TOTAL CURVATURE AND ISOTOPY OF GRAPHS IN $\mathbb{R}^3$

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ABSTRACT. Knot theory is the study of isotopy classes of embeddings of the circle $S^1$ into a 3-manifold, specifically $\mathbb{R}^3$. The Fáry-Milnor Theorem says that any curve of total curvature less than $4\pi$ is unknotted. More generally, a (finite) graph consists of a finite number of edges and vertices. Given a topological type of graphs $\Gamma$, what limitations on the isotopy class of $\Gamma$ are implied by a bound on total curvature? What does “total curvature” mean for a graph? We define a natural notion of net total curvature $N(\Gamma)$ of a graph $\Gamma \subset \mathbb{R}^3$ (see definition (4) below), and prove that if $\Gamma$ is homeomorphic to the $\theta$-graph, then $N(\Gamma) \geq 3\pi$; and if $N(\Gamma) < 4\pi$, then $\Gamma$ is isotopic in $\mathbb{R}^3$ to a planar $\theta$-graph. Further, $N(\Gamma) = 3\pi$ only when $\Gamma$ is a convex plane curve plus a chord.

1. INTRODUCTION: CURVATURE OF A GRAPH

The celebrated Fáry-Milnor theorem states that a curve in $\mathbb{R}^n$ of total curvature at most $4\pi$ is unknotted. In the present paper, we shall emphasize the knotting dimension $n = 3$. As a key step in his 1950 proof, John Milnor showed that for a smooth Jordan curve $\Gamma$ in $\mathbb{R}^3$, the total curvature equals half the integral over $e \in S^2$ of the number $\mu(e)$ of local maxima of the linear “height” function $\langle e, \cdot \rangle$ along $\Gamma$ [Mi]. This is then taken as the definition of total curvature when $\Gamma$ is only continuous. The Fáry-Milnor theorem (even for $C^0$ curves) follows, since total curvature less than $4\pi$ implies there is a unit vector $e_0 \in S^2$ so that $\langle e_0, \cdot \rangle$ has a unique local maximum, and therefore that this linear function is increasing on an interval of $\Gamma$ and decreasing on the complement. Without changing the pointwise value of this “height” function, $\Gamma$ can be topologically untwisted to a standard embedding of $S^1$ into $\mathbb{R}^3$. The Fenchel theorem, that any curve in $\mathbb{R}^3$ has total curvature at least $2\pi$, also follows from Milnor’s key step, since for all $e \in S^2$, the linear function $\langle e, \cdot \rangle$ assumes its maximum somewhere along $\Gamma$, implying $\mu(e) \geq 1$. Milnor’s proof is independent from the proof of Istvan Fáry, published earlier, which takes a different approach [Fa].

We would like to extend these results, replacing the simple closed curve by a finite graph $\Gamma$ in $\mathbb{R}^3$, $\Gamma$ consists of a finite number of points, called vertices, and a finite number of simple arcs, called edges, which each have as endpoints two of the vertices. We shall assume $\Gamma$ is connected. The valence of a vertex $q$ is the number $d(q)$ of edges which have $q$ as an endpoint. For convenience, we shall assume in the first part of this paper that each edge is of class $C^2$ up to its endpoints. In section 5 below, we shall extend our main theorem to graphs which are merely continuous. We include a regularity result (Lemma 5) implying that curves of finite
total curvature have one-sided tangent vectors at each point; we have not found this result in the literature.

The first difficulty, and a substantial one, in extending the results of Fáry and Milnor, is to understand the contribution to total curvature at a vertex of valence $d(q) \geq 3$. For a smooth closed curve $\Gamma$, the total curvature is

$$C(\Gamma) = \int_{\Gamma} |\vec{k}| \, ds,$$

where $s$ denotes arc length along $\Gamma$ and $\vec{k}$ is the curvature vector. If $x(s) \in \mathbb{R}^3$ denotes the position of the point measured at arc length $s$ along the curve, then $\vec{k} = \frac{dx}{ds}$. For a piecewise smooth curve, that is, a graph with vertices $q_1, \ldots, q_N$ having always valence $d(q_i) = 2$, the total curvature is readily generalized to

$$C(\Gamma) = \sum_{i=1}^{N} c(q_i) + \int_{\Gamma_{\text{reg}}} |\vec{k}| \, ds,$$

where the integral is taken over the separate $C^2$ edges of $\Gamma$ without their endpoints; and where $c(q_i) \in [0, \pi]$ is the exterior angle formed by the two edges of $\Gamma$ which meet at $q_i$. That is, $\cos(c(q_i)) = \langle T_1, -T_2 \rangle$, where $T_1 = \frac{dx}{ds}(q_i^+)$ and $T_2 = -\frac{dx}{ds}(q_i^-)$ are the unit tangent vectors at $q_i$ pointing into the two edges which meet at $q_i$. The exterior angle $c(q_i)$ is the correct contribution to total curvature, since any sequence of smooth curves converging to $\Gamma$ in $C^0$, with $C^1$ convergence on compact subsets of each open edge, includes a small arc near $q_i$ along which the tangent vector changes from near $\frac{dx}{ds}(q_i^-)$ to near $\frac{dx}{ds}(q_i^+)$. The greatest lower bound of the contribution to total curvature of this disappearing arc along the smooth approximating curves equals $c(q_i)$.

**Definitions of Total Curvature for Graphs**

When we turn our attention to a graph $\Gamma$, we find the above definition for curves (valence $d(q) = 2$) does not generalize in any obvious way to higher valence (see [G]). The ambiguity of the general formula (1) is resolved if we specify the definition of $c(0)$ when $\Gamma$ is the cone over a finite set $\{T_1, \ldots, T_d\}$ in the unit sphere $S^2$.

A rather natural definition of total curvature of graphs was given by Taniyama in [T]. We have called this maximal total curvature $\mathcal{MC}(\Gamma)$ in [G]; the contribution to total curvature at a vertex $q$ of valence $d$ is

$$mc(q) := \sum_{1 \leq i < j \leq d} \arccos \langle T_i, -T_j \rangle.$$

In the case $d(q) = 2$, the sum above has only one term, the exterior angle $c(q)$ at $q$. However, as may become apparent to readers of the present paper, for $d \geq 3$, this notion of total curvature offers certain problems for questions of the isotopy type of a graph in $\mathbb{R}^3$. 
In our earlier paper [GY] on the density of an area-minimizing two-dimensional rectifiable set $\Sigma$ spanning $\Gamma$, we found that it was very useful to apply the Gauss-Bonnet formula to the cone over $\Gamma$ with a point $p$ of $\Sigma$ as vertex. The relevant notion of total curvature in that context is \textbf{cone total curvature} $C_{\text{tot}}(\Gamma)$, defined using $tc(q)$ as the choice for $c(q)$ in equation (1):

\begin{equation}
(2) \quad tc(q) := \sup_{e \in S^2} \left\{ \sum_{i=1}^{d} \left( \frac{\pi}{2} - \arccos\langle T_i, e \rangle \right) \right\}.
\end{equation}

Note that in the case $d(q) = 2$, the supremum above is assumed at vectors $e$ lying in the smaller angle between the tangent vectors $T_1$ and $T_2$ to $\Gamma$, so that $tc(q)$ is then the exterior angle $c(q)$ at $q$. The main result of [GY] is that $2\pi$ times the area density of $\Sigma$ at any of its points is at most equal to $C_{\text{tot}}(\Gamma)$. The same result had been proved by Eckholm, White and Wienholtz for the case of a simple closed curve [EWW]. Taking $\Sigma$ to be the branched immersion of the disk given by Douglas [D1] and Radó [R], it follows that if $C(\Gamma) \leq 4\pi$, then $\Sigma$ is embedded, and therefore $\Gamma$ is unknotted. Thus [EWW] provided an independent proof of the Fáry-Milnor theorem. However, $C_{\text{tot}}(\Gamma)$ may be small for graphs which are far from the simplest isotopy types of a graph $\Gamma$.

