Examples of hypersurfaces flowing by curvature in a Riemannian manifold

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Abstract

This paper gives some examples of hypersurfaces \( \varphi_t(M^n) \) evolving in time with speed determined by functions of the normal curvatures in an \((n + 1)\)-dimensional hyperbolic manifold; we emphasize the case of flow by harmonic mean curvature. The examples converge to a totally geodesic submanifold of any dimension from 1 to \( n \), and include cases which exist for infinite time. Convergence to a point was studied by Andrews, and only occurs in finite time. For dimension \( n = 2 \), the destiny of any harmonic mean curvature flow is strongly influenced by the genus of the surface \( M^2 \).

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1 Background

Unless otherwise mentioned, all Riemannian manifolds in this article are connected and complete. Let \( M^n \) be a smooth, connected, orientable compact manifold of dimension \( n \geq 2 \), without boundary, and let \((N^{n+1}, g^N)\) be a smooth connected Riemannian manifold. \( \sigma^N \) is any sectional curvature of \( N^{n+1} \), \( \mathcal{R} \) is the Riemann tensor of \( N^{n+1} \), and \( \nabla^N \) is the Levi-Civita connection corresponding to \( g^N \). For a hyperbolic manifold, \( \sigma^N = -1 \). When an index such as \( i \) is repeated in one term of an expression, summation \( 1 \leq i \leq n \) is indicated.
Suppose \( \varphi_0 : M^n \to N^{n+1} \) is a smooth immersion of an oriented manifold \( M^n \) into \( N^{n+1} \); write \( \vec{v} \) for the induced normal vector to \( \varphi_0(M) \). The second fundamental form of \( M \) is a covariant tensor which we represent at each point by a matrix \( A \), where the entry \( A_{ij} = h_{ij} = \langle \nabla_N^N \vec{v}, \frac{\partial}{\partial x_j} \rangle_{g^N} \). The Weingarten tensor is given by the matrix \( W \), whose entry \( \omega_{ki} = h_{ij}g^{jk} \) and \( \{g^{jk}\} \) is the pointwise inverse matrix of \( \{g_{jk}\} \).

We seek a solution \( \varphi : M^n \times [0, T) \to N^{n+1} \) to an equation

\[
\frac{\partial}{\partial t} \varphi(x, t) = -f(\lambda(W(x, t)))\vec{v}(x, t) \tag{1}
\]
\[
\varphi(x, 0) = \varphi_0(x)
\]

where \( F(x, t) = f(\lambda(W(x, t))) \) and \( f \) is a smooth symmetric function, where \( \vec{v}(x, t) \) is the outward normal vector to \( \varphi(M^n, t) \). \( W(x, t) \) is the Weingarten matrix of \( \varphi(M^n, t) \) in \( N^{n+1} \), and \( \lambda(W) \) is the set of eigenvalues \( (\lambda_1, \ldots, \lambda_n) \) of \( W \). Define \( \varphi_t(x) = \varphi(x, t) \), then \( (\lambda_1, \ldots, \lambda_n) \) are the principal curvatures of the hypersurface \( M_t \triangleq \varphi_t(M) \subset N \).

For example, (1) becomes Mean Curvature Flow when \( f(\lambda) = \sum_i \lambda_i \) (see [4], [8]).

Consider the solution \( \varphi : M^n \times [0, T) \to N^{n+1} \) of the following equations:

\[
\frac{\partial}{\partial t} \varphi(x, t) = -\left( \sum_i \lambda_i^{-1} \right)^{-1} \vec{v}(x, t) \tag{2}
\]
\[
\varphi(x, 0) = \varphi_0(x)
\]

Such a solution \( \varphi(x, t) \) is Harmonic Mean Curvature Flow; \( f(\lambda) = \left( \sum_i \lambda_i^{-1} \right)^{-1} \) is the harmonic mean of the numbers \( \lambda_1, \ldots, \lambda_n \).

It has been noted that the mean curvature flow of hypersurfaces in a Riemannian \( (n+1) \)-dimensional manifold, \( n \geq 2 \), does not have all the desirable properties satisfied for \( n = 1 \) [3]. For some purposes, harmonic mean curvature flow (2) may be the preferred way to extend curve-shortening flow to \( n \geq 2 \).

Andrews proved the following theorem in [2]:

**Theorem 1.** Let \( M^n \) and \( \varphi_0 \) be assumed as at the beginning of this paper, and that the Riemannian manifold \( (N^{n+1}, g^N) \) satisfies the following conditions:

\[-K_1 \leq \sigma^N \leq K_2, \quad |\nabla^N R^N|_{g^N} \leq L\]
for some nonnegative constants $K_1$, $K_2$ and $L$.
Assume every principal curvature $\lambda_i$ of $\varphi_0$ satisfies the following condition:
$\lambda_i > \sqrt{K_1}$

Then there exists a unique smooth solution to (2) on a maximal time interval $[0, T)$, $T < \infty$, and the immersion $\varphi_t$ converges uniformly to a round point $p$ in $N^{n+1}$ as $t$ approaches $T$.

Also, we have the following theorem, to appear in [7]:

**Theorem 2.** Let $M^n$ be a smooth, connected, orientable compact manifold of dimension $n \geq 2$, without boundary. Assume $N^{n+1}$ is a non-positively curved, simply-connected smooth manifold, and suppose $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion of $M^n$. Assume every principal curvature of $\varphi_0(M)$ is positive. Then there exists a unique smooth solution to (2) on a maximal time interval $[0, T)$, $T < \infty$, and the immersion $\varphi_t$ converges uniformly to a round point $p$ in $N^{n+1}$ as $t$ approaches $T$.

