Math 4428: Mathematical Modeling  
Spring 2010  
Project III: Dynamic Models  
Due Monday, 5 April 2010  

Instructions: You may use a computer algebra system (Mathematica, Maple, etc.) for your computations, such as computing derivatives of functions or solving systems of equations. However, you must clearly explain all the major steps of your solution and give all needed justifications. (“This is the answer that Black Box software gave” is not an adequate justification.) You are encouraged to discuss the project with other students, but you will have to work out and write down the solutions yourself. 

There are a total of 16 points.

1. (6 pts.) Consider the space docking problem of Examples 4.3 (p. 124) and 5.2 (p. 143) of Meerschaert. You may use the approximation \( \Delta v_n \sim -kwv_n \) of Meerschaert p. 129 if you prefer. In this case, for full credit on parts (a) and (b), you must carry out a sensitivity analysis of the optimal value \( k_0 \) with respect to the value of \( c \) in \( \Delta v_n = a_{n-1}c + a_nw \): compute \( S(k_0, 1 + c) \).
   
   (a) Take \( w = 10, c = 5 \), and assume \( 0 < k \leq 0.02 \) is a parameter. Corrected version: \( v_n \) converges to 0 with the approximate value \( C|\lambda|^n \), where \( C \) is a constant and \( \lambda \) is the eigenvalue of largest absolute value of a suitable matrix. Find the value of \( 0 < k \leq 0.02 \) which gives the fastest rate of convergence.
   
   (b) Same as part (a), but with \( 0.028 < k \leq 0.37 \).
   
   (c) Take the values \( k = 0.02 \) and \( w = 10 \). Would it increase the convergence rate if the control adjustment time \( c \) is decreased slightly?

2. (4 pts.) Reconsider the hardwood-softwood problem from Example 4.1 (p. 114) and Example 5.1 (p. 137). Use the same values of \( r_1, r_2, a_1 \) and \( a_2 \), but assume that \( b_1 = 0.5a_1 \) and \( b_2 = sa_2 \) where \( 0 < s \leq 0.5 \) is a parameter.
   
   (a) For each value of \( s \), find all equilibria in the state space \( \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\} \).
   
   (b) Determine if there is a stable coexistence equilibrium \( (H = x_1 > 0, S = x_2 > 0) \) for all values of \( 0 < s \leq 0.5 \).

3. (6 pts.) In this problem we will model the coexistence of blue whales and krill, one of the whale’s favorite foods, a small creature related to shrimp. Fin whales are not considered in this model. Assume logistic growth of whale and krill populations and model the interaction by a Lotka-Volterra system of differential equations, similar to the two differential equations at the top of p. 48. Use data for blue whales as given in Example 4.2 (p. 119). For krill, use the intrinsic growth rate of \( r_2 = 0.25 \) and the carrying capacity \( K_2 = 500 \) tons/acre. The whales benefit from the krill population level \( y \): if \( y \) increases by 500, then the growth rate \( \frac{dx}{dt} / x \) for the whale population \( x \) is 2% greater. Also, if \( x \) is increased by 15000 then the annual krill growth rate \( \frac{dy}{dt} / y \) decreases by 10%.

Determine whether there is a coexistence equilibrium, and whether it is stable.