In this paper, we introduce the notion of \textbf{net total curvature} $\mathcal{N}(\Gamma)$, which is the appropriate definition for generalizing — to graphs — Milnor’s approach to isotopy and total curvature of curves. For each unit tangent vector $T_i$ at $q$, $1 \leq i \leq d = d(q)$, let $\chi_i : S^2 \to \{-1, +1\}$ be equal to $-1$ on the hemisphere with center at $T_i$, and $+1$ on the opposite hemisphere (values along the equator, which has measure zero, are arbitrary). We then define

\begin{equation}
(3) \quad \text{nc}(q) := \frac{1}{4} \int_{S^2} \left[ \sum_{i=1}^{d} \chi_i(e) \right]^+ dA_{S^2}(e).
\end{equation}

In the case $d(q) = 2$, the integrand is positive (and equals 2) only on the set of unit vectors $e$ which have a negative inner product with both $T_1$ and $T_2$, ignoring $e$ in a set of measure zero. This set is a lune bounded by semi-great circles orthogonal to $T_1$ and to $T_2$, and has spherical area equal to twice the exterior angle. So in this case, $\text{nc}(q)$ is the exterior angle. We then define the net total curvature of a piecewise $C^2$ graph $\Gamma$ with vertices $\{q_1, \ldots, q_N\}$ as

\begin{equation}
(4) \quad \mathcal{N}(\Gamma) := \sum_{i=1}^{N} \text{nc}(q_i) + \int_{\Gamma_{\text{reg}}} |\tilde{k}| \, ds.
\end{equation}

We would like to explain how the net total curvature $\mathcal{N}(\Gamma)$ of a graph is related to more familiar notions of total curvature. Recall that a graph $\Gamma$ has an Euler circuit if and only if its vertices all have even valence, by a theorem of Euler. An Euler circuit is a closed, connected path which traverses each edge of $\Gamma$ exactly once. Of course, we do not have the hypothesis of even valence. We can attain that hypothesis by passing to the \textit{double} $\tilde{\Gamma}$ of $\Gamma$: let $\tilde{\Gamma}$ be the graph with the same vertices as $\Gamma$, but with two copies of each edge of $\Gamma$. Then at each vertex $q$, the
Valence as a vertex of $\tilde{\Gamma}$ is $\tilde{d}(q) = 2d(q)$, which is even. By Euler’s theorem, there is an Euler circuit $\Gamma'$ of $\tilde{\Gamma}$, which may be thought of as a closed path which traverses each edge of $\Gamma$ exactly twice. Now at each of the points $\{q_1, \ldots, q_d\}$ along $\Gamma'$ which are mapped to $q \in \Gamma$, we may consider the exterior angle $c(q_i)$. One-half the sum of these exterior angles, however, depends on the choice of the Euler circuit $\Gamma'$. For example, if $\Gamma$ is the union of the $x$-axis and the $y$-axis in Euclidean space $\mathbb{R}^3$, then one might choose $\Gamma'$ to have four right angles, or to have four straight angles, or something in between, with completely different values of total curvature. In order to form a version of total curvature at a vertex $q$ which only depends on the original graph $\Gamma$ and not on the choice of Euler circuit $\Gamma'$, it is necessary to consider some of the exterior angles as partially balancing others. In the example just considered, where $\Gamma$ is the union of two orthogonal lines, opposite right angles will be considered to balance each other completely, so that $nc(q) = 0$, regardless of the choice of Euler circuit of the double.

It will become apparent to the reader that the precise character of an Euler circuit of $\tilde{\Gamma}$ is not necessary in what follows. Instead, we shall refer to a parameterization of the double $\tilde{\Gamma}$, which is a mapping from a 1-dimensional manifold, which need not be connected; the mapping is assumed to cover each edge of $\tilde{\Gamma}$ once.

The nature of $nc(q)$ is clearer when it is localized on $S^2$, analogously to [Mi]. In the case $d(q) = 2$, Milnor showed that the exterior angle at the vertex $q$ equals $\frac{1}{2}$ the area of those $e \in S^2$ such that the linear function $\langle e, \cdot \rangle$, restricted to $\Gamma$, has a local maximum at $q$. In our case, we would like to describe $nc(q)$ as one-half the integral over the sphere of the number of net local maxima, which balances local maxima and local minima against each other. Along the parameterization $\Gamma'$ of the double of $\Gamma$, the linear function $\langle e, \cdot \rangle$ may have a local maximum at some of the vertices $q_1, \ldots, q_d$, and may have a local minimum at others. In our construction, each local minimum balances against one local maximum. If there are more local minima than local maxima, the number $nlm(e, q)$, the net number of local maxima, will be negative; however, we will use only the positive part $[nlm(e, q)]^+$. We need to show that

$$\int_{S^2} [nlm(e, q)]^+ dA_{S^2}(e)$$

is independent of the choice of parameterization, and in fact is equal to $2nc(q)$; this will follow from another way of computing $nlm(e, q)$, in the next section (see Corollary 2 below).

2. Some Combinatorics

**Definition 1.** Let a parameterization $\Gamma'$ of the double of $\Gamma$ be given. Then a vertex $q$ of $\Gamma'$ corresponds to a number of vertices of $\Gamma'$, this number being exactly the valence $d(q)$ of $q$ as a vertex of $\Gamma$. If $q \in \Gamma$ is not a vertex, then as the need arises, we may consider $q$ as a vertex of valence $d(q) = 2$. Let $l_{\text{max}}(e, q)$ be the number of local maxima of $\langle e, \cdot \rangle$ along $\Gamma'$ at points over $q$, and similarly let $l_{\text{min}}(e, q)$ be the number of local minima. Finally, we define the number of net local maxima of
\[ \langle e, \cdot \rangle \text{ at } q \text{ to be} \]
\[ \nlm(e, q) = \frac{1}{2}[\lmax(e, q) - \lmin(e, q)] \]

**Remark 1.** *The definition of \( \nlm(e, q) \) appears to depend not only on \( \Gamma \) but on a choice of the parameterization \( \Gamma' \) of the double of \( \Gamma \); but \( \Gamma' \) is not unique, and indeed \( \lmax(e, q) \) and \( \lmin(e, q) \) may depend on the choice of \( \Gamma' \). However, we shall see in Corollary 1 below that the number of net local maxima \( \nlm(e, q) \) is in fact independent of \( \Gamma' \).***

**Remark 2.** *We have included the factor \( \frac{1}{2} \) in the definition of \( \nlm(e, q) \) in order to agree with the difference of the numbers of local maxima and minima along a parameterization of \( \Gamma \) itself; in those cases (namely if \( d(q) \) is even) where these numbers may be defined.*

We shall *assume* for the rest of this section that a unit vector \( e \) has been chosen, and that the linear “height” function \( \langle e, \cdot \rangle \) has only a finite number of critical points along \( \Gamma \); this excludes \( e \) belonging to a subset of \( S^2 \) of measure zero. We shall also assume that the graph \( \Gamma \) is subdivided to include among the vertices all critical points of the linear function \( \langle e, \cdot \rangle \), with of course valence \( d(q) = 2 \) if \( q \) is an interior point of one of the original edges of \( \Gamma \).

**Definition 2.** *Choose a unit vector \( e \). At a point \( q \in \Gamma \) of valence \( d = d(q) \), let the up-valence \( d^+ = d^+(e, q) \) be the number of edges of \( \Gamma \) with endpoint \( q \) on which \( \langle e, \cdot \rangle \) is greater (“higher”) than \( \langle e, q \rangle \), the “height” of \( q \). Similarly, let the down-valence \( d^-(e, q) \) be the number of edges along which \( \langle e, \cdot \rangle \) is less than its value at \( q \). Note that \( d(q) = d^+(e, q) + d^-(e, q) \), for almost all \( e \) in \( S^2 \).***

**Lemma 1.** *(Combinatorial Lemma) \( \nlm(e, q) = \frac{1}{2}[d^-(e, q) - d^+(e, q)] \).***

**Proof.** Let a parameterization \( \Gamma' \) of the double of \( \Gamma \) be chosen, with respect to which \( \lmax(e, q) \), \( \lmin(e, q) \), \( \nlm(e, q) \) and \( d^\pm = d^\pm(e, q) \) are defined. Recall the assumption above, that \( \Gamma \) has been subdivided so that along each edge, the linear function \( \langle e, \cdot \rangle \) is strictly monotone.

The parameterization \( \Gamma' \) of the double of \( \Gamma \) has \( 2d = 2d(q) \) edges with an endpoint among the points \( q_1, \ldots, q_d \) which are mapped to \( q \in \Gamma \). There are two copies of each edge of \( \Gamma \), and on \( 2d^+ \), resp. \( 2d^- \) of these, \( \langle e, \cdot \rangle \) is greater resp. less than \( \langle e, q \rangle \). But for each \( 1 \leq i \leq d \), the parameterization \( \Gamma' \) has exactly two edges which meet at \( q_i \). Depending on the up/down character of both edges of \( \Gamma' \) which meet at \( q_i \), \( 1 \leq i \leq d \), we can count:

(+) If \( \langle e, \cdot \rangle \) is greater than \( \langle e, q \rangle \) on both edges, then \( q_i \) is a local minimum point; there are \( \lmin(e, q) \) of these among \( q_1, \ldots, q_d \).

(-) If \( \langle e, \cdot \rangle \) is less than \( \langle e, q \rangle \) on both edges, then \( q_i \) is a local maximum point; there are \( \lmax(e, q) \) of these.

(0) In all remaining cases, the linear function \( \langle e, \cdot \rangle \) is greater than \( \langle e, q \rangle \) along one edge and less along the other, in which case \( q_i \) is not counted in computing \( \lmax(e, q) \) or \( \lmax(e, q) \); there are \( d(q) - \lmax(e, q) - \lmin(e, q) \) of these.
Now count the individual edges of $\Gamma'$:

(+) There are $l_{\text{min}}(e, q)$ pairs of edges, each of which is part of a local minimum, both of which are counted among the $2d^+(e, q)$ edges of $\Gamma'$ with $\langle e, \cdot \rangle$ greater than $\langle e, q \rangle$. 