In the rest of this paper, except for Section 6, and unless otherwise mentioned, we consider harmonic mean curvature flow and let $f(\lambda) = (\sum_i \lambda_i^{-1})^{-1}$. We provide two specific examples of harmonic mean curvature flow for infinite time: in section 2, with dimension reduction in the limit, and in section 3, with the limit manifold of the same dimension as $M$. Note these examples in section 2 and section 3 provide barriers for harmonic mean curvature flow in Riemannian manifolds; further applications will be addressed in [7]. We discuss the limit behavior of the harmonic mean curvature flow at infinite time in section 4. Then we treat the special consequences of the Gauss-Bonnet theorem for 2-dimensional surfaces in section 5, and turn to examples of more general flows by functions of normal curvatures in section 6.

We would like to thank Gerhard Huisken for interesting discussions, and in particular for the observation that there are no examples in the literature for convergence of a compact hypersurface flowing by harmonic mean curvature in infinite time to a set of positive dimension. And we also would like to thank the referee for pointing out a gap in the earlier version of this paper.

2 The dimension-reduction example

In this section, we give an example where $\varphi_t$ converges to $\varphi_\infty$ in the $C^\infty$ topology but the dimension of $M_\infty = \varphi_\infty(M)$ is less than the dimension of $M_t$; i.e. there is **dimension reduction**.
Theorem 3. Let $N^3$ be a hyperbolic manifold containing an embedded closed geodesic $M_\infty$. Then there is a flow $\varphi_t : M^2 \to N^3$ by harmonic mean curvature, where $M^2$ is a torus, which converges to $M_\infty$ as $t \to +\infty$. The flow consists of immersions $\varphi_t$, which become embedded for $t$ sufficiently large.

For example, we may let the ambient manifold $N$ be $H^3/\mathbb{Z}$, where $H^3$ is hyperbolic space, represented as the Poincaré half space $(\mathbb{R}^3)^+ = \{(x,y,z) | (x,y,z) \in \mathbb{R}^3, z > 0\}$ with the metric $g_{ij}^N = \frac{1}{z^2} \delta_{ij}$ ($\delta_{ij} = \delta_{ji}$ = Kronecker delta), and the $\mathbb{Z}$ action $f : \mathbb{Z} \times H^3 \to H^3$ is defined as:

$$f(k)(x,y,z) = 2^k(x,y,z).$$

Recall that $f(k)$ is an isometry of $H^3$ for each $k \in \mathbb{Z}$.

Now we let $N$ be the quotient manifold of $H^3$ under the $\mathbb{Z}$-action, with fundamental domain $\{(x,y,z) | 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2\}$. Then $M_\infty$ = the positive $z$-axis, modulo $f(1)$, is a closed geodesic in $N$.

Proof: Let $\psi_0 : \mathbb{S}^1 \to N$ be an embedding as the given closed geodesic curve $M_\infty$ in $N$. We choose a unit vector field $w(x)$ in $(T_x \psi_0)$⊥. Then for $r > 0$, we define

$$\psi(x, \theta, r) = \psi_r(x, \theta) : \mathbb{S}^1 \times \mathbb{S}^1 \to N^3$$

by

$$\psi(x, \theta, r) = \psi_r(x, \theta) = \gamma(x, \theta, r),$$

where $\gamma(x, \theta, \cdot)$ is the unit-speed geodesic in $N$ with $\gamma(x, \theta, 0) = \psi_0(x)$ and $\frac{d}{dr}\gamma(x, \theta, r) = \vec{N}(x, \theta)$ at $r = 0$. Here $\vec{N}(x, \theta)$ is the unit tangent vector in $T_{\psi_0(x)} N^3$ which is perpendicular to $T_x \psi_0$ and makes the angle $\theta$ with $w(x)$. Then $\psi_r(\mathbb{S}^1 \times \mathbb{S}^1)$ has two principal curvatures:

$$\lambda_1(r) \equiv \tanh r, \quad \lambda_2(r) \equiv \coth r.$$

In fact, for $i = 1, 2$, $\lambda_i(r)$ is the logarithmic derivative of the length of a Jacobi field, and hence satisfies the Ricatti equation $\lambda'_i(r) + (\lambda_i(r))^2 = 1$.

We have constructed a one-parameter family of immersions $\psi_r : M \to N$, $-\infty < r < \infty$, with two principal curvatures: $\lambda_1(r) \equiv \tanh r$ and $\lambda_2(r) \equiv \coth r$. It may be observed that $\psi_r$ is an embedding for $r$ sufficiently small.

Now consider the harmonic mean curvature flow $\varphi_t = \psi_{r(t)} : M \to N$, with initial conditions $\varphi_0 = \psi_{r_0}$, $r(0) = r_0$, where $r_0$ is some fixed positive
constant. The speed must satisfy:

\[
\frac{\partial r}{\partial t} = \left\langle \frac{\partial \gamma}{\partial r}, \vec{v} \right\rangle = \left\langle \frac{\partial \gamma(x, r)}{\partial t}, \vec{v} \right\rangle
\]

\[
= \left\langle \frac{\partial \psi(x, r)}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \phi(x, t)}{\partial t}, \vec{v} \right\rangle
\]

\[
= \left\langle -F \vec{v}, \vec{v} \right\rangle = -F(\lambda_1, \lambda_2)
\]

In the first equation we use the fact \( \frac{\partial \gamma}{\partial r} = \vec{v} \); in the third equation we use the definition of \( \psi_r \), where \( \vec{v} = \vec{N}(x, \theta) \) is the outward normal vector of \( \psi_r(M) \) at \((x, \theta) \in S^1 \times S^1\).

Solving, we find

\[
r(t) = \frac{1}{2} \sinh^{-1} \left( e^{-t} \sinh 2r_0 \right).
\]

Note that \( r(t) \to 0 \) as \( t \to \infty \). □

3 The no-dimension-reduction example

In this section, we give an example in which \( M_t \) converges to \( M_\infty \) in the \( C^\infty \) topology and the dimension of \( M_\infty \) is the same as the dimension of \( M_t \), i.e. there is no dimension reduction.

