This review sheet is only part of a complete review. A complete review consists
of study of all previously posted review materials along with homework, worksheets,
and past midterms. An answer key for this review sheet will be posted on the course
web page Wednesday, May 1, 2019 in late afternoon or evening.

Amplitudes, phase angles and all that

Give answers accurate to four decimal places.

Cartesian-to-polar conversion. For the points with the Cartesian coordinates

\((1, 2), (-2, 3), (-5, -8), (1, -3)\)

find polar coordinates \((r, \theta)\) with \(r \geq 0\) and \(0 \leq \theta < 2\pi\).

Phase-angle/amplitude form. Write each of the functions

\(\cos(3t) + 2\sin(3t), -2\cos(5t) + 3\sin(5t), -5\cos(2t) - 8\sin(2t),\cos(7t) - 3\sin(7t)\)

in the form \(C\cos(\omega t - \delta)\) where \(C, \omega, \delta \geq 0\) and \(\delta < 2\pi\).

Solution

Solution of cartesian to polar conversion. For the point with cartesian coordinates \((x, y) = (1, 2)\) which is in quadrant I the polar coordinates \((r, \theta)\) are

\((\sqrt{1^2 + 2^2}, \arctan(2/1)) = (2.2361, 1.1071)\).

For the point with cartesian coordinates \((x, y) = (-2, 3)\) which is in quadrant II
the polar coordinates \((r, \theta)\) are

\((\sqrt{(-2)^2 + 3^2}, \arctan(3/(-2)) + \pi) = (3.6056, 2.1588)\).

For the point with cartesian coordinates \((x, y) = (-5, -8)\) which is in quadrant III
the polar coordinates \((r, \theta)\) are

\((\sqrt{(-5)^2 + (-8)^2}, \arctan((-8)/(-5)) + \pi) = (9.4340, 4.1539)\).

Finally, for the point with cartesian coordinates \((x, y) = (1, -3)\) which is in quadrant IV the polar coordinates \((r, \theta)\) are

\((\sqrt{1^2 + (-3)^2}, \arctan((-3)/1) + 2\pi) = (3.1623, 5.0341)\).
Solution of phase-angle/amplitude form.

\[
\begin{align*}
\cos(3t) + 2\sin(3t) &= 2.2361 \cos(3t - 1.1071), \\
-2\cos(5t) + 3\sin(5t) &= 3.6056 \cos(5t - 2.1588), \\
-5\cos(2t) - 8\sin(2t) &= 9.4340 \cos(2t - 4.1539), \\
\cos(7t) - 3\sin(7t) &= 3.1623 \cos(7t - 5.0341).
\end{align*}
\]

LINEAR ALGEBRA

Reading rref. Consider the system of equations

\[
\begin{align*}
x_1 + 3x_2 + x_5 &= 7, \\
x_3 + 3x_5 &= 8, \\
x_4 - x_5 &= 17.
\end{align*}
\]

(i) Write out the augmented matrix for this system of equations. (ii) Solve the system of equations, presenting your answer in the same form as in Example 7 on pp. 139-140 of our textbook. (Look at the next-to-last displayed line in the example.)

Answer. (i) The augmented matrix is

\[
\begin{bmatrix}
1 & 3 & 0 & 0 & 1 & 7 \\
0 & 0 & 1 & 0 & 3 & 8 \\
0 & 0 & 1 & -1 & 17
\end{bmatrix},
\]

and you should see that this matrix is already in reduced row echelon form. (There is no need to push the rref button.) (ii) The nonpivot columns in the augmented matrix are columns 2 and 5 leading to two parameters in the description of the solution. The final answer is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} =
\begin{bmatrix}
7 - 3s - t \\
s \\
8 - 3t \\
17 + t \\
t
\end{bmatrix} =
\begin{bmatrix}
7 \\
0 \\
8 \\
17 \\
0
\end{bmatrix} + s
\begin{bmatrix}
-3 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} + t
\begin{bmatrix}
-1 \\
0 \\
+3 \\
+1 \\
1
\end{bmatrix},
\]

where the parameters \(s\) and \(t\) run through all real values.