(-) There are $l_{\text{max}}(e, q)$ pairs of edges, each of which is part of a local maximum; these are counted among the number $2d^-(e, q)$ of edges of $\Gamma'$ with $\langle e, \cdot \rangle$ less than $\langle e, q \rangle$. Finally, 

(0) there are $d(q) - l_{\text{max}}(e, q) - l_{\text{min}}(e, q)$ edges of $\Gamma'$ which are not part of a local maximum or minimum, with $\langle e, \cdot \rangle$ greater than $\langle e, q \rangle$; and an equal number of edges with $\langle e, \cdot \rangle$ less than $\langle e, q \rangle$.

Thus, the total number of these edges of $\Gamma'$ with $\langle e, \cdot \rangle$ greater than $\langle e, q \rangle$ is

$$2d^+ = 2 l_{\text{min}} + (d - l_{\text{max}} - l_{\text{min}}) = d + l_{\text{min}} - l_{\text{max}}.$$

Similarly,

$$2d^- = 2 l_{\text{max}} + (d - l_{\text{max}} - l_{\text{min}}) = d + l_{\text{max}} - l_{\text{min}}.$$

Subtracting gives the conclusion:

$$n_{\text{lm}}(e, q) := l_{\text{max}}(e, q) - l_{\text{min}}(e, q) = \frac{d - \langle e, q \rangle - d^+(e, q)}{2}.$$

**Corollary 1.** The number of net local maxima $n_{\text{lm}}(e, q)$ is independent of the choice of parameterization $\Gamma'$ of the double of $\Gamma$.

**Proof.** Given a direction $e \in S^2$, the up-valence and down-valence $d^\pm(e, q)$ at a vertex $q \in \Gamma$ are defined independently of the choice of $\Gamma'$.

**Corollary 2.** For any $q \in \Gamma$, we have $n_{\text{c}}(q) = \frac{1}{2} \int_{S^2} \left[ n_{\text{lm}}(e, q) \right]^+ dA_{S^2}$.

**Proof.** Consider $e \in S^2$. In the definition (3) of $n_{\text{c}}(q)$, $\chi_i(e) = \pm 1$ whenever $\pm \langle e, T_i \rangle < 0$. But the number of $1 \leq i \leq d$ with $\pm \langle e, T_i \rangle < 0$ equals $d^+(e, q)$, so that

$$\sum_{i=1}^{d} \chi_i(e) = d^-(e, q) - d^+(e, q) = 2 n_{\text{lm}}(e, q)$$

by Lemma 1.

**Definition 3.** For a graph $\Gamma$ in $\mathbb{R}^3$ and $e \in S^2$, define the multiplicity at $e$ as

$$\mu(e) = \mu_{\Gamma}(e) = \sum\{n_{\text{lm}}^+(e, q) : q \text{ a vertex of } \Gamma \text{ or a critical point of } \langle e, \cdot \rangle\}.$$

Note that $\mu(e)$ is a half-integer. Note also that in the case when $\Gamma$ has no topological vertices, or equivalently, when $d(q) \equiv 2$, $\mu(e)$ is exactly the quantity $\mu(\Gamma, e)$, the number of local maxima of $\langle e, \cdot \rangle$ along $\Gamma$ as defined in [Mi], p. 252. It was shown in Theorem 3.1 of that paper that, in this case, $C(\Gamma) = \frac{1}{2} \int_{S^2} \mu(e) dA_{S^2}$.

We may now extend this result to graphs:
Theorem 1. For a (piecewise $C^2$) graph $\Gamma$, the net total curvature has the following representation:

$$\mathcal{N}(\Gamma) = \frac{1}{2} \int_{S^2} \mu(e) \, dA_{S^2}(e).$$

Proof. We have $\mathcal{N}(\Gamma) = \sum_{j=1}^{N} nc(q_j) + \int_{\Gamma_{\text{reg}}} |\vec{k}| \, ds$, where $q_1, \ldots, q_N$ are the vertices of $\Gamma$, including vertices of valence $d(q_j) = 2$, and where $nc(q) := \frac{1}{2} \int_{S^2} \left[ \sum_{i=1}^{d} \chi_i(e) \right] \, dA_{S^2}(e)$ by Definition 3. By Milnor’s result, $C(\Gamma_{\text{reg}}) = \frac{1}{2} \int_{S^2} \mu_{\Gamma_{\text{reg}}}(e) \, dA_{S^2}$. But $\mu(\Gamma) = \mu_{\Gamma_{\text{reg}}}(e) + \sum_{j=1}^{N} \text{nlm}^+(e, q_j)$, and the theorem follows. 

In section 5, we will have need of the monotonicity of $\mathcal{N}(P)$ under refinement of polygonal graphs $P$. This follows from Theorem 1 and the following Proposition.

Proposition 1. Let $P$ and $\tilde{P}$ be polygonal graphs in $\mathbb{R}^3$, having the same topological vertices, and homeomorphic to each other. Suppose that every vertex of $P$ is also a vertex of $\tilde{P}$. Then for almost all $e \in S^2$, the multiplicity $\mu_{\tilde{P}}(e) \geq \mu_P(e)$.

Proof. We may assume, as an induction step, that $\tilde{P}$ is obtained from $P$ by replacing the edge having endpoints $q_0, q_2$ with two edges, one having endpoints $q_0, q_1$ and the other having endpoints $q_1, q_2$. Choose $e \in S^2$. We consider several cases:

If the new vertex $q_1$ satisfies $\langle e, q_0 \rangle < \langle e, q_1 \rangle < \langle e, q_2 \rangle$, then $\text{nlm}_{\tilde{P}}(e, q_1) = \text{nlm}_P(e, q_1)$ for $i = 0, 2$ and $\text{nlm}_{\tilde{P}}(e, q_1) = 0$, hence $\mu_{\tilde{P}}(e) = \mu_P(e)$.

If $\langle e, q_0 \rangle < \langle e, q_2 \rangle < \langle e, q_1 \rangle$, then $\text{nlm}_{\tilde{P}}(e, q_0) = \text{nlm}_P(e, q_0) = \text{nlm}_P(e, q_1) = 1$. The vertex $q_2$ requires more careful counting: the up- and down-valence $d^\pm_{\tilde{P}}(e, q_2) = d^\pm_P(e, q_2) \pm 1$, so that by Lemma 1, $\text{nlm}_{\tilde{P}}(e, q_2) = \text{nlm}_P(e, q_2) - 1$. Meanwhile, for each of the polygonal graphs, $\mu(e)$ is the sum over $q$ of $\text{nlm}^+(e, q)$, so the change from $\mu_P(e)$ to $\mu_{\tilde{P}}(e)$ depends on the value of $\text{nlm}_P(e, q_2)$:

(a) if $\text{nlm}_P(e, q_2) \leq 0$, then $\text{nlm}^+_{\tilde{P}}(e, q_2) = \text{nlm}^+_P(e, q_2) = 0$;

(b) if $\text{nlm}_P(e, q_2) = \frac{1}{2}$, then $\text{nlm}^+_{\tilde{P}}(e, q_2) = \text{nlm}^+_P(e, q_2) - \frac{1}{2}$;

(c) if $\text{nlm}_P(e, q_2) \geq 1$, then $\text{nlm}^+_{\tilde{P}}(e, q_2) = \text{nlm}^+_P(e, q_2) - 1$.

Since the new vertex $q_1$ does not appear in $P$, recalling that $\text{nlm}_{\tilde{P}}(e, q_1) = 1$, we have $\mu_{\tilde{P}}(e) - \mu_P(e) = +1, +\frac{1}{2}$ or 0 in the respective cases (a), (b) or (c). In any case, $\mu_{\tilde{P}}(e) \geq \mu_P(e)$.

The reverse inequality $\langle e, q_1 \rangle < \langle e, q_2 \rangle < \langle e, q_0 \rangle$ leads to a similar case-by-case argument. This time, $\text{nlm}_{\tilde{P}}(e, q_1) = -1$, so $q_1$ does not contribute to $\mu_{\tilde{P}}(e)$. We have $\text{nlm}_{\tilde{P}}(e, q_0) - \text{nlm}_P(e, q_0) = 0$, while $\text{nlm}_{\tilde{P}}(e, q_2) - \text{nlm}_P(e, q_2) = \frac{1}{2} [d^-_{\tilde{P}} - d^-_P - d^+_P + d^+_P] = 1$. Now depending whether $\text{nlm}_P(e, q_2)$ is $\leq -1, = -\frac{1}{2}$ or $\geq 0$, we find that $\mu_{\tilde{P}}(e) - \mu_P(e) = \text{nlm}^+_{\tilde{P}}(e, q_2) - \text{nlm}^+_P(e, q_2) = 0, \frac{1}{2},$ or 1. In any case, $\mu_{\tilde{P}}(e) \geq \mu_P(e)$.
Note that these arguments are unchanged if $q_0$ is switched with $q_2$. This covers all cases except those in which equality occurs between $\langle e, q_i \rangle$ and $\langle e, q_j \rangle$ ($i \neq j$). The set of such unit vectors $e$ form a set of measure zero in $S^2$.