**Theorem 4.** There is a compact surface \( M^2 \) of genus 2, a hyperbolic manifold \( N^3 \) diffeomorphic to \( M \times \mathbb{R} \), a totally geodesic embedding \( \psi_0 : M \to N \) and a flow by harmonic mean curvature \( \varphi_t : M \to N \) such that as \( t \to +\infty \), \( \varphi_t(M) \to \psi_0(M) \) smoothly.

**Proof:** Let \( \Omega \) be a regular geodesic octagon in the hyperbolic plane \( H^2 \), with angles \( \pi/2 \), and thus area \( 4\pi \). Label the edges as

\[\beta_1, \alpha_1', -\beta_1', -\alpha_1, \beta_2, \alpha_2', -\beta_2', -\alpha_2,\]

in that order, where the signs indicate orientation. Let \( A_1 \) be the orientation-preserving isometry of \( H^2 \) which maps the oriented geodesic segments \( \alpha_1 \) to \( \alpha_1' \); \( A_2 \) maps \( \alpha_2 \) to \( \alpha_2' \); \( B_1 \) maps \( \beta_1 \) to \( \beta_1' \); and \( B_2 \) maps \( \beta_2 \) to \( \beta_2' \). The group \( G \) of isometries of \( H^2 \) generated by \( A_1, A_2 \) and \( B_1 \) also includes \( B_2 \). \( G \) is
isomorphic to the fundamental group of the compact surface of genus 2. (See pp. 95–98 in Katok [6] for the arithmetic properties of the group $G$.)

Let $\psi_0 : H^2 \to H^3$ be an embedding as a totally geodesic surface in $H^3$. The isometries in $G$ extend in a well-known fashion to isometries of $H^3$, leaving the distance from $\psi_0(H^2)$ invariant.

Choose a unit normal vector field $\vec{N}$ to $\psi_0(H^2)$. Define $\psi_r : H^2 \to H^3$ by $\psi(r,\gamma(x)) = \psi_0(x)$, where $\gamma(x,\cdot)$ is the unit-speed geodesic in $H^3$ with $\gamma(x,0) = x$ and $\frac{\partial}{\partial r}\gamma(x,0) = \vec{N}(x)$.

Then $\psi_r(H^2)$ is totally umbilic, with normal curvatures $\lambda(r) \equiv \tanh r$. In fact, $\lambda(r)$ satisfies the Ricatti equation $\lambda'(r) + (\lambda(r))^2 = 1$, with the initial condition $\lambda(0) = 0$.

Now let the group $G$ act by isometries on $H^2$ and on $H^3$. The quotient $H^2/G = M^2$ is a compact surface of genus 2, with fundamental domain $\Omega$, and the quotient $H^3/G = N^3$ is a noncompact hyperbolic manifold diffeomorphic to $M \times \mathbb{R}$. The group $G$ acting on $N$ preserves each of the hypersurfaces $\psi_r(H^2)$. We have constructed a one-parameter family of totally umbilic embeddings $\psi_r : M \to N$, $-\infty < r < \infty$, with normal curvatures $\equiv \tanh r$.

Now consider the harmonic mean curvature flow $\varphi_t : M \to N$, with initial conditions $\varphi_0 = \psi_{r_0}$, where $r_0$ is some fixed positive constant. The speed must satisfy

$$\frac{\partial r}{\partial t} = \langle \frac{\partial \gamma}{\partial r} \frac{\partial r}{\partial t}, \vec{v} \rangle = \langle \frac{\partial \gamma(x,r)}{\partial t}, \vec{v} \rangle = \langle \frac{\partial \psi(x,r)}{\partial t}, \vec{v} \rangle$$

$$= \langle \frac{\partial \varphi(x,t)}{\partial t}, \vec{v} \rangle = \langle -F \vec{v}, \vec{v} \rangle = -F(\lambda_1,\lambda_2)$$

$$= -\frac{1}{\lambda_1^{-1} + \lambda_2^{-1}} = -\frac{1}{2} \tanh r.$$ 

In the first equation we use the fact $\frac{\partial \gamma}{\partial r} = \vec{N}(x) = \vec{v}$. In the third equation we use the definition of $\psi_r$, where $\vec{v}$ is the outward normal vector of $\psi_r$.

Solving, we find

$$r(t) = \sinh^{-1} \left( e^{-t/2} \sinh r_0 \right).$$

Note that $r(t) \to 0$ as $t \to \infty$. □
4 The limit behavior of harmonic mean curvature flow at infinite time

In this section, we will give a sufficient condition where harmonic mean curvature flow will exist forever, and discuss the limit behavior. Let \( \varphi_t : M \to N \) be an immersion of \( M^n \) into a hyperbolic manifold \( N^{n+1} \).

**Definition 5.** We define the following notation:

\[
\dot{F}_{kl} = \frac{\partial F}{\partial h_{kl}}, \quad \ddot{F}_{kl,pq} = \frac{\partial^2 F}{\partial h_{kl} \partial h_{pq}}, \quad \dot{H}_k = \frac{\partial H}{\partial \omega_k}, \quad \ddot{H}_{r,k} = \frac{\partial^2 H}{\partial \omega_k^r \partial \omega_k^s}, \quad \mathcal{R}_{ij} = \mathcal{R}_{i0j0},
\]

where 0 appearing as a tensor index represents the normal vector \( \vec{v} \) of \( \varphi(M) \) in \( N \). For any \( W : M \to \mathbb{R} \), we define:

\[
\mathcal{L}(W) = \dot{F}_{kl} \nabla_k \nabla_l W.
\]

Recall from Andrews [2] that \( \mathcal{L} \) is elliptic as long as \( \varphi_t(M) \) remains locally strictly convex.