Row-reduction practice. Find the reduced row echelon form of the matrix

\[
\begin{bmatrix}
1 & 1 & -2 & -3 \\
0 & 2 & 0 & 4 \\
1 & 3 & 3 & 16 \\
1 & -7 & 1 & -10
\end{bmatrix}
\]

by hand describing each row operation with notation of the form \(R_2^* = R_3 - R_2\) as in the textbook.

Solution.

\[
\begin{bmatrix}
1 & 1 & -2 & -3 \\
0 & 2 & 0 & 4 \\
1 & 3 & 3 & 16 \\
1 & -7 & 1 & -10
\end{bmatrix}
\xrightarrow{R_2^* = R_3 - R_2, \; R_4^* = R_4 - R_1}
\begin{bmatrix}
1 & 1 & -2 & -3 \\
0 & 2 & 0 & 4 \\
0 & 2 & 5 & 19 \\
0 & -8 & 3 & -7
\end{bmatrix}
\]
Linear combinations. Write each of the vectors
\[
\begin{bmatrix}
-3 \\
4 \\
16 \\
-10
\end{bmatrix}, \quad
\begin{bmatrix}
-2 \\
4 \\
16 \\
-10
\end{bmatrix}
\]
as a linear combination of the vectors
\[
\begin{bmatrix}
1 \\
0 \\
1 \\
-7
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
2 \\
3 \\
-7
\end{bmatrix}, \quad
\begin{bmatrix}
-2 \\
0 \\
3 \\
1
\end{bmatrix}, \quad
\text{if possible.}
\]

Answer. The system of equations
\[
x \cdot \begin{bmatrix}
1 \\
0 \\
1 \\
1
\end{bmatrix} + y \cdot \begin{bmatrix}
1 \\
2 \\
3 \\
-7
\end{bmatrix} + z \cdot \begin{bmatrix}
-2 \\
0 \\
3 \\
1
\end{bmatrix} = \begin{bmatrix}
-3 \\
4 \\
16 \\
-10
\end{bmatrix}
\]
has the unique solution \( x = 1, y = 2 \) and \( z = 3 \). The system of equations
\[
x \cdot \begin{bmatrix}
1 \\
0 \\
1 \\
1
\end{bmatrix} + y \cdot \begin{bmatrix}
1 \\
2 \\
3 \\
-7
\end{bmatrix} + z \cdot \begin{bmatrix}
-2 \\
0 \\
3 \\
1
\end{bmatrix} = \begin{bmatrix}
-2 \\
4 \\
16 \\
-10
\end{bmatrix}
\]
has no solution.

Practice with determinants. For what values of \( c \) does the system of equations
\[
\begin{bmatrix}
1 & 0 & 3 & 4 \\
0 & 3 & 9 & 0 \\
c & 2 & 2 & -7 \\
2 & 1 & 4 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
have infinitely many solutions? Do all your calculations by hand. If you make the right choice of methods the calculations are easy.
Answer. We need to calculate the determinant of the coefficient matrix and solve for values of $c$ making the determinant equal zero. Remember the “checkerboards”

\[
\begin{vmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & + \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
+ & - & + \\
- & + & - \\
+ & - & + \\
\end{vmatrix}
\]

We compute the determinant as follows. We follow what seems to us to be the easiest path. Of course other paths are possible.

\[
\begin{vmatrix}
1 & 0 & 3 & 4 \\
0 & 3 & 9 & 0 \\
c & 2 & 2 & -7 \\
2 & 1 & 4 & 0 \\
\end{vmatrix} = -4 \begin{vmatrix}
0 & 3 & 9 \\
2 & 2 & +7 \\
2 & 1 & 4 \\
\end{vmatrix}
\]

\[
= 4c \begin{vmatrix}
3 & 9 & -8 & 3 & 9 \\
1 & 4 & 2 & 2 & +7 & 1 & 4 & +14 & 0 & 3 \\
\end{vmatrix}
\]

\[
= 12c + 96 + 21 - 126 = 12c - 9
\]

Thus the determinant of the coefficient matrix of the given system of equations is $12c - 9$. This noted, the answer to this question is the unique value of $c$ satisfying $12c - 9 = 0$, namely $c = 3/4$.

