REVIEW PROBLEMS FOR MIDTERM I
MATH 2373, SPRING 2019
UNIVERSITY OF MINNESOTA
ANSWER KEY

This list of problems is not guaranteed to be a complete review. For a complete review make sure that you know how to do all the homework assigned in the course up to and including that due on Tuesday, February 19, and all the worksheets up to and including all those of Week 4. Strive to solve all the problems on this review sheet by hand, except perhaps for the Xenia, Yolanda and Zephyr problem which involves too many decimal places. In general you are expected to solve problems on an exam by hand unless specific instructions to the contrary are supplied on the exam. On the first midterm exam we will permit use of the calculator only sparingly (just for scientific calculator functions like \( +, -, \times, \div, \sqrt{}, \log, \exp \)) but on later exams we will ease up and permit calculator use for row reduction and determinants in many cases. However, on all exams, for algebra and calculus work not involving matrices, we insist that you perform and show all the steps needed to justify your final answer by hand calculation.

An answer key for this review sheet will be posted on the course web page

www.math.umn.edu/~gwanders/Math2373

by 5pm on Friday, Feb. 15. There is a link to this URL on the Moodle page.

PROBLEM 1

Standing in line at the supermarket I see Xenia, Yolanda and Zephyr ahead of me in the express check-out lane. Xenia buys 2 bags of chips, 3 diet sodas, 1 bag of circus peanuts and spends $5.54. Yolanda buys 3 bags of chips, 1 diet soda, 3 bags of circus peanuts and spends $7.13. Zephyr buys 1 bag of chips, 1 diet soda, 1 bag of circus peanuts and spends $2.97. Find the price for a bag of chips, for a diet soda and for a bag of circus peanuts. You may use your calculator for the numerical work. Use the \( \vec{x} = A^{-1}\vec{b} \) method of solution. Identify \( A \), \( \vec{x} \) and \( \vec{b} \) clearly. Also show the numerical value of \( A^{-1} \).

Solution

We have to solve \( A\vec{x} = \vec{b} \) where

\[
A = \begin{bmatrix}
2 & 3 & 1 \\
3 & 1 & 3 \\
1 & 1 & 1
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

\( x \) price of bag of chips, price of can of diet soda, price of bag of circus peanuts

\[
\vec{b} = \begin{bmatrix}
Xenia’s bill \\
Yolanda’s bill \\
Zephyr’s bill
\end{bmatrix} = \begin{bmatrix}
5.54 \\
7.13 \\
2.97
\end{bmatrix},
\]

\( \vec{b} \) contains the three bill amounts.

\[
A^{-1} = \begin{bmatrix}
1 & 1 & -4 \\
0 & -1/2 & 3/2 \\
-1 & -1/2 & 7/2
\end{bmatrix}.
\]
The final answer is

\[
\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \bar{b} = \begin{bmatrix} 1 & 1 & -4 \\ 0 & -1/2 & 3/2 \\ -1 & -1/2 & 7/2 \end{bmatrix} \begin{bmatrix} 5.54 \\ 7.13 \\ 2.97 \end{bmatrix} = \begin{bmatrix} .79 \\ .89 \end{bmatrix}.
\]

**Problem 2**

Solve the system of equations

\[
\begin{align*}
-2w &- x + 5z = 1 \\
2w &+ y + z = 3
\end{align*}
\]

using the “standard method” taught in class, which means: write out the augmented matrix for the system, apply row operations to the augmented matrix to get to reduced row echelon form, and present the final answer in the format of Example 7 on pp. 139-140 (see the next-to-last displayed line in the example).

Do your work entirely by hand. Explain each row operation using notation like \(R_1^* = R_1 - 3R_3\), as in the textbook.