**Theorem 6.** If \( N^{n+1} \) is a hyperbolic manifold, \( F(x) < \frac{1}{n} \) for any \( x \in M \), then \( \varphi_t(M) \) remains locally convex and \( F(x,t) < \frac{1}{n} \) for any \( x \in M, t \in [0, +\infty) \), \( \lim_{t \to -\infty} F(x,t) = 0 \), and the harmonic mean curvature flow exists for all \( t \) in \( [0, +\infty) \).

**Proof:** By Andrews [2], using a curvature coordinate system at one point, we have the following formula:

\[
\frac{\partial F}{\partial t} = \mathcal{L}(F) + F < \dot{F}, (\mathcal{R}^2) > + F < \dot{F}^{ij}, (\mathcal{R}_{ij}) >
\]

\[
= \mathcal{L}(F) + \sum_i F \frac{\partial f}{\partial \lambda_i} (\lambda_i^2 + \mathcal{R}_{ii})
\]

\[
\leq \mathcal{L}(F) + F^3 (n - \sum_i \lambda_i^{-2})
\]

\[
\leq \mathcal{L}(F) + F^3 \left( n - \frac{1}{n} F^{-2} \right).
\]

Consider the ODE

\[
\frac{\partial \tilde{F}}{\partial t} = \tilde{F}^3 (n - \frac{1}{n} \tilde{F}^{-2}),
\]

\[
\tilde{F}(0) = \max_{x \in M} F(x,0).
\]
Solving the above ODE, we get \( \tilde{F}(t)^{-2} - n^2 = (\tilde{F}(0)^{-2} - n^2) e^{2t/n} \). Because
\( 0 < \tilde{F}(0) = \max_{x \in \mathbb{M}^n} F(x, 0) < \frac{1}{n} \), we get \( \lim_{t \to \infty} \tilde{F}(t) = 0 \).

By the maximum principle, \( F(x, t) \leq \tilde{F}(t) < \frac{1}{n} \), for all \( x \in \mathbb{M}, t \in [0, +\infty) \), and therefore \( \lim_{t \to \infty} F(x, t) = 0 \).

On the other hand, we have the following estimate by the above evolution equation of \( F \):

\[
\frac{\partial F}{\partial t} \geq \mathcal{L}(F) + F^3(-\sum_{i} \lambda_i^{-2}) \geq \mathcal{L}(F) - F.
\]

Now consider the ODE

\[
\frac{\partial \tilde{F}}{\partial t} = -\tilde{F},
\]

\( \tilde{F}(0) = \min_{x \in \mathbb{M}} F(x, 0) \).

Then by the maximum principle again, we get for all \( x \in \mathbb{M}, t \in [0, +\infty) \):

\[
F(x, t) \geq \tilde{F}(t) = \min_{x \in \mathbb{M}} F(x, 0) e^{-t} > 0
\]

In particular, \( \varphi_t(M) \) remains convex for all \( t \).

Finally, we have the following estimate of \( H \). By Andrews’[2]:

\[
\frac{\partial}{\partial t} \omega_i^r = \hat{F}^{kl} \nabla_k \nabla_l \omega_i^r + \hat{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq}) g^{jr}
\]

\[
+ \hat{F}^{kl}(h_{ml} \omega_k^m) \omega_i^r + \hat{F}^{st} \hat{\mathcal{R}}_{st} h_{ij} g^{jr} + 2 \hat{F}^{pm} g^{tr} \omega_i^m \hat{\mathcal{R}}_{pqt}
\]

\[
- \hat{F}^{pq}(g^{tr} \omega_i^s \hat{\mathcal{R}}_{pqt} + g^{ts} \omega_s \hat{\mathcal{R}}_{pqt}) + \hat{F}^{pq} g^{tr} (\nabla_i \hat{\mathcal{R}}_{tpq0} - \nabla_p \hat{\mathcal{R}}_{qt0})
\]

Now referring to the last five terms above, we define:

\( (I) = \hat{H}_r^i \hat{F}^{kl}(h_{ml} \omega_k^m) \omega_i^r \), \( (II) = \hat{H}_r^i \hat{F}^{st} \hat{\mathcal{R}}_{st} h_{ij} g^{jr} \)

\( (III) = 2 \hat{H}_r^i \hat{F}^{pm} g^{tr} \omega_i^m \hat{\mathcal{R}}_{pqt} \), \( (IV) = -\hat{H}_r^i (\hat{F}^{pq} g^{tr} \omega_i^s \hat{\mathcal{R}}_{pqt} + \hat{F}^{pq} g^{ts} \omega_s \hat{\mathcal{R}}_{pqt}) \)

\( (V) = \hat{H}_r^i \hat{F}^{pq} g^{tr} (\nabla_i \hat{\mathcal{R}}_{tpq0} - \nabla_p \hat{\mathcal{R}}_{qt0}) \)

then

\[
\frac{\partial}{\partial t} H = \hat{H}_r^i \left( \frac{\partial}{\partial t} \omega_i^r \right)
\]

\[
= \hat{H}_r^i (\hat{F}^{kl} \nabla_k \nabla_l \omega_i^r) + \hat{H}_r^i \hat{F}^{kl,pq}(\nabla_i h_{kl})(\nabla_j h_{pq}) g^{jr} + (I) + \cdots + (V)
\]