### 3. Valence Three or Four

Before proceeding to the relation between $N(\Gamma)$ and isotopy, we shall illustrate some properties of net total curvature $N(\Gamma)$ in a few relatively simple cases.

#### 3.1. Minimum curvature for valence three.

**Proposition 2.** If a vertex $q$ has valence $d(q) = 3$, then $nc(q) \geq \pi/2$, with equality if and only if the three tangent vectors $T_1, T_2, T_3$ at $q$ are coplanar but do not lie in any open half-plane.

**Proof.** At a vertex of valence 3, we have the simplification that the local parameterization $\Gamma'$ of the double $\tilde{\Gamma}$ is unique. Write $\alpha_i \in [0, \pi]$ for the angle between $T_i$ and $T_{i+1}$ (subscripts modulo 3). Then $nc(q) = \frac{1}{2} \sum_{i=1}^{3} (\pi - \alpha_i) = \frac{3}{2} \pi - \frac{1}{2} \sum_{i=1}^{3} \alpha_i$; but $\sum_{i=1}^{3} \alpha_i \leq 2\pi$, with strict inequality if the vectors are not coplanar or lie in an open half-plane. Otherwise, equality holds.

#### 3.2. Non-monotonicity of $N$ for subgraphs.

**Observation 1.** It might be assumed that if $\Gamma_0$ is a subgraph of a graph $\Gamma$, then $N(\Gamma_0) \leq N(\Gamma)$. However, this is not the case.

For a simple polyhedral counterexample, we may consider the “butterfly” graph $\Gamma$ in the plane with six vertices: $q_0^+ = (0, \pm 1)$, $q_1^+ = (1, \pm 3)$, and $q_2^+ = (-1, \pm 3)$. Three vertical edges $L_0, L_1$ and $L_2$ are the line segments joining $q_i^-$ to $q_i^+$; four additional edges are the line segments from $q_0^+$ to $q_1^+$ and from $q_0^-$ to $q_2^+$. The small angle $2\alpha$ at $q_0^+$ has $\tan \alpha = 1/2$, so that $\alpha < \pi/4$.

The subgraph $\Gamma_0$ will be $\Gamma$ minus the interior of $L_0$. Then $\Gamma_0$ is a simple closed curve, so that at each vertex $q_i^\pm$, we have $nc_{\Gamma_0}(q) = c(q) \in [0, \pi]$, the exterior angle. The edges are all straight, so the net total curvature has contributions only at the six vertices; we compute

$$N(\Gamma_0) = C(\Gamma_0) = 4(\pi - \alpha) + 2(\pi - 2\alpha) = 6\pi - 8\alpha.$$  

Meanwhile, as we have seen in Proposition 2, the three (coplanar) edges of $\Gamma$ at each of the two vertices $q_0^\pm$ determine $nc_{\Gamma}(q_0^\pm) = \pi/2$, so that

$$N(\Gamma) = 4(\pi - \alpha) + 2(\pi/2) = 5\pi - 4\alpha.$$  

Since $\alpha < \pi/4$, this implies that $N(\Gamma_0) > N(\Gamma)$.

#### 3.3. Simple description of net total curvature for valence 3.

**Proposition 3.** If $\Gamma$ is a graph having vertices only of valence two or three, then $N(\Gamma) = \frac{1}{2} C(\Gamma')$ for any parameterization $\Gamma'$ of the double of $\Gamma$. 


Proof. Since $\Gamma'$ covers each edge of $\Gamma$ twice, we need only show, for every vertex $q$ of $\Gamma$, having valence $d(q) \in \{2, 3\}$, that

$$2 \text{nc}_\Gamma(q) = \sum_{i=1}^{d} c_{\Gamma'}(q_i).$$

If $d = 2$, then $\text{nc}_\Gamma(q) = c_{\Gamma'}(q_1) = c_{\Gamma'}(q_2)$, so equation (5) clearly holds. For $d = 3$, write both sides of equation (5) as integrals over $S^2$, using the definition (3) of $\text{nc}_\Gamma(q)$:

$$2 \int_{S^2} [\chi_1 + \chi_2 + \chi_3]^+ \ dA_{S^2} = \int_{S^2} [\chi_1 + \chi_2]^+ \ dA_{S^2} +$$

$$+ \int_{S^2} [\chi_2 + \chi_3]^+ \ dA_{S^2} + \int_{S^2} [\chi_3 + \chi_1]^+ \ dA_{S^2},$$

where at each direction $e \in S^2$, $\chi_i(e) = \pm 1$ is the sign of $\langle -e, T_i \rangle$. But the integrands are equal at almost every point $e$ of $S^2$:

$$2 [\chi_1 + \chi_2 + \chi_3]^+ = [\chi_1 + \chi_2]^+ + [\chi_2 + \chi_3]^+ + [\chi_3 + \chi_1]^+, $$

as may be confirmed by cases: $6 = 6$ if $\chi_1 = \chi_2 = \chi_3 = +1$; $2 = 2$ if exactly one of the $\chi_i$ equals $-1$ and $0 = 0$ in the remaining cases.

3.4. Simple description of net total curvature fails, $d \geq 4$.

Observation 2. We have seen, for any parameterization $\Gamma'$ of the double $\bar{\Gamma}$ of a graph $\Gamma$, that $\mathcal{N}(\Gamma) \leq \frac{1}{2} \mathcal{C}(\Gamma')$, the total curvature in the usual sense of the parameterized curve $\Gamma'$. Moreover, for graphs with vertices of valence $\leq 3$, equality holds, by Proposition 3. A natural suggestion would be that $\mathcal{N}(\Gamma)$ might be half the infimum of total curvature of all parameterizations of the double. However, in some cases, we have the strict inequality $\mathcal{N}(\Gamma) < \inf_{\Gamma'} \frac{1}{2} \mathcal{C}(\Gamma')$.

In light of Proposition 3, we choose an example of a vertex $q$ of valence four.

Suppose that for a small positive angle $\alpha$, ($\alpha \leq 1$ radian would suffice) the four unit tangent vectors at $q$ are $T_1 = (1, 0, 0)$; $T_2 = (0, 1, 0)$; $T_3 = (-\cos \alpha, 0, \sin \alpha)$; and $T_4 = (0, -\cos \alpha, -\sin \alpha)$. Write $\theta_{ij} = \pi - \arccos \langle T_i, T_j \rangle$. Then each of the possible parameterizations of the double $\bar{\Gamma}$ has total curvature equal to the sum of any four of the $\theta_{ij}$, where each of the subscripts 1, 2, 3 and 4 appears twice. The particular parameterization where the four connected components are criss-crossing at the origin, which has total exterior curvature $2\theta_{13} + 2\theta_{24} = 4\alpha$, realizes the minimum value of the exterior curvature, as it can be seen as a perturbation of the simple “X-crossing” case ($\alpha = 0$) in which case there is no curvature contribution at the vertex. This shows that $\inf_{\Gamma'} \frac{1}{2} \mathcal{C}(\Gamma') = 2\alpha$.

However, $\text{nc}(q)$ is strictly less than $2\alpha$. We have written it as an integral over the unit sphere $S^2$:

$$\text{nc}(q) := \frac{1}{2} \int_{S^2} \left[ \sum_{i=1}^{d} \chi_i(e) \right]^+ \ dA_{S^2}(e)$$
(see Definition (3)). Note that $\chi_1(e)$ cancels $\chi_3(e)$, and $\chi_2(e)$ cancels $\chi_4(e)$, except on regions bounded by two pairs of great circles: $T_{1\perp}^1$ and $T_{3\perp}^1$; and $T_{2\perp}^1$ and $T_{4\perp}^1$. Each of the four lune-shaped sectors has area $2\alpha$ (see Figure 1).

Inside the sector partly bounded by the meridian $T_{1\perp}^1 \cap \{ z < 0 \}$, the sum $\chi_1 + \chi_3 = 2$, with the opposite sign on the antipodal sector. The sum $\chi_2 + \chi_4$ has a congruent pattern, but rotated by a right angle so that the sector where $\chi_1 + \chi_3$ is positive is crossed by the sector where $\chi_2 + \chi_4$ is negative (see Figure 1). Thus, since we compute the integral only of the positive part of the sum, there is partial cancellation $\sim \alpha^2$ when $\alpha$ is small. It follows that $\text{nc}(q) < 2\alpha$.