**Solution**

The augmented matrix of the system is

\[
\begin{bmatrix}
-2 & -1 & 0 & 5 & 1 \\
2 & 0 & 1 & 1 & 3
\end{bmatrix}
\]

Applying \(R_1^* = -R_1/2\), we get

\[
\begin{bmatrix}
1 & 1/2 & 0 & -5/2 & -1/2 \\
2 & 0 & 1 & 1 & 3
\end{bmatrix}
\]

Applying \(R_2^* = R_2 - 2R_1\), we get

\[
\begin{bmatrix}
1 & 1/2 & 0 & -5/2 & -1/2 \\
0 & -1 & 1 & 6 & 4
\end{bmatrix}
\]

Applying \(R_2^* = -R_2\), we get

\[
\begin{bmatrix}
1 & 1/2 & 0 & -5/2 & -1/2 \\
0 & 1 & 1 & -6 & -4
\end{bmatrix}
\]

which is in row echelon form. Finally, applying \(R_1^* = R_1 - R_2/2\), we get

\[
\begin{bmatrix}
1 & 0 & 1/2 & 1/2 & 3/2 \\
0 & 1 & -1 & -6 & -4
\end{bmatrix}
\]

which is in reduced row echelon form.

The last matrix above corresponds to the pair of equations

\[
\begin{align*}
w &+ y/2 + z/2 = 3/2 \\
x &- y - 6z = -4
\end{align*}
\]

The variables \(y\) and \(z\) are free since they appear in columns without pivots in the reduced row echelon matrix. Let

\[
y = s \quad \text{and} \quad z = t.
\]

Then, solving for the remaining variables \(w\) and \(x\) in terms of the parameters \(s\) and \(t\), we get

\[
w = -s/2 - t/2 + 3/2 \quad \text{and} \quad x = s + 6t - 4.
\]
The final answer written in the officially accepted manner is
\[
\begin{bmatrix}
  w \\
  x \\
  y \\
  z
\end{bmatrix} = s \begin{bmatrix}
  1 \\
  1 \\
  0
\end{bmatrix} + t \begin{bmatrix}
  -1/2 \\
  6 \\
  0
\end{bmatrix} + \begin{bmatrix}
  3/2 \\
  -4 \\
  0
\end{bmatrix},
\]
where \( s \) and \( t \) may take any real values.

**Problem 3**

The following matrix is not quite in row echelon form.
\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 1 & 0 & 1 & 1 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

(i) Reduce the matrix to row echelon form. (At most two more steps are required.) Clearly identify which matrix is supposed to be in row echelon form. (ii) Then reduce the matrix all the way to reduced row echelon form. The last matrix you write down should be exactly in reduced row echelon form. Furthermore, every step of row-reduction should be explained by using notation like \( R_2^* = R_2 - 3R_1 \), as in the textbook. (We just want to see row operations; no equations are to be solved.)

**Solution**

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 1 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 1 & 0 & 1 & 1 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \quad \text{(given matrix)}
\]

\[ R_4^* = R_4 - R_3, \quad R_4^* = -R_4 \]
\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 0 \\
  0 & 0 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 1 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad \text{(row echelon form)}
\]

\[ R_3^* = R_3 - R_4, \quad R_2^* = R_2 - R_4, \quad R_1^* = R_1 - R_4 \]
\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 0 & -1 \\
  0 & 0 & 1 & 1 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

\[ R_2^* = R_2 - R_3, \quad R_1^* = R_1 - R_3 \]
\[
\begin{bmatrix}
  1 & 1 & 1 & 0 & 0 & 0 & -1 \\
  0 & 0 & 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix} \quad \text{(reduced row echelon form)}
\]
\[ R_1^* = R_1 - R_2 \]
**Problem 4**

Expand the following determinant along the third row. You need not evaluate the resulting 3 by 3 determinants, just get all the numbers in the right places, and get all the signs right. Then in similar fashion expand along the second column.

\[
\begin{vmatrix}
2 & -1 & 6 & -7 \\
-3 & 4 & 1 & 0 \\
12 & -6 & 0 & 8 \\
5 & 11 & -4 & 10 \\
\end{vmatrix}
\]

**Solution**

column 2  
\[
\downarrow
\]
checkerboard:  
+ - + -  
- + - +  
+ - + - \leftarrow \text{row 3}  
- + - +  

expansion along row 3:

\[
\begin{vmatrix}
-1 & 6 & -7 \\
4 & 1 & 0 \\
11 & -4 & 10 \\
\end{vmatrix}
\]