Note

\[
\hat{F}^{kl} \nabla_k \nabla_l H = \hat{F}^{kl} \nabla_k (\hat{H}_r^i \nabla_l \omega_i^r) = \hat{F}^{kl} \hat{H}_r^i (\nabla_k \omega_i^r)(\nabla_l \omega_i^r) + \hat{F}^{kl} \hat{H}_r^i \nabla_k \nabla_l \omega_i^r.
\]
Define

$$(J) = \dot{H}_i \dot{F}^{kl,q}(\nabla_i h_{kl})(\nabla_j h_{pq})g^{jr} - \ddot{F}^{kl} \dddot{H}_i(\nabla_k \omega^r_i)(\nabla_l \omega^r_j);$$

we get

$$\frac{\partial}{\partial t} H = \mathcal{L}(H) + (J) + (I) + \cdots + (V).$$

It is straightforward to get

$$(I) + (II) = H[<\dot{F}, (\mathcal{W})^2 > + \dot{\mathcal{R}}_{i0j0}] \leq n F^2 H \leq \frac{1}{n} H$$

and

$$(V) = \frac{\partial f}{\partial \lambda_i} (\nabla_j \mathcal{R}_{ji0} - \nabla_i \mathcal{R}_{ij0}) = 0.$$

Choose a curvature coordinate system around one point; then we could do the following calculation:

$$(J) = \dot{F}^{kl,q}(\nabla_i h_{kl})(\nabla_j h_{pq})$$

But by the Lemma 2.22 in [1], we know $F$ is concave from the fact that $f$ is concave. So we get $(J) \leq 0$.

Now

$$(III) + (IV) = 2 \dot{H}_i \dot{F}^{pm} g^{tr} \omega^q_{pq} \mathcal{R}_{pqt} - \ddot{H}_i (\dot{F}^{pq} g^{tr} \omega^s_{pq} \mathcal{R}_{psqt} + \ddot{F}^{ps} g^{tr} \omega^s_{pq} \mathcal{R}_{psqt})$$

$$= 2 \delta_i^l \frac{\partial f}{\partial \lambda_p} \delta^m_q \delta^r_s \delta^t_k \mathcal{R}_{pqt} - \delta_i^l \left( \frac{\partial f}{\partial \lambda_p} \delta^m_q \delta^r_s \delta^t_k \mathcal{R}_{pqt} + \frac{\partial f}{\partial \lambda_p} \delta^m_q \delta^r_s \delta^t_k \mathcal{R}_{pqt} \right)$$

$$= 2 \mathcal{R}_{prpr} \left( \frac{\partial f}{\partial \lambda_p} \right)^2 = 2 \sum_{p<r} \mathcal{R}_{prpr} \left( \frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_r} \right)(\lambda_p - \lambda_r)$$

$$= \left( \sum_{k} \lambda_k^{-1} \right)^2 \sum_{i,j} \mathcal{R}_{ijij} \cdot (\lambda_i - \lambda_j)^2 (\lambda_i + \lambda_j) \cdot \lambda_i^{-2} \lambda_j^{-2}$$

$$\leq \sum_{i,j} (\lambda_i + \lambda_j) \cdot \left( \frac{\lambda_i^{-1} - \lambda_j^{-1}}{\sum_{k} \lambda_k^{-1}} \right)^2 \leq \sum_{i,j} (\lambda_i + \lambda_j) = 2n H$$

We have the following inequality for $H$ by the above estimates:

$$\frac{\partial H}{\partial t} \leq \mathcal{L}(H) + \left( 2n + \frac{1}{n} \right) H.$$

Now consider the ODE

$$\frac{\partial \dot{H}}{\partial t} = \left( 2n + \frac{1}{n} \right) \dot{H},$$
\[
\hat{H}(0) = \max_{x \in M} H(x, 0).
\]

Then by the maximum principle again, we get for all \(x \in M, t \in [0, +\infty)\):

\[
H(x, t) \leq \hat{H}(t) = \max_{x \in M} H(x, 0) e^{(2n+\frac{1}{n})t} < +\infty.
\]

This shows that the harmonic mean curvature flow exists on \([0, +\infty)\). □

In the rest of this section, we do not assume the ambient manifold \(N^{n+1}\) is a hyperbolic manifold.

**Proposition 7.** Assume \(N^{n+1}\) is a smooth \(n+1 \geq 3\) dimensional manifold which is convex at infinity, the maximal existence time of the harmonic mean curvature flow \(\varphi : M \times [0, T) \to N\) is \(T = +\infty\), and as \(t \to +\infty\), \(M_t = \varphi(M, t)\) converges to a smooth \(n\) dimensional submanifold \(M_\infty\) of \(N\) in the \(C^\infty\)-topology; then

\[
\max_{x \in M, t \in [0, +\infty)} \{|F(x, t)|, |\nabla F(x, t)|, |\nabla^2 F(x, t)|\} \leq C,
\]

where \(C\) is a constant depending on \(M_0, N^{n+1}\) and \(M_\infty\).

**Proof:** Straightforward from the assumptions. □

**Proposition 8.** Assume \(N\) and \(M_t \to M_\infty\) are as in the hypotheses of Proposition 7. Then

\[
\lim_{t \to \infty} \int_{M_t} F^2 d\mu_t = 0.
\]

**Proof:** By Theorem 1.1 in [5], we have the formula \(\frac{\partial}{\partial t}(\int_{M_t} d\mu_t) = -\int_{M_t} FH d\mu_t\). Because \(\int_{M_t} d\mu_t \to \mu(M_\infty)\) as \(t \to \infty\), we could find an \(\epsilon\)-dense set \(\{t_k\}_{k=1}^\infty\) for any positive constant \(\epsilon > 0\) such that

\[
\lim_{k \to \infty} t_k = \infty
\]

and

\[
\lim_{k \to \infty} \int_{M_{t_k}} FH d\mu_{t_k} = 0.
\]

Then using the inequality \(H \geq n^2 F\), we get \(\lim_{k \to \infty} \int_{M_{t_k}} F^2 d\mu_{t_k} = 0\). 