**3.5. Net total curvature $\neq$ cone total curvature $\neq$ Taniyama’s total curvature.**

It is not difficult to construct a vertex $q$ and three unit vectors $T_1, T_2, T_3$ such that the values of $\text{nc}(q)$, $\text{tc}(q)$ and $\text{mc}(q)$, with these vectors as the $d(q) = 3$ tangent vectors to a graph, have quite different values. For example, we may take $T_1, T_2$ and $T_3$ to be three unit vectors in a plane, making equal angles $\frac{2\pi}{3}$. According to Proposition 2, we have the contribution to net total curvature $\text{nc}(q) = \pi/2$. But the contribution to cone total curvature is $\text{tc}(q) = 0$. Namely, $\text{tc}(q) := \sup_{e \in S^2} \sum_{i=1}^3 \left( \frac{\pi}{2} - \arccos \langle T_i, e \rangle \right)$. In this supremum, we may choose $e$ to be normal to the plane of $T_1, T_2$ and $T_3$, and $\text{tc}(q) = 0$ follows. Meanwhile, $\text{mc}(q)$ is the sum of the exterior angles formed by the three pairs of vectors, each equal to $\pi/3$, so that $\text{mc}(q) = \pi$.

A similar computation for valence $d$ and coplanar vectors making equal angles gives $\text{tc}(q) = 0$, and $\text{mc}(q) = \pi \left[ \frac{(d-1)^2}{2} \right]$ (brackets denoting integer part), while $\text{nc}(q) = \pi/2$ for $d$ odd, $\text{nc}(q) = 0$ for $d$ even.

We may apply this to the problem of isotopy types of graphs of a specified homeomorphism type. Any isotopy type may be represented in the (plane or) the sphere by a piecewise smooth diagram in which at most two edges cross at any point, with the over/under character of each crossing indicated. Wherever a vertex has odd valence, let the diagram be given on the sphere with equal angles. Now let the isotopy class be represented by the graph of a smooth function in central

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**Figure 1.** An individual lune and overlapping lunes from above.
projection as indicated by the diagram, with the function multiplied by a small positive constant \( \varepsilon \) before adding the constant 1. As \( \varepsilon \to 0 \), the contribution to cone total curvature at each vertex tends to 0, but any vertex of odd valence \( d \) contributes \( \frac{\pi}{2d} \) in the limit to net total curvature. Meanwhile, \( mc(q) \) is roughly \( \frac{\pi}{2d^2} \).

For a graph \( \Gamma \) of a complicated isotopy type, \( C_{\text{tot}}(\Gamma) \) will be approximately as large as the total curvature of a similarly complicated knot; but \( \mathcal{N}(\Gamma) \) and \( \mathcal{MC}(\Gamma) \) may be much larger.

4. ISOTOPY OF THETA GRAPHS

We shall consider in this section one of the simpler homeomorphism types of graphs, the **theta graph**. A theta graph has only two vertices \( q^\pm \), and three edges, each of which connects \( q^+ \) to \( q^- \). The **standard theta graph** is the isotopy class in \( \mathbb{R}^3 \) of a plane circle plus a diameter, or of the lower-case Greek letter \( \theta \).

We may observe that there are nonstandard theta graphs in \( \mathbb{R}^3 \). For example, the union of two edges might form a knot.

Using the notion of net curvature, we may extend the theorems of Fenchel [Fen] and of Fáry-Milnor ([Fa],[Mi]), for curves homeomorphic to \( S^1 \), to graphs homeomorphic to the theta graph. We would like to thank Jaigyoung Choe and Rob Kusner for their comments about the case \( \mathcal{N}(\Gamma) = 3\pi \).

**Theorem 2.** Suppose \( \Gamma \subset \mathbb{R}^3 \) is a (piecewise \( C^2 \)) theta graph. Then \( \mathcal{N}(\Gamma) \geq 3\pi \). If \( \mathcal{N}(\Gamma) < 4\pi \), then \( \Gamma \) is isotopic in \( \mathbb{R}^3 \) to the standard theta graph. Moreover when \( \mathcal{N}(\Gamma) = 3\pi \), the graph is a planar convex curve plus a straight chord.

Recall Observation 1: for a subgraph \( \Gamma_0 \) of \( \Gamma \), \( \mathcal{N}(\Gamma_0) \) may be greater than \( \mathcal{N}(\Gamma) \). Thus, the unknottedness of subgraphs needs proof.

**Proof.** We first show the Fenchel-type **lower bound** \( 3\pi \). For any \( e \in S^2 \), consider the linear function \( \langle \cdot, e \rangle \) along \( \Gamma \). We need to show that \( \mu(e) \geq 3/2 \), from which Theorem 1 implies that \( \mathcal{N}(\Gamma) \geq 3\pi \). Write the two vertices of \( \Gamma \) as \( q^\pm \), each with valence \( d(q^\pm) = 3 \).

If the maximum \( t_{\text{max}} \) along \( \Gamma \) of \( \langle \cdot, e \rangle \) occurs at a vertex, say \( q^+ \), then \( \text{n}l\text{m}(e, q^+) = 3/2 \). Namely, at each of the three vertices \( q_1^+, q_2^+, q_3^+ \) of \( \Gamma' \) corresponding to \( q^+ \), \( \langle \cdot, e \rangle \) has a (global hence local) maximum. It follows in this case from Definitions 1 and 3 that \( \mu(e) \geq 3/2 \).

Otherwise, the maximum occurs at an interior point \( q_{\text{max}} \) of one of the original edges of \( \Gamma \); then \( \text{n}l\text{m}(e, q_{\text{max}}) = 1 \). By disregarding a set of \( e \in S^2 \) of measure zero, we may assume \( q_{\text{max}} \) is the unique maximum point. Then for values of \( t \) slightly smaller than \( t_{\text{max}} = \langle q_{\text{max}}, e \rangle \), there are exactly two points, which will be close to \( q_{\text{max}} \), where \( \langle \cdot, e \rangle \) takes the value \( t \). As \( t \) decreases towards \( t_{\text{min}} := \text{min}_\Gamma \langle \cdot, e \rangle \), the cardinality of the fibers of \( \langle \cdot, e \rangle \) must change from two to at least three, since otherwise \( \Gamma \) would be homeomorphic to \( S^1 \). If this cardinality increases at another critical point \( q_{\text{crit}} \neq q^+ \), then \( q_{\text{crit}} \) is a local maximum point and \( \mu(e) \geq \text{n}l\text{m}(e, q_{\text{max}}) + \text{n}l\text{m}(e, q_{\text{crit}}) = 2 > 3/2 \), as we wished to show. The remaining possibility is that the cardinality increases at a vertex, say \( q^+ \). Since the cardinality increases as \( t \) decreases through \( \langle q^+, e \rangle \), we have the down-valence \( d^-(e, q^+) \) strictly greater than the up-valence \( d^+(e, q^+) \). By
the Combinatorial Lemma 1, \( nlm(e, q^+) \geq 1/2 \), and in this case also, we have 
\( \mu(e) \geq nlm(e, q_{\text{max}}) + nlm(e, q^+) \geq 3/2 \). This shows that for any theta graph \( \Gamma \),
and almost any \( e \in S^2 \), \( \mu(e) \geq 3/2 \), hence by Theorem 1, the net total curvature 
\( \mathcal{N}(\Gamma) \geq 3\pi \).

The equality \( \mathcal{N}(\Gamma) = 3\pi \) specifies the geometry of the theta graph \( \Gamma \) as follows.
The theta graph consists of three arcs \( a_1, a_2 \) and \( a_3 \), meeting at \( q^+ \) and \( q^- \). We consider the two exclusive cases: the first is when the three arcs are coplanar. The second is when they are not.

We first make two observations about the case of Jordan curves, i.e. graphs \( \Gamma_0 \) homeomorphic to the circle.

(i) When \( \Gamma_0 \) is not planar, there exist four non-coplanar points \( p_i \), occurring in order, so that there are four disjoint arcs of \( \Gamma_0 \) (disjoint except for endpoints) joining \( p_1 \) to \( p_2 \), \( p_2 \) to \( p_3 \), \( p_3 \) to \( p_4 \) and \( p_4 \) to \( p_1 \). Now let an oriented plane \( P_0 \) be chosen to contain \( p_1 \) and \( p_2 \), rotated until both \( p_2 \) and \( p_4 \) are above \( P_0 \) (strictly, since no three of the points can be collinear). Then the cardinality of the set \( P_t \cap \Gamma_0 \) is greater than or equal to four, for \( 0 < t < \delta \ll 1 \). since each plane \( P_t \) parallel to \( P_0 \) at distance \( t \) meets each of the four disjoint arcs. This remains true when the normal vector \( e_0 \) to \( P_0 \) is replaced by any nearby \( e \in S^2 \).

(ii) When \( \Gamma_0 \) is planar and nonconvex, there exists a plane \( P_0 \) perpendicular to the plane containing \( \Gamma_0 \) which supports \( \Gamma_0 \) and touches \( \Gamma_0 \) at two distinct points. Then the cardinality of the set \( P_t \cap \Gamma_0 \) is greater than or equal to four, since the planes \( P_t \) parallel to \( P_0 \) meet each of the four disjoint arcs, for \( 0 < t < \delta \ll 1 \). Here once again there is a set of positive measure of \( e \in S^2 \) such that each plane \( P_t = \{ x : \langle e, x \rangle = t \} \) meets \( \Gamma_0 \) in at least four points, for \( t \) in a nonempty open interval.

Coming back to the theta graph, as we have seen, if \( \mathcal{N}(\Gamma) = 3\pi \) then for almost all \( e \in S^2 \), the value of \( \mu(e) \) equals 3/2.