**Linear independence.** Which sets of three of the vectors

\[
\begin{bmatrix}
30 \\
20 \\
11 \\
\end{bmatrix}, \begin{bmatrix}
20 \\
14 \\
8 \\
\end{bmatrix}, \begin{bmatrix}
11 \\
8 \\
5 \\
\end{bmatrix}, \begin{bmatrix}
50 \\
34 \\
19 \\
\end{bmatrix}
\]

are linearly independent? and why?

Answer. Call the vectors $v_1, v_2, v_3, v_4$, respectively. The sets

\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_1, v_3, v_4\}

are linearly independent. The set \{v_1, v_2, v_4\} is not. Each of the four cases is settled the same way. For example, to show \{v_1, v_2, v_4\} is not linearly independent, we have to show that the system of equations

\[
x \begin{bmatrix}
30 \\
20 \\
11 \\
\end{bmatrix} + y \begin{bmatrix}
20 \\
14 \\
8 \\
\end{bmatrix} + z \begin{bmatrix}
50 \\
34 \\
19 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

has more than one solution. We can see this because

\[
\begin{bmatrix}
30 & 20 & 50 & 0 \\
20 & 14 & 34 & 0 \\
11 & 8 & 19 & 0 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
30 & 20 & 50 \\
20 & 14 & 34 \\
11 & 8 & 19 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

In the other three cases the result of hitting the \texttt{rref} button is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix},
\]

leading to the conclusion of linear independence.
Cramer’s Rule. Consider the system of equations

\[
\begin{bmatrix}
1 & 0 & 3 & 4 \\
0 & 3 & 9 & 0 \\
7 & 2 & 2 & -7 \\
2 & 1 & 4 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
=
\begin{bmatrix}
1 \\
4 \\
9 \\
16 \\
\end{bmatrix}
\]

Use Cramer’s Rule to express \(x_2\) as a quotient of four-by-four determinants. You do not have to evaluate the determinants. Just get all the numbers in the right places. Take for granted that the determinant of the coefficient matrix is not zero.

\[
x_2 = \frac{\begin{vmatrix}
1 & 1 & 3 & 4 \\
0 & 4 & 9 & 0 \\
7 & 9 & 2 & -7 \\
2 & 16 & 4 & 0 \\
\end{vmatrix}}{\begin{vmatrix}
1 & 0 & 3 & 4 \\
0 & 3 & 9 & 0 \\
7 & 2 & 2 & -7 \\
2 & 1 & 4 & 0 \\
\end{vmatrix}}
\]

Diagonalization. Diagonalize the following matrices.

\[
\begin{bmatrix}
147 & -100 \\
210 & -143 \\
\end{bmatrix},
\begin{bmatrix}
74 & -52 \\
100 & -70 \\
\end{bmatrix},
\begin{bmatrix}
-5 & 12 & -9 \\
-6 & 22 & -18 \\
-6 & 24 & -20 \\
\end{bmatrix}
\]

For the three-by-three matrix we tell you that the eigenvalues are \(-2, -2,\) and \(1\). Yes, \(-2\) is a double root. But we assure you that there are enough eigenvectors to diagonalize. Note for Spring 2019: You may ignore the middle matrix which has complex eigenvalues. Diagonalization of matrices with complex eigenvalues is not covered on the final exam.)

\[
\begin{bmatrix}
147 & -100 \\
210 & -143 \\
\end{bmatrix}
\begin{bmatrix}
100 & 100 \\
140 & 150 \\
\end{bmatrix}
=\begin{bmatrix}
100 & 100 \\
140 & 150 \\
\end{bmatrix}
\begin{bmatrix}
7 & 0 \\
0 & -3 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
74 & -52 \\
100 & -70 \\
\end{bmatrix}
\begin{bmatrix}
52 & 52 \\
72 - 4i & 72 + 4i \\
\end{bmatrix}
=\begin{bmatrix}
52 & 52 \\
72 - 4i & 72 + 4i \\
\end{bmatrix}
\begin{bmatrix}
2 + 4i & 0 \\
0 & 2 - 4i \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-5 & 12 & -9 \\
-6 & 22 & -18 \\
-6 & 24 & -20 \\
\end{bmatrix}
\begin{bmatrix}
1 & 4 & -3 \\
2 & 1 & 0 \\
2 & 0 & 1 \\
\end{bmatrix}
=\begin{bmatrix}
1 & 4 & -3 \\
2 & 1 & 0 \\
2 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2 \\
\end{bmatrix}
\]

