Math 5616H: Introduction to Analysis II. Spring 2012
Homework 3. Problems and Solutions.

#1 (Ch. 7: #13(a)). Assume that \{f_n\} is a sequence of monotonically increasing functions on \(\mathbb{R}^1\) with \(0 \leq f_n(x) \leq 1\) for all \(x\) and all \(n\). Prove that there is a function \(f\) and a sequence \(\{n_k\}\) such that
\[
f(x) = \lim_{k \to \infty} f_{n_k}(x) \quad \text{for every} \quad x \in \mathbb{R}^1.
\] (1)

Proof. First we apply Theorem 7.23 with \(E := Q\) - the countable set of all rational real numbers. By this theorem, \(\{f_m\}\) has a subsequence \(\{f_{m_j}\}\) such that there exists
\[
f(q) := \lim_{j \to \infty} f_{m_j}(q) \quad \text{for every} \quad q \in Q.
\]

From monotonicity of \(f_m\) it follows that \(f\) is also monotonically increasing on \(Q\). We can extend \(f\) from \(Q\) to \(\mathbb{R}^1\) by the formula
\[
f_0(x) := \sup_{\{q \in Q : q \leq x\}} f(q) \quad \text{for every} \quad x \in \mathbb{R}^1.
\]

By Theorem 4.30, the set \(Q_0\) of all point of discontinuity of \(f_0\) is at most countable. Using Theorem 7.23 again with \(E := Q_0\), one can guarantee that \(\{f_{m_j}\}\) contains a subsequence \(\{f_{n_k}\} := \{f_{m_{j_k}}\} \subseteq \{f_n\}\) which converges at every point \(x \in Q_0\), in addition to every point in \(Q\) by the previous selection. This means that there exists the limit \(f(x)\) in (1) for every \(x \in Q \cup Q_0\).

In the remaining case \(x \not\in Q \cup Q_0\), we can approximate \(x\) by sequences \(q_j' \to x\), \(q_j'' \to x\), such that \(q_j', q_j'' \in Q\), and \(q_j' < x < q_j''\) for all \(j\). Then from the equalities
\[
f_{n_k}(q_j') \leq f_{n_k}(x) \leq f_{n_k}(q_j'')
\]
it follows
\[
f_0(q_j') = f(q_j') \leq \liminf_{k \to \infty} f_{n_k}(x) \leq \limsup_{k \to \infty} f_{n_k}(x) \leq f(q_j'') = f_0(q_j').
\]

Since \(f_0\) is continuous at \(x\), we have \(\lim f_0(q_j') = \lim f_0(q_j'') = f_0(x)\), hence
\[
f_0(x) = \liminf_{k \to \infty} f_{n_k}(x) = \limsup_{k \to \infty} f_{n_k}(x) = \lim_{k \to \infty} f_{n_k}(x),
\]
i.e. we have (1) with \(f(x) := f_0(x)\).

#2 (Ch. 7: #15). Suppose \(f\) is a real continuous function on \(\mathbb{R}^1\), \(f_n(t) = f(nt)\) for \(n = 1, 2, 3, \ldots\), and \(\{f_n\}\) is equicontinuous on \([0, 1]\). What conclusion can you draw about \(f\)?

Solution. We must have \(f(x) \equiv f(0)\) for all \(x > 0\). Indeed, suppose \(f(x) \neq f(0)\) for some \(x > 0\). Fix such \(x\) and take \(0 < \varepsilon < |f(x) - f(0)|\). For an arbitrary \(\delta > 0\), choose a natural \(n > x/\delta\). Then \(t_1 := 0\) and \(t_2 := x/n\) satisfy
\[
|t_2 - t_1| = \frac{x}{n} < \delta, \quad \text{and} \quad |f_n(t_2) - f_n(t_1)| = |f(x) - f(0)| > \varepsilon.
\]
By Definition 7.22, this means that the family \(\mathcal{F} := \{f_n\}\) is not equicontinuous on \([0, 1]\). Therefore, we have \(f(x) \equiv f(0)\) for all \(x > 0\). Note that there are no assumptions on the values of \(f(x)\) for \(x < 0\) in addition to the continuity of \(f\) on \(\mathbb{R}^1\).
#3 (Ch. 7: #18). Let \( \{f_n\} \) be a uniformly bounded sequence of functions which are Riemann-integrable on \([a,b]\), and put
\[
F_n(x) = \int_a^x f_n(t) \, dt \quad \text{for} \quad a \leq x \leq b.
\]
Prove that there exists a subsequence \( \{F_{n_k}\} \) which converges uniformly on \([a,b]\).

**Proof.** We have
\[
|f_n| \leq M < \infty \quad \text{on} \quad [a,b] \quad \text{for all} \quad n.
\]
This implies
\[
|F_n(x)| \leq \int_a^x |f_n(t)| \, dt \leq M \cdot (b-a) \quad \text{on} \quad [a,b] \quad \text{for all} \quad n.
\]
Moreover,
\[
|F_n(x) - F_n(y)| = \left| \int_y^x f_n(t) \, dt \right| \leq M \cdot |x-y| \quad \text{for all} \quad x, y \in [a,b],
\]
so that the family \( \mathcal{F} := \{F_n\} \) is equicontinuous on \([a,b]\). By Theorem 7.25, there exists a uniformly convergent subsequence \( \{F_{n_k}\} \) on \([a,b]\).