We first consider the case when the three arcs are all coplanar. We claim that the three Jordan curves \( \Gamma_1 := a_2 \cup a_3, \Gamma_2 := a_3 \cup a_1 \) and \( \Gamma_3 := a_2 \cup a_3 \) are all convex. If not, say \( \Gamma_1 \) is not convex. Then there is a set of positive measure of \( e \in S^2 \) such that each plane \( P_t \) perpendicular to \( e \) meets \( \Gamma_1 \) in at least four points, for \( t \) in a nonempty open interval \( (t_1, t_2) \). The same will be true of the theta graph \( \Gamma \). But this implies the value of \( \mu(e) \) is at least two on a set of positive measure. Namely, the sum of \( d^-(e, q) - d^+(e, q) \) over all vertices \( q \) of \( \Gamma \) with \( \langle e, q \rangle \geq t_2 \) is at least 4. Thus by Lemma 1 and Definition 3, \( \mu(e) \geq 2 \). This contradicts the equality \( \mu = 3/2 \) a.e.. Hence each \( \Gamma_i \) is convex. Then it follows that the middle arc of the theta graph \( \Gamma \) is a line segment, as it needs to be a shared piece of two convex curves bounding disjoint open sets in the plane. The conclusion is that if \( \Gamma \) lies in a plane, it must be a convex Jordan curve plus a chord.

The second case is when the three arcs \( a_1, a_2 \) and \( a_3 \) are not coplanar. Then at least one of the Jordan curves, say \( \Gamma_1 \), is not planar. We refer to the observation (i) above to conclude that there is a set of positive measure of \( e \in S^2 \) such that each plane \( P_t \) orthogonal to \( e \) meets \( \Gamma_1 \), and hence \( \Gamma \), in at least four points, for \( t \) in a nonempty open interval. This leads as above to \( \mu(e) \geq 2 \), a contradiction to the
equality \( \mu = 3/2 \) almost everywhere. This concludes the argument in the case of equality \( \mathcal{N}(\Gamma) = 3\pi \).

We finally turn our attention to the Fáry-Milnor type upper bound, to ensure that a \( \theta \)-graph is isotopically standard: we shall assume that \( \mathcal{N}(\Gamma) < 4\pi \). By Theorem 1, since \( S^2 \) has area \( 4\pi \), it follows that there exists \( e_0 \in S^2 \) with \( \mu(e_0) < 2 \). Since \( \mu(e_0) \) is a half-integer, and since \( \mu(e) \geq 3/2 \), as we have shown in the first part of this proof, we have \( \mu(e_0) = 3/2 \) exactly.

We shall show that there are at most two points \( q \in \Gamma \), either vertices or critical points, with \( \text{nlm}(e_0, q) > 0 \). The “height” function \( \langle \cdot, e_0 \rangle \) has a maximum at a point \( q_{\text{max}} \), with \( \text{nlm}(e_0, q_{\text{max}}) = 1 \) or \( 3/2 \) according as the valence \( d(q_{\text{max}}) = 2 \) or \( 3 \). If there is a second point \( q \) with \( \text{nlm}(e_0, q) > 0 \), then \( \text{nlm}(e_0, q_{\text{max}}) = 1 \) and \( \text{nlm}(e_0, q) = \frac{1}{2} \), which implies that \( q \) is a vertex, say \( q^+ \), with \( d^+(q^+) = 1 \) and \( d^-(q^+) = 2 \) (recall that \( d(q^+) = 3 \)).

Observe that for any \( e \in S^2 \), \( \sum_q \text{nlm}(e, q) = 0 \); in fact the number of local maxima along any parameterization \( \Gamma' \) of \( \tilde{\Gamma} \) is equal to the number of local minima. It follows that \( \mu(-e_0) = \sum \text{nlm}(e_0, q)^- = 3/2 \). We may apply the argument above, replacing \( e_0 \) with \(-e_0 \), to show that either \( \text{nlm}(e_0, q_{\text{min}}) = -3/2 \) and \( \text{nlm}(e_0, q) \geq 0 \) elsewhere; or \( \text{nlm}(e_0, q_{\text{min}}) = -1 \) and there is a second point, which must be a vertex, say \( q^- \), with \( \text{nlm}(e_0, q^-) = -\frac{1}{2} \).

Thus, according to Lemma 1, there are four cases, depending on the valences \( d(q_{\text{max}}) \) and \( d(q_{\text{min}}) \) \in \{2, 3\}, for the cardinality \( \#(e_0, t) \) of the fiber \( \{ q \in \Gamma : \langle e_0, q \rangle = t \} \) as \( t \) decreases from \( t_{\text{max}} \) to \( t_{\text{min}} \). Write \( t^\pm := (q^\pm, e_0) \). The four cases, listed by \( (d(q_{\text{max}}), d(q_{\text{min}})) \), are:

1. \( (3,3) \): \#(\( e_0, t \)) \equiv 3, \( t_{\text{min}} = t^- < t < t_{\text{max}} = t^+ \).
2. \( (3,2) \): \#(\( e_0, t \)) = 3 for \( t^- < t < t_{\text{max}} = t^+ \); \#(\( e_0, t \)) = 2 for \( t_{\text{min}} < t < t^- \).
3. \( (2,3) \): \#(\( e_0, t \)) = 2 for \( t^- < t < t_{\text{max}} \); \#(\( e_0, t \)) = 3 for \( t_{\text{min}} = t^- < t < t^+ \).
4. \( (2,2) \): \#(\( e_0, t \)) = 2 for \( t^- < t < t_{\text{max}} \) and for \( t_{\text{min}} < t < t^- \); \#(\( e_0, t \)) = 3 for \( t^- < t < t^+ \).

In each of these four cases, we shall show that \( \Gamma \) is isotopic in \( \mathbb{R}^3 \) to a planar \( \theta \)-graph via an isotopy which does not change the values of \( \langle e_0, \cdot \rangle \). Let us consider the fourth case \( (2, 2) \) in detail, and observe that the other three cases follow in a similar fashion.

Write \( P(t) \) for the “horizontal” plane of \( \mathbb{R}^3 \) defined by \( \{ x \in \mathbb{R}^3 : \langle e_0, x \rangle = t \} \). Then \#(\( e_0, t \)) is the cardinality of \( P(t) \cap \Gamma \).

In the fourth case \( (2, 2) \), for \( t^- < t < t^+ \), there are \#(\( e_0, t \)) = 3 \) points in \( \Gamma \cap P(t) \). One of the three edges of \( \Gamma \), say \( a_2 \), lies entirely in the closed slab between \( P(t^-) \) and \( P(t^+) \). As \( t \to t^\pm \), two of the points converge from a well-defined direction, since the three unit tangent vectors at \( q^\pm \) are distinct. Letting \( t \) decrease from \( t = t^+ \), there is an isotopy of the plane \( P(t) \), varying continuously with \( t \), so that the three points of \( \Gamma \cap P(t) \) become collinear, with the points \( a_1 \cap P(t) \), \( a_2 \cap P(t) \) and \( a_3 \cap P(t) \) appearing in that order, or the reverse order, along a line.

After a further isotopy in \( \mathbb{R}^3 \) which translates and rotates each plane \( P(t) \) rigidly, we may achieve that \( P(t) \cap \Gamma \subset Q \) for some plane \( Q \) in \( \mathbb{R}^3 \) transverse to the planes \( P(t) \).
For \( t > t^+ \) and for \( t < t^- \), we may continuously rotate and translate the planes \( P(t) \) so that this “top” portion and this “bottom” portion of \( \Gamma \) each lie inside the same plane \( Q \). Thus, \( \Gamma \) is isotopic to a planar graph.

We may now find a further isotopy of the plane \( Q \), extendible to \( \mathbb{R}^3 \), which deforms \( \Gamma \) into a circle with one diameter. Therefore, \( \Gamma \) is isotopic in \( \mathbb{R}^3 \) to the standard \( \theta \) graph.

It should be noted that the last part of our proof does not hold when \( \Gamma \) has the homeomorphism type of a circle with two disjoint, parallel chords. Of course, the Fenchel-type lower bound for \( N(\Gamma) \) is larger than \( 3\pi \), namely \( 4\pi \). However, this graph may be embedded into \( \mathbb{R}^3 \) so that it is not isotopic to a planar graph, but so that a particular vector \( e_0 \) has the minimum value \( \mu(e_0) = 2 \): think of twisting the two chords of the standard graph an even number of times about each other, without introducing any critical points of \( \langle e_0, t \rangle \), and leaving the rest of the graph alone. Similarly, one may construct nonisotopic graphs with \( N(\Gamma) \) slightly greater than \( 4\pi \). Nonetheless, we conjecture that any graph \( \Gamma \) of this homeomorphism type with \( N(\Gamma) = 4\pi \) is a convex plane curve with two disjoint chords.

There remains a challenging question to determine the Fáry-Milnor type lower bound for the net total curvature of graphs in \( \mathbb{R}^3 \) of any specific homeomorphism type which are not isotopic to a standard embedding of that graph.