Solving linear equations. (This problem is pure plain vanilla.) Solve if possible the following system of equations:

\[
\begin{align*}
x_1 + 3x_2 + 2x_3 &= 0 \\
3x_1 + 11x_2 + 8x_4 &= -10 \\
-2x_1 - 10x_2 + 9x_3 - 13x_4 &= 25 \\
6x_2 - 21x_3 + 19x_4 &= -41
\end{align*}
\]

Start by writing out the augmented matrix of the system carefully.
Answer. The augmented matrix is
\[
\begin{bmatrix}
1 & 3 & 2 & 0 & 0 \\
3 & 11 & 0 & 8 & -10 \\
-2 & -10 & 9 & -13 & 25 \\
0 & 6 & -21 & 19 & -41
\end{bmatrix}
\]
and after hitting the rref button we get
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 5 \\
0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

Differential equations

Brine tank. A very large tank contains 47 lbs of salt dissolved in 900 gallons of water. Brine that contains 2 lb/gal of salt starts entering the tank at time \( t = 0 \) at the rate of 10 gal/min. The mixture leaves the tank at the lower rate of 7 gal/min. Find an expression for the amount of salt in the tank at time \( t \).

Answer.
\[
\frac{dy}{dt} = 2 \text{ lb/gal} \cdot 10 \text{ gal/min} - \frac{y}{900 + (10 - 7)t} \text{ gal} \times 7 \text{ gal/min}, \quad y(0) = 47.
\]
\[
y' + \frac{7}{3} \frac{y}{t + 300} = 20, \quad y(0) = 47.
\]
The integrating factor is \( \exp\left(\frac{7}{3} \int \frac{dt}{t+300}\right) = (t + 300)^{7/3} \).

\[
\frac{d}{dt} (t + 300)^{7/3} y = 20(t + 300)^{7/3} \Rightarrow (t + 300)^{7/3} y = 6(t + 300)^{10/3} + C
\]

\[
300^{7/3} \cdot 47 = 6 \cdot 300^{10/3} + C \Rightarrow C = 300^{7/3} \cdot 47 - 6 \cdot 300^{10/3}.
\]
\[
y = 6(t + 300) + C/(t + 300)^{7/3} \quad (\text{with } C \text{ as above})
\]

Logistic-type equation. Consider the autonomous differential equation
\[
y' = (2 + y)(8 - y).
\]

(i) Determine the equilibrium solutions and also determine which equilibrium solution is stable. Justify your answer by sketching the direction field for the differential equation. (ii) Solve the initial value problem
\[
\frac{dy}{dt} = (2 + y)(8 - y), \quad y(0) = 10.
\]

Answer. (i) \( y = -2 \) and \( y = 8 \) are the equilibrium solutions (solutions of the differential equation that are constant). The slopes of the direction field are negative for \( y > 8 \), positive for \( -2 < y < 8 \) and negative for \( y < -2 \). Thus \( y = 8 \) is stable and \( y = -2 \) is unstable. (A drawing of the direction field appears on the next page of the answer key.) (ii) \( y = \frac{48e^{10t} + 2}{6e^{10t} - 1} \)
$y'' = (2+y)(8-y)$

Arrow of slope 1.0
Mass-spring system. Suppose we have a spring mass system with an object having a mass of 2 kilograms. The system has a damping constant of 6 newtons/(meters/second). A force of 60 newtons will stretch the spring 3 meters beyond its natural length. There is an external force applied to the system of $8 \sin(3t)$ newtons. The object is started in motion 4 meters above its equilibrium position with an initial velocity of 3 meters/second in the downward direction. Set up the initial value problem which describes the motion of this object. Also say if the system is over-, critically- or underdamped. Then STOP. You need not solve the initial value problem. In this problem “up” is the positive direction. Let $y$ denote the height of the mass above the equilibrium position.

Answer. The IVP is as follows:

$$2y'' + 6y' + \frac{60}{3}y = 2y'' + 6y' + 20y = 8 \sin(3t), y(0) = 4, y'(0) = -3$$

We have $b^2 - 4ac = 6^2 - 4 \cdot 2 \cdot 20 = 36 - 160 < 0$ so the system is underdamped.

