frac{x}{x+2y}$, which implies that $y = \pm x$. If $y = +x$, then $g(x,y) = g(x,x) = x^2 + x^2 + x^2 = 3$, so $x = \pm 1$ and $f(x,y) = f(x,x) = x^2 = 1$. On the other hand, if $y = -x$, then $g(x,y) = g(x,-x) = x^2 - x^2 + x^2 = 3$, so $x = \pm\sqrt{3}$ and $f(x,y) = f(x,-x) = -x^2 = -3$. That is, the maximum of f along the ellipse is $f(1,1) = f(-1,-1) = +1$, and the minimum is $f(\sqrt{3}, -\sqrt{3}) = f(-\sqrt{3}, \sqrt{3}) = -3$.

7. (22 points) Compute the surface area $A(S)$ of the parabolic hyperboloid $S = \{(x,y,z) : x^2 + y^2 \le 1, z = y^2 - x^2\}$. (*Hint:* polar coordinates.)

ANSWER: Write $D = \{(x,y) : x^2 + y^2 \le 1\}$ and $f(x,y) = y^2 - x^2$. Then $A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$. Compute $f_x = -2x$ and $f_y = 2y$, so $A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy$. In polar coordinates, this integral becomes computable: $A(S) = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\,r\,dr\,d\theta = 2\pi\frac{1}{8}\int_1^5 \sqrt{u}\,du$, where $u = 1 + 4r^2$. This yields $A(S) = \frac{\pi}{6}(5\sqrt{5} - 1)$.