5. Nowhere-smooth graphs

Milnor extended his proof of the Fáry-Milnor Theorem to continuous knots in [Mi]; we shall carry out an analogous extension for graphs.

Up to this point, we have treated graphs which are piecewise \( C^2 \), so that the definition (4) of net total curvature:

\[
N(\Gamma) := \sum_{i=1}^{N} nc(q_i) + \int_{\Gamma_{\text{reg}}} |\vec{k}| \, ds
\]

makes sense. Recall that Milnor [Mi] defines the total curvature of a continuous simple closed curve \( C \) as the supremum of the total curvature of all the polygons inscribed in \( C \). In analogy to Milnor, we define total curvature of a continuous graph \( \Gamma \) to be the supremum of the total curvature of all polygonal graphs \( P \) suitably inscribed in \( \Gamma \) as follows. For a given continuous graph \( \Gamma \), we say a polygonal graph \( P \subset \mathbb{R}^3 \) is \( \Gamma \)-approximating, provided that its topological vertices (those of valence \( \neq 2 \)) are exactly the topological vertices of \( \Gamma \), and having the same valences; and that an arc of \( P \) between two topological vertices corresponds to each edge of \( \Gamma \) between those two vertices, the vertices of valence 2 of that arc lying in order along the corresponding edge of \( \Gamma \). It follows that \( P \) is homeomorphic to \( \Gamma \).

Recall that according to Proposition 1, if \( P \) and \( \tilde{P} \) are \( \Gamma \)-approximating polygonal graphs, and \( \tilde{P} \) is a refinement of \( P \), then \( N(\tilde{P}) \geq N(P) \).

**Definition 4.** Define the net total curvature of \( \Gamma \) by

\[
N(\Gamma) := \sup_P N(P)
\]
where the supremum is taken over all \( \Gamma \)-approximating polygonal graphs \( P \).

Here the net total curvature \( \mathcal{N}(P) \) is given as in the definition (4) by

\[
\mathcal{N}(P) := \sum_{i=1}^{N} \text{nc}(q_i)
\]

where \( q_1, \ldots, q_N \) are the vertices of \( P \).

Definition 4 is consistent with Definition 3 in the case where both may be applied, namely in the case of a piecewise \( C^2 \) graphs \( \Gamma \). In fact, Milnor showed that the total curvature \( \mathcal{C}(\Gamma_0) \) of a smooth curve is the supremum of the total curvature of inscribed polygons ([Mi], p. 251). At a vertex \( q \) of \( \Gamma \), as inscribed polygons \( P_k \) become arbitrarily fine, a vertex \( q_k \) of \( P_k \) corresponding to \( q \) has unit tangent vectors converging in \( S^2 \) to the unit tangent vectors to \( \Gamma \) at \( q \). Thus \( \text{nc}_{P_k}(q_k) \to \text{nc}_\Gamma(q) \).

**Definition 5.** We say a point is critical relative to \( e \in S^2 \) when \( e, \cdot > \) is not monotone in any open interval of \( \Gamma \) containing the point.

When \( \mathcal{N}(\Gamma) \) is finite, we shall show that the number of critical points is finite for almost all \( e \in S^2 \). (See Lemma 4 below.)

**Lemma 2.** Let \( \Gamma \) be a continuous, finite graph in \( \mathbb{R}^3 \), and choose a sequence \( \hat{P}_k \) of \( \Gamma \)-approximating polygonal graphs with \( \mathcal{N}(\Gamma) = \lim_{k \to \infty} \mathcal{N}(\hat{P}_k) \). Then for each \( e \in S^2 \), there is a refinement \( P_k \) of \( \hat{P}_k \) such that \( \lim_{k \to \infty} \mu_{P_k}(e) \) exists in \([0, \infty] \).

**Proof.** As a first step, we refine \( \hat{P}_k \) to include all vertices of \( P_{k-1} \). Then for all \( e \in S^2 \), \( \mu_{\hat{P}_k}(e) \geq \mu_{\hat{P}_{k-1}}(e) \). As the second step, we refine \( \hat{P}_k \) so that the arc of \( \Gamma \) corresponding to each edge of \( \hat{P}_k \) has diameter \( \leq 1/k \). As the third step, given a particular \( e \in S^2 \), for each edge \( \hat{E}_k \) of \( \hat{P}_k \), we add 0, 1 or 2 points from \( \Gamma \) as vertices of \( \hat{P}_k \) so that \( \max_{\hat{E}_k} \langle e, \cdot \rangle = \max_E \langle e, \cdot \rangle \) where \( E \) is the closed arc of \( \Gamma \) corresponding to \( \hat{E}_k \); and similarly so that \( \min_{\hat{E}_k} \langle e, \cdot \rangle = \min_E \langle e, \cdot \rangle \). Write \( P_k \) for the result of this three-step refinement. Note that all vertices of \( P_{k-1} \) appear among the vertices of \( P_k \). Then by Proposition 1,

\[
\mathcal{N}(\hat{P}_k) \leq \mathcal{N}(P_k) \leq \mathcal{N}(\Gamma),
\]

so we still have \( \mathcal{N}(\Gamma) = \lim_{k \to \infty} \mathcal{N}(P_k) \).

Now compare the values of \( \mu_{P_k}(e) = \sum_{q \in P_k} \text{nlm}_{P_k}(e, q) \) with the same sum for \( P_{k-1} \). Since \( P_k \) is a refinement of \( P_{k-1} \), we have \( \mu_{P_k}(e) \geq \mu_{P_{k-1}}(e) \) by Proposition 1.

Therefore the values \( \mu_{P_k}(e) \) are non-decreasing in \( k \), which implies they are either convergent or properly divergent; in the latter case we write \( \lim_{k \to \infty} \mu_{P_k}(e) = \infty \).

**Definition 6.** For a continuous graph \( \Gamma \), define the multiplicity at \( e \in S^2 \) as

\[
\mu_{\Gamma}(e) := \lim_{k \to \infty} \mu_{P_k}(e) \in [0, \infty],
\]

where \( P_k \) is a sequence of refined \( \Gamma \)-approximating polygonal graphs as given in Lemma 2.
Remark 3. Note that any two $\Gamma$-approximating polygonal graphs have a common refinement. Hence, from the proof of Lemma 2, any two choices of sequences \{\(P_k\)\} of refined $\Gamma$-approximating polygonal graphs lead to the same value $\mu_\Gamma(e)$.

Lemma 3. Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$. Then $\mu_\Gamma : S^2 \to [0, \infty]$ takes its values in the half-integers, or $\infty$. If $\mathcal{N}(\Gamma) < \infty$, then $\mu_\Gamma$ is integrable, hence finite almost everywhere on $S^2$, and

$$
\mathcal{N}(\Gamma) = \int_{S^2} \mu_\Gamma(e) \, dA_{S^2}(e).
$$

For almost all $e \in S^2$, a sequence $P_k$ of $\Gamma$-approximating polygonal graphs may be chosen (depending on $e$) so that each local extreme point $q$ of $\langle e, \cdot \rangle$ along $\Gamma$ occurs as a vertex of $P_k$ for sufficiently large $k$.

Proof. The half-integer-valued functions $\mu_{P_k}$ are non-negative, integrable on $S^2$ with bounded integrals since $\mathcal{N}(\Gamma) < \infty$, and monotone increasing in $k$. Thus for almost all $e \in S^2$, $\mu_{P_k}(e) = \mu_\Gamma(e)$ for $k$ sufficiently large. It follows that if $\mu_\Gamma(e)$ is finite, it must be a half-integer.

Since the functions $\mu_{P_k}$ are non-negative and pointwise non-decreasing, it now follows from the Monotone Convergence Theorem that

$$
\int_{S^2} \mu_\Gamma(e) \, dA_{S^2}(e) = \lim_{k \to \infty} \int_{S^2} \mu_{P_k}(e) \, dA_{S^2}(e) = \mathcal{N}(\Gamma),
$$

and in particular, $\mu_\Gamma(e)$ is a half-integer for almost all $e \in S^2$.

Finally, given $e \in S^2$, choose $\ell$ sufficiently large that $\mu_{P_k}(e) = \mu_\Gamma(e)$ for all $k \geq \ell$. Then for $k \geq \ell$, along any edge $E_k$ of $P_k$ with corresponding arc $E$ of $\Gamma$, the maximum and minimum values of $\langle e, \cdot \rangle$ along $E$ occur at the endpoints, which are also the endpoints of $E_k$. Otherwise, as $P_k$ is further refined, new interior points of $E$ would contribute a new, positive value to $\mu_{P_k}(e)$ as $k$ increases. Since the diameter of $E$ tends to zero as $k \to \infty$, any local maximum or local minimum of $\langle e, \cdot \rangle$ must become an endpoint of some edge of $P_k$ for $k$ sufficiently large. \qed

Lemma 4. Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$, with $\mathcal{N}(\Gamma) < \infty$. Then for almost all $e \in S^2$,

$$
\mu_\Gamma(e) = \sum_q (\text{nlm}(e, q))^+, \tag{7}
$$

where the sum is over the finite number of topological vertices of $\Gamma$ and critical points $q$ of $\langle e, \cdot \rangle$ along $\Gamma$. Further, for each $q$, $\text{nlm}(e, q) = \frac{1}{2}[d^-(e, q) - d^+(e, q)]$. All of these critical points which are not topological vertices are local extrema of $\langle e, \cdot \rangle$ along $\Gamma$.