Now to get our conclusion we only need to show $\frac{\partial}{\partial t} \int_{M_t} F^2 \, d\mu_t$ is uniformly bounded. First, we know from Proposition 7 that $|F|$, $|\nabla F|$ and $|\nabla^2 F|$ are uniformly bounded. So we have

$$\frac{\partial}{\partial t} \left( \int_{M_t} F^2 \, d\mu_t \right) = \int 2FF_t + F^2(-FH) \, d\mu_t$$

$$= \int \left[ 2F \mathcal{L}(F) + \sum_{i=1}^n F \left( \frac{\partial f}{\partial \lambda_i} \right) (\lambda_i^2 + \mathcal{R}_{ii}) \right] - F^3 H \, d\mu_t$$

(where we use equation (3))

$$= \int 2nF^4 + 2F^4 \left( \sum_{i=1}^n \lambda_i^{-2} \mathcal{R}_{ii} \right) + 2F \mathcal{L}(F) - F^3 H \, d\mu_t$$

$$\leq \int 2F^4 K_2 \left( \sum_{i=1}^n \lambda_i^{-2} \right) \, d\mu_t + \int 2F \mathcal{L}(F) \, d\mu_t$$

$$\leq C \int F^2 \, d\mu_t + 2 \int F \mathcal{L}(F) \, d\mu_t,$$

where the first inequality uses the following facts:

$M_t$ is always contained in some compact set of $N^{n+1}$, since $N^{n+1}$ is convex at infinity, so its sectional curvature is bounded above by some constant $K_2$; and $HF^{-1} = (\sum_{i=1}^n \lambda_i) (\sum_{i=1}^n \lambda_i^{-1}) \geq n^2 \geq 2n$.

Next, since we know the volume of $M_t$ is always non-increasing and $|F|$ is uniformly bounded, we get

$$C \int_{M_t} F^2 \, d\mu_t \leq C_1,$$

where $C_1$ is some constant depending only on $M_0$, $N$ and $M_\infty$.

Since $|\nabla^2 F|$ is uniformly bounded, we get

$$2 \int F \mathcal{L}(F) \, d\mu_t \leq 2n^2 \int F|\nabla^2 F| \, d\mu_t \leq C_2,$$

where $C_2$ is some constant depending on $M_0$, $N$ and $M_\infty$.

By all the above we get

$$\frac{\partial}{\partial t} \left( \int F^2 \, d\mu_t \right) \leq C_3.$$
where $C_3$ is another constant depending on $M_0$, $N$ and $M_\infty$. Therefore
\[
\lim_{t \to \infty} \int_{M_t} F^2 \, d\mu_t = 0.
\]
□

**Corollary 9.** Assume $N$ and $M_t \to M_\infty$ are as assumed for Proposition 7. Then we have
\[
\lim_{t \to \infty} \left( \max_{x \in M} F(x,t) \right) = 0.
\]

**Proof:** By Proposition 8, we have
\[
0 = \lim_{t \to \infty} \int_{M_t} F^2 \, d\mu_t = \int_{M_\infty} \lim_{t \to \infty} F^2(x,t) \, d\mu_\infty,
\]
so the corollary follows. □

By the above results, assume $N$ and $M_t \to M_\infty$ are as in the hypotheses of Proposition 7, we know that $F \equiv 0$ on the limit surface $M_\infty$, if $M_\infty$ is the smooth limit of the harmonic mean curvature flow, which implies that $\det \mathcal{H} = 0$ on $M_\infty$.

### 5 Classification of harmonic mean curvature flow on surfaces

In this section, we consider harmonic mean curvature flow for $n = 2$, where $M^2$ is an orientable surface, $N^3$ is a hyperbolic manifold, and the harmonic mean $f(\lambda) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$. As before, we assume that $\varphi_0(M)$ is locally strictly convex.

In the following we always assume $F(x, 0) < \frac{1}{2}$, i.e. $\lambda_1^{-1} + \lambda_2^{-1} > 2$, which will guarantee the harmonic mean curvature flow exists forever by Lemma 6. Note that, for example, $f(\lambda_1, \lambda_2) < \frac{1}{2}$ for the examples of Theorems 3 and 4, and that the horospheres have $f(\lambda_1, \lambda_2) \equiv \frac{1}{2}$.

We define $C_0 = 2\pi \chi(M_0) = \int_{M_t} (K - 1) \, d\mu_t$, where the second equation is true for any $M_t$ because of the Gauss-Bonnet theorem, where $\chi(M_0)$ is the Euler number of $M_0$; $K(x, t) = \lambda_1(x, t) \lambda_2(x, t)$, $\lambda_1(x, t)$ and $\lambda_2(x, t)$ are the
principal curvatures at the point $x$ on $M_t$ in the ambient hyperbolic manifold $N^3$; and the Gauss equation, which implies the Gauss curvature $= K - 1$.

First, define $V(t) = \int_{M_t} 1 d\mu_t$, the area of $M_t$. Then using the formula

$$\frac{\partial}{\partial t} d\mu_t = -FH d\mu_t$$

we get

$$\frac{d}{dt} V(t) = \int_{M_t} \frac{\partial}{\partial t} d\mu_t = \int_{M_t} (-FH) d\mu_t = \int_{M_t} (-K) d\mu_t$$

$$= -\int_{M_t} (K - 1) d\mu_t - \int_{M_t} 1 d\mu_t = -C_0 - V(t).$$

Solving the above ODE, we get

$$V(t) = (V(0) + C_0)e^{-t} - C_0.$$ 

This shows that the area of $M_t$ is determined by its genus and the area $V(0)$ of the initial surface $M_0$.

There are three cases: $C_0 < 0$, $C_0 = 0$, $C_0 > 0$, corresponding to the surfaces with genus $g > 1$ (Case I), $g = 1$ (Case II) and $g = 0$ (Case III).

(I). Let us first consider the case $C_0 = 2\pi\chi(M_0) < 0$. In this case, we have

$$\lim_{t \to \infty} V(t) = -C_0 > 0$$

which means the limit surface has non-zero volume. We conjecture that in a hyperbolic manifold $N^3$, the limit surface will be the totally geodesic surface, if there is one in the homotopy class of $M_0$. This behavior is seen in Theorem 4.