General solutions of some differential equations. Find the general solutions of the following differential equations:

1. $y'' + 2y' + 10y = 78 \cos(2t)$.
2. $y'' + 16y = 12 \cos(4t)$.
3. $t \frac{dy}{dt} = 3y + 10t^4 \sin(5t)$.

Answers.

1. $y = 9 \cos(2t) + 6 \sin(2t) + C_1 e^{-t} \cos(3t) + C_2 e^{-t} \sin(3t)$.
2. $y = \frac{3}{2} t \sin(4t) + C_1 \cos(4t) + C_2 \sin(4t)$.
3. $y = C t^3 - 2 t^3 \cos(5t)$

Newton’s Law of Heating and Cooling. A steel ball is heated to a temperature of 200°C and at time $t = 0$ is placed in water maintained at 20°C. At $t = 5$ minutes the temperature of the ball is 110°C. (i) Find the temperature $y$ of the ball at time $t$. Start from the differential equation. (ii) Also find the time at which the temperature of the steel ball equals 50°C, reporting your answer to 4 decimal places of accuracy.

Answer. (i) The differential equation is $dy/dt = k(20 - y)$. The temperature at time $t$ is

$$y(t) = 20 + 180 \left( \frac{1}{2} \right)^{t/5} = 20 + 180 e^{-0.138629 t}.$$ 

Either form of the answer is okay. You do not have to give both. (ii) The solution of the equation $y(t) = 50$ is $t = 12.9248$.

The J-method. Solve the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -74 & 52 \\ -100 & 70 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 11 \\ -7 \end{bmatrix}$$

by the J-method or the Laplace transform method. (In fact the eigenvalues of the coefficient matrix are complex so these are the only options we studied.)
Solution by *J*-method.

\[
0 = \begin{vmatrix} -74 - \lambda & 52 \\ -100 & 70 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda + 20 = (\lambda + 2)^2 + 4^2 \Rightarrow \lambda = -2 \pm 4i.
\]

\[
J = \frac{1}{4} \begin{bmatrix} -74 & 52 \\ -100 & 70 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -18 & 13 \\ -25 & 18 \end{bmatrix},
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -7 \end{bmatrix} e^{-2t} \cos(4t) + \begin{bmatrix} -289 \\ -401 \end{bmatrix} e^{-2t} \sin(4t).
\]

Solution by Laplace transforms+Cramer's Rule.

\[
\begin{bmatrix} -74 - s & 52 \\ -100 & 70 - s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 11 \\ -7 \end{bmatrix}.
\]

\[
X = \begin{vmatrix} -11 & 52 \\ 7 & 70 - s \end{vmatrix} = \frac{11s - 1134}{(s + 2)^2 + 4^2} = \frac{11(s + 2) - 1156}{(s + 2)^2 + 4^2}
\]

\[
Y = \begin{vmatrix} -74 - s & -11 \\ -100 & 7 \end{vmatrix} = \frac{-7s - 1618}{(s + 2)^2 + 4^2} = \frac{-7(s + 2) - 1604}{(s + 2)^2 + 4^2}
\]

(same final answer as above of course)

**A linear iterative system.** Solve the linear iterative system

\[
\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 32 & -10 \\ 105 & -33 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

**Solution.** For \( \begin{bmatrix} 32 & -10 \\ 105 & -33 \end{bmatrix} \) we have eigenvalue/eigenvector pairs \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \leftrightarrow 2 \) and \( \begin{bmatrix} 2 \\ 7 \end{bmatrix} \leftrightarrow -3 \). So the general solution of the problem is

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = c_1 2^n \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 (-3)^n \begin{bmatrix} 2 \\ 7 \end{bmatrix}.
\]

We then have

\[
c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}.
\]

The final answer is

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 9 \cdot 2^n \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 4 \cdot (-3)^n \begin{bmatrix} 2 \\ 7 \end{bmatrix}.
\]

After simplifying as much as possible the answer can be rewritten as

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 2^n \begin{bmatrix} 9 \\ 27 \end{bmatrix} + (-3)^n \begin{bmatrix} -8 \\ -28 \end{bmatrix}.
\]