Proof. We have seen in the proof of Lemma 3 that for almost all $e \in S^2$, there is a sequence \{\(P_k\)\} of $\Gamma$-approximating polygonal graphs with $\mu_\Gamma(e) = \mu_{P_k}(e)$ for $k$ sufficiently large. We consider such a unit vector $e$, chosen so that, further, $\langle e, \cdot \rangle$ is not constant along any open arc of $\Gamma$. Recall that $\mu_{P_k}(e) = \sum_q \text{nlm}_{P_k}^+(e, q)$, where the sum is over certain vertices of $P_k$, and the refined inscribed polygonal graph $P_k$ includes as vertices the maximum and minimum points of $\langle e, \cdot \rangle$ along the.
closed arc $E$ of $\Gamma$ corresponding to each edge $E_k$ of $P_k$. Thus, since each edge of $P_k$ corresponds to an arbitrarily small arc of $\Gamma$, each local maximum point $q$ for $\langle e, \cdot \rangle$ along $\Gamma$ provides a non-negative term $\operatorname{nlm}^+(e, q)$ in the sum for $\mu_{P_k}$, if $k$ is large enough. Since we have already chosen $k$ large enough that $\mu_{\Gamma}(e) = \mu_{P_k}(e)$, no further local maximum points will be encountered as $k$ increases further. Thus all local maximum points of $\langle e, \cdot \rangle$ along $\Gamma$ already occur among the vertices of $P_k$.

Recall that for a critical point $q$ relative to $e$, $\langle e, \cdot \rangle$ is not monotone on any neighborhood of $q$. We assume $q$ is not a topological vertex of $\Gamma$. Choose an ordering of the edge of $\Gamma$ containing $q$, and consider the interval of points $< q$. Then by an induction argument, there exist increasing sequences $p_n \to q$, $q_n \to q$, and $r_n \to q$ of points of the edge such that for each $n$, $r_n < p_n < q_n < r_n < q$, but the value $\langle e, q_n \rangle$ lies outside of the closed interval between $\langle e, p_n \rangle$ and $\langle e, r_n \rangle$. It follows that there is a local extremum $s_n \in (p_n, r_n)$. Since $r_n - 1 < p_n$, the $s_n$ are all distinct. Between any two local minimum points, there is a local maximum point. But as we have just shown, all local maximum points, specifically $s_n$, of $\langle e, \cdot \rangle$ along $\Gamma$ occur among the finite number of vertices of $P_k$, a contradiction. This shows that $\langle e, \cdot \rangle$ is monotone on an interval to the left of $q$. A similar argument shows that $\langle e, \cdot \rangle$ is monotone on an interval to the right. By our definition of critical point, the sense of monotonicity must be opposite on the two sides of $q$. Therefore every critical point $q$, which is not a topological vertex, is a local extremum.

Now fix $k$ such that $\mu_{\Gamma}(e) = \mu_{P_k}(e)$. Then for any edge $E_k$ of $P_k$, the function $\langle e, \cdot \rangle$ is monotone along the corresponding arc $E$ of $\Gamma$, as well as along $E_k$. Thus for each $t \in \mathbb{R}$, the cardinality $\#(e, t)$ of the fiber $\{q \in \Gamma : \langle e, q \rangle = t\}$ is the same for $P_k$ as for $\Gamma$. It follows from Lemma 1 that for each vertex or critical point $q$, $\operatorname{nlm}(e, q) = \frac{1}{2}[d^-(e, q) - d^+(e, q)]$, where $\operatorname{nlm}(e, q)$ and $d^\pm(e, q)$ have the same values for $\Gamma$ as for $P_k$. Finally, the formula $\mu_{\Gamma}(e) = \sum_q \{\operatorname{nlm}(e, q)\}^+$ now follows from the same formula for $P_k$, for almost all $e \in S^2$.

We may employ a similar proof as for Lemma 4 above to show

**Lemma 5.** Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$, with $N(\Gamma) < \infty$. Then at each vertex $q_0$ of $\Gamma$ of valence $d = d(q_0) \geq 1$, and for each edge $E_i$ of $\Gamma$ with $q_0$ as endpoint, the unit tangent vector

\[
T_i(q_0) := \lim_{q \to q_0, q \in E_i} \frac{q - q_0}{|q - q_0|}
\]

exists in $S^2$, $1 \leq i \leq d$.

**Proof.** Since $N(\Gamma)$ is finite, we have by Lemma ?? that $\mu(e) < \infty$ almost everywhere on $S^2$. Let $E$ be an open edge of $\Gamma$ with endpoint $q_0$, or, if $q_0$ is not a topological vertex of $\Gamma$, let $E$ be an open arc of $\Gamma$ lying on one side of $q_0$. In either case, $E \cup \{q_0\}$ is homeomorphic to $[0, 1)$. Consider a sequence $q_n \in E$, $q_n \to q_0$ monotonically. We claim that the limit, as $n \to \infty$, of

\[
T^n := \frac{q_n - q_0}{|q_n - q_0|}
\]
exists in $S^2$; the lemma will then follow.

Otherwise, by compactness of $S^2$, there are subsequences converging to distinct limits $v, w \in S^2$. By passing to further subsequences, we may assume the odd subsequence $T^{2k+1}$ converges to $v$ and the even subsequence $T^{2k}$ converges to $w$. There is an open lune $\subset S^2$ of $e \in S^2$ with $\langle e, v \rangle > 0 > \langle e, w \rangle$; further restricting to an open subset $U$ of the lune, if $\varepsilon$ is sufficiently small,

$$U := \{ e \in S^2 : \langle e, \tilde{v} \rangle > 0 > \langle e, \tilde{w} \rangle \forall \tilde{v} \in B_e^S(v), \tilde{w} \in B_e^S(w) \}$$

is nonempty. For some $\ell$, if $k \geq \ell$ and $e \in U$ then $\langle e, T^{2k} \rangle < 0 < \langle e, T^{2k+1} \rangle$. Therefore $\langle e, q_{2k} \rangle < \langle e, q_0 \rangle < \langle e, q_{2k+1} \rangle$. Choose $e \in U$ with $\mu(e) < \infty$. Now for each $n$, construct a $\Gamma$-approximating polygonal graph $P_n$ which includes $q_{2k}, q_{2k+1}, \ldots, q_{2\ell}$ as a set of consecutive vertices. Then $\langle e, \cdot \rangle$ has a local maximum along $P_n$ at $q_{2k+1}$, so $\mu_{P_n}(e) \geq (n - \ell) \to \infty$ as $n \to \infty$. But $\mu_{P_n}(e) \leq \mu_{\Gamma}(e)$. This contradicts $\mu_{\Gamma}(e) < \infty$.

**Corollary 3.** Let $\Gamma$ be a continuous, finite graph in $\mathbb{R}^3$, with $N(\Gamma) < \infty$. Then for each topological vertex $q$ of $\Gamma$, the contribution at $q$ to net total curvature is given by equation (3), where for $e \in S^2$, $\chi_i(e) =$ the sign of $(-T_i, e)$.

**Proof.** According to Lemma 5, $T_1(q), \ldots, T_d(q)$ are defined at $q$. If $P_n$ is a sequence of $\Gamma$-approximating polygonal graphs with maximum edgelength tending to 0, then the corresponding unit tangent vectors $T_{iP_n} \to T_i^\Gamma$ as $n \to \infty$. For each $P_n$, we have

$$\nu_{P_n}(q) = \frac{1}{2} \int_{S^2} \left[ \sum_{i=1}^d \chi_{P_n}(e) \right]^+ dA_{S^2}(e),$$

and $\chi_{iP_n} \to \chi_i^\Gamma$ at each point of $S^2$. Since the sum is uniformly bounded, the integrals for $P_n$ converge to those for $\Gamma$, which is equation (3).

We are ready to state the formula for the net total curvature, generalizing Theorem 1:

**Theorem 3.** For a continuous graph $\Gamma$, the net total curvature has the following representation:

$$N(\Gamma) = \frac{1}{2} \int_{S^2} \mu(e) dA_{S^2}(e),$$

where, for almost all $e \in S^2$, $\mu(e)$ is given as the finite sum (7).

**Proof.**

**Theorem 4.** Suppose $\Gamma \subset \mathbb{R}^3$ is a continuous theta graph. Then $N(\Gamma) \geq 3\pi$. If $N(\Gamma) < 4\pi$, then $\Gamma$ is isotopic in $\mathbb{R}^3$ to the standard theta graph. Moreover when $N(\Gamma) = 3\pi$, the graph is a planar convex curve plus a straight chord.

**Proof.**
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