#4 (Ch. 7: #20). If \( f \) is continuous on \([0,1]\) and if
\[
\int_0^1 f(x) \, x^n \, dx = 0 \quad \text{for all} \quad n = 0, 1, 2, \ldots,
\]
prove that \( f(x) \equiv 0 \) on \([0,1]\).

**Proof.** From the given assumption it follows that
\[
\int_0^1 P_n(x) f(x) \, dx = 0 \quad \text{for every polynomial} \quad P_n(x).
\]
By Theorem 7.26, there exists a sequence of polynomials \( P_n \) such that
\[
\varepsilon_n := \sup_{[0,1]} |f - P_n| \to 0 \quad \text{as} \quad n \to \infty.
\]
Therefore,
\[
\int_0^1 f^2 \, dx = \int_0^1 (f - P_n)f \, dx \leq \varepsilon_n \cdot \sup_{[0,1]} |f| \to 0 \quad \text{as} \quad n \to \infty.
\]
Since the left side is non-negative and does not depend on \( n \), it must be equal to 0. Finally, since \( f \) is continuous on \([0,1]\) we get \( f(x) \equiv 0 \) on \([0,1]\) (Problem #4 in Homework #1).

#5 (Ch. 8: #1). Define
\[
f(x) = \begin{cases} 
  e^{-1/x^2} & (x \neq 0), \\
  0 & (x = 0).
\end{cases}
\]
Prove that \( f \) has derivatives of all orders at \( x = 0 \), and that \( f^{(n)}(0) = 0 \) for \( n = 1, 2, 3, \ldots \).

**Proof.** It is easy to prove by induction that
\[
f^{(n)}(x) = f(x) \cdot P_n(x) \cdot x^{-3n} \quad \text{for all} \quad x \neq 0 \quad \text{and} \quad n = 1, 2, 3, \ldots,
\]
where \( P_n(x) \) is a polynomial of some degree. Indeed, by the chain rule, \( f'(x) = 2x^{-3}f(x) \) for \( x \neq 0 \), i.e. (2) holds true for \( n = 1 \) with \( P_1 \equiv 2 \). Assuming that (2) is true for some \( n \geq 1 \), we get

\[
f^{(n+1)}(x) = \left( f(x) \cdot P_n(x) \cdot x^{-3n} \right)' = f(x) \cdot P_{n+1}(x) \cdot x^{-3n-3},
\]

where \( P_{n+1}(x) := 2P_n(x) + x^3P'_n(x) - 3nx^2P_n(x) \). By induction, we conclude that (2) is true for each \( n = 1, 2, 3, \ldots \), where \( P_n(x) \) is a polynomial of degree \( 2n - 2 \).

Next, from Taylor’s expansion

\[
e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} > \frac{t^k}{k!} \quad \text{for all } k = 1, 2, \ldots \text{ with } t = x^{-2} > 0
\]

it follows

\[
0 < f(x) = e^{-t} < \frac{k!}{t^k} = k! \cdot x^{2k} \quad \text{for all } x \neq 0 \text{ and } k = 1, 2, \ldots.
\]

Using this estimate with \( k = 2 \), we derive

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 0.
\]

Now suppose that \( f^{(n)}(0) = 0 \) for some natural \( n \). Combining (2) and the previous estimate with \( 2k > 3n + 1 \), we also get

\[
f^{(n+1)}(0) = \left| \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} \right| \leq \lim_{x \to 0} k! \cdot |P_n(x)| \cdot |x|^{2k-3n-1} = 0.
\]

By induction, we conclude that that \( f^{(n)}(0) = 0 \) for all natural \( n \).

**#6 (Ch. 8: #5).** Find the following limits

(a) \( \lim_{x \to 0} \frac{e^{-1 + x^{1/x}}}{x} \), \quad (b) \( \lim_{n \to \infty} \frac{n}{\ln n} (n^{1/n} - 1) \).

**Solution.** (a) We write

\[
(1 + x)^{1/x} = e^{\frac{\ln(1 + x)}{x}} = e^{1 + \alpha(x)}, \quad \text{where } \alpha(x) := \frac{\ln(1 + x)}{x} - 1 \to 0 \text{ as } x \to 0.
\]

Then using L’Hospital’s rule, we get

\[
\lim_{x \to 0} \frac{e - (1 + x)^{1/x}}{x} = e \lim_{x \to 0} \frac{1 - e^{\alpha(x)}}{\alpha(x)} = e \lim_{x \to 0} \frac{e^{\alpha} - 1}{\alpha} \cdot \lim_{x \to 0} \frac{\ln(1 + x) - x}{x^2} = e \cdot \lim_{x \to 0} \frac{(1 + x)^{-1} - 1}{2x} = e \cdot \lim_{x \to 0} \frac{1}{2(1 + x)} = \frac{e}{2}.
\]

(b) By L’Hospital’s rule,

\[
\alpha_n := \frac{\ln n}{n} \text{ satisfies } \lim_{n \to \infty} \alpha_n = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{x^{-1}}{1} = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \frac{n}{\ln n} (n^{1/n} - 1) = \lim_{n \to \infty} \frac{e^{\alpha_n} - 1}{\alpha_n} = \lim_{\alpha \to 0} \frac{e^{\alpha} - 1}{\alpha} = 1.
\]