(II). When $C_0 = 2\pi\chi(M_0) = 0$, we have

$$\lim_{t \to \infty} V(t) = -C_0 = 0$$

which means the limit surface has zero volume. In fact we could prove the following:

**Proposition 10.** If $N^3$ is a hyperbolic manifold, $F(x, 0) < \frac{1}{2}$ for all $x \in M$ and the genus of $M = 0$, then

$$\lim_{t \to \infty} \left(\max_{x \in M_t} H(x, t)\right) = +\infty.$$
Proof: Because \( \int_{M_t} (K-1) \, d\mu = C_0 = 0 \), we have \( \max_{x \in M_t} K(x,t) \geq 1 \). We also have \( \lim_{t \to \infty} (\max_{x \in M_t} F(x,t)) = 0 \), using the assumption \( F(x,0) < \frac{1}{2} \), by Theorem Then for any \( x \in M_t, \ t > 0 \), we have the following:
\[
K(x,t) = H(x,t)F(x,t) \leq F(x,t) \left( \max_{x \in M_t} H(x,t) \right). 
\]
Taking the maximum on the both sides of the above inequality, we have
\[
1 \leq \max_{x \in M_t} K(x,t) \leq (\max_{x \in M_t} F(x,t)) \left( \max_{x \in M_t} H(x,t) \right).
\]
So
\[
\max_{x \in M_t} H(x,t) \geq \frac{1}{\max_{x \in M_t} F(x,t)}.
\]
Taking the limit on both sides, we get
\[
\lim_{t \to \infty} \left( \max_{x \in M_t} H(x,t) \right) \geq \frac{1}{\lim_{t \to \infty} (\max_{x \in M_t} F(x,t))} = +\infty.
\]

The above proposition means that there exists at least one blowup point on the limit set; the example of Theorem 3 blows up at every point.

(III) Finally, when \( C_0 = 2\pi \chi(M_0) > 0 \), we have an interesting geometric result. In this case, because
\[
V(t) = (V(0) + C_0)e^{-t} - C_0,
\]
there exists some \( T_0, \ 0 < T_0 < +\infty \), such that \( V(T_0) = 0 \). That means the harmonic mean curvature flow stops in finite time. But we already proved that the flow will exist forever if \( F < \frac{1}{2} \). So under the assumption \( F < \frac{1}{2} \), this surface will not exist.

Remark 11. Observe that the nonexistence of the initial surfaces in Case (III) above may also be proven by lifting the simply-connected surface \( M_0 \) to the universal cover \( \mathbb{H}^3 \) of \( N^3 \) and applying the comparison principle with shrinking spheres centered at a point: the sphere of radius \( r \) has \( F = \frac{1}{2} \coth r > \frac{1}{2} \).
6 General geometric flows

In this section we give examples for a general geometric flow (1) in a hyperbolic manifold \(N^{n+1}\) which will exist forever or for a computable finite time, and converge to a given totally geodesic submanifold \(P^k\) of any codimension. In this section, we always assume the existence of a totally geodesic submanifold \(P^k\) in \(N^{n+1}\).

Firstly, by similar methods to those of section 2 and section 3, we may prove a theorem for general dimensions and codimensions:

**Theorem 12.** Assume \(P^k\) is a compact totally geodesic submanifold of the hyperbolic manifold \(N^{n+1}\), where \(1 \leq k \leq n\). Let \(M\) be diffeomorphic to the unit sphere bundle of the normal bundle \(\perp P\) when \(k < n\); we choose \(M\) to be one of the two connected components of the unit sphere bundle of the normal bundle \(\perp P\) when \(k = n\). Then we have a flow by harmonic mean curvature \(\varphi_t : M \to N\) such that as \(t \to +\infty\), \(\varphi_t(M) \to P\).

**Proof:** We only sketch the proof. We find the second fundamental form matrix of \(\psi_r(M)\) with respect to a basis of curvature directions is the following:

\[
\Psi = \begin{pmatrix}
I_k \tanh r & 0_{k \times (n-k)} \\
0_{(n-k) \times k} & I_{n-k} \coth r
\end{pmatrix}
\]

Then we find

\[
\frac{\partial r}{\partial t} = -F = -\frac{\tanh r}{k + (n-k)(\tanh r)^2}
\]

Solving this ODE, we get

\[
(sinh r(t))^k(cosh r(t))^{n-k} = Ce^{-t}
\]

where \(C = (\sinh r_0)^k(cosh r_0)^{n-k}\) is a fixed positive constant. This shows that \(\varphi_t := \psi_{r(t)}\) is a solution of harmonic mean curvature flow.

Note that \(r(t) \to 0\) as \(t \to +\infty\). \(\square\)

Now let \(M^n\) be diffeomorphic to (one connected component of) the unit sphere normal bundle of \(P^k\) in \(N^{n+1}\), and let \(\psi_r : M \to N\) define the hypersurface at distance \(r > 0\) from \(P^k\). We consider flow by an arbitrary symmetric function of the normal curvatures:
Theorem 13. For the symmetric function $f(\lambda_1, \cdots, \lambda_n)$, define
\[ h(r) = f(\tanh r, \cdots, \coth r), \]
where $\tanh r$ is repeated $k$ times and $\coth r$ is repeated $n-k$ times. Choose $r_0 > 0$ and define
\[ T_0 = \int_0^{r_0} \frac{1}{h(r)} \, dr, \quad 0 < T_0 \leq +\infty. \]

Then we may construct a flow
\[ \frac{\partial}{\partial t} \varphi(\cdot, t) = f(\lambda(W(x,t))) \vec{v}(x,t) \] (5)
with initial condition $\varphi(\cdot, 0) = \psi_{r_0}$, which exists for time $0 \leq t \leq T_0 \leq \infty$, and $\varphi(\cdot, t)$ converges to the totally geodesic $k$-dimensional submanifold $P^k$ as $t \to T_0$.

Proof: The hypersurface defined by $\varphi(\cdot, t) := \psi_{r(t)}$ flows by (5) if
\[ \frac{\partial r}{\partial t} = -F(x,t) \equiv -h(r) \] (6)

\[ \implies \int_{r(0)}^{r(T_0)} \frac{1}{h(r)} \, dr = \int_0^{T_0} -1 \, dt \]
\[ \implies T_0 = \int_0^{r_0} \frac{1}{h(r)} \, dr. \]

The conclusion now follows from the proof of Theorem 12, replacing equation (4) with equation (6). □

Remark 14. Note that the flow (5) is parabolic if $\frac{\partial f}{\partial \lambda_i} > 0$ ($1 \leq i \leq n$); parabolic for backwards time if $\frac{\partial f}{\partial \lambda_i} < 0$ ($1 \leq i \leq n$); and is a first-order PDE if $f$ is constant.

The following corollary is a generalization of both mean curvature flow ($m = 1, \ell = 0$) and of harmonic mean curvature flow ($m = n, \ell = n-1$).

Corollary 15. Assume $P^k$ is a compact totally geodesic submanifold of $N^{n+1}$, where $1 \leq k \leq n$. Let $M$ be diffeomorphic to the unit sphere bundle of the normal bundle $\perp P$ when $k < n$; $M$ is one of the two components of the unit sphere bundle of $\perp P$ when $k = n$. 

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For integers $0 \leq m, \ell \leq n$, let $S_m, S_\ell$ be the elementary symmetric functions of degree $m, \ell$ respectively, of the principal curvatures $\lambda_1, \ldots, \lambda_n$ of $M_t$. We have a flow by curvature function

$$F(x,t) = \frac{S_m(\lambda_1, \ldots, \lambda_n)}{S_\ell(\lambda_1, \ldots, \lambda_n)},$$

for time $0 \leq t < \infty$, such that $\varphi(t) : M \to N$, and $\varphi_* (M) \to P$ as $t \to +\infty$; assuming that the integers $m, \ell$ satisfy $|m - (n - k)| < |\ell - (n - k)|$.

**Remark 16.** Theorem 13 also may be applied to prove a partial converse of Corollary 15: assuming $P^k$ and $N^{n+1}$ are as in Corollary 15, if the opposite condition $|m - (n - k)| \geq |\ell - (n - k)|$ holds, then the same construction yields a flow of hypersurfaces by the curvature function $F = \frac{S_m}{S_\ell}$ which converges to the totally geodesic submanifold $P^k$ in finite time $T_0$.

**Proof:** In the following, we fix an arbitrary positive constant $r(0) = r_0$. Firstly we have

$$S_m = \sum_{\substack{p + q = m \\ 0 \leq p \leq \ell}} C^p_k (\tanh r)^p C^q_{n-k} (\coth r)^q = \sum_{\substack{p + q = m \\ 0 \leq q \leq n-k}} C^p_k C^q_{n-k} (\coth r)^q,$$

where $C^p_k$ is the combinatorial coefficient $\frac{k!}{p!(k-p)!}$.

Since $\coth r \geq 1$, it is easy to see

$$S_m \sim \begin{cases} (\coth r)^m & \text{if } m \leq n - k \\ (\coth r)^{2(n-k)-m} & \text{if } m > n - k \end{cases}$$

where the notation $S_m \sim (\coth r)^{i}$ means that there exist positive constants $C_1, C_2$ such that $C_1 (\coth r)^{i} \leq S_m \leq C_2 (\coth r)^{i}$. Here $C_1$ and $C_2$ will depend only on $m, n, k, \ell$ and $r_0$.

Similarly, we have

$$S_\ell \sim \begin{cases} (\coth r)^\ell & \text{if } \ell \leq n - k \\ (\coth r)^{2(n-k)-\ell} & \text{if } \ell > n - k. \end{cases}$$

Therefore

$$F = \frac{S_m}{S_\ell} \sim \begin{cases} (\coth r)^{m-\ell} & \text{if } m, \ell \leq n - k \\ (\coth r)^{\ell-m} & \text{if } m, \ell > n - k \\ (\coth r)^{2(n-k)-m-\ell} & \text{if } \ell \leq n - k < m \\ (\coth r)^{m+\ell-2(n-k)} & \text{if } m \leq n - k < \ell. \end{cases}$$
By Theorem 13, we obtain that the flow exists forever if and only if the power of \( \coth r \) is negative in the asymptotic estimate for \( F \) above. That is, if and only if \( m \) and \( \ell \) satisfy one of the following conditions:

\[
\begin{align*}
&\begin{cases} 
m < \ell & \text{if } m, \ell \leq n - k \\
\ell < m & \text{if } m, \ell > n - k \\
2(n - k) < m + \ell & \text{if } \ell \leq n - k < m \\
m + \ell < 2(n - k) & \text{if } m \leq n - k < \ell.
\end{cases}
\end{align*}
\]

It is straightforward to see the above inequalities are equivalent to the inequality \(|m - (n - k)| < |\ell - (n - k)|\), which is our conclusion. \(\square\)

**Remark 17.** In particular, the case \( k = n, \ m = 1, \ \ell = 0 \) is the first example we are aware of in the literature of a locally convex compact hypersurface flowing by mean curvature and converging smoothly to a submanifold in infinite time. And the case \( k = n - 1, \ m = 0, \ \ell = 1 \) gives an example of (backwards parabolic) inverse mean curvature flow existing forever and converging to a totally geodesic hypersurface. After reversing time to obtain parabolicity, this example of \(-1/\ell \) flow is properly divergent as \( t \to \infty \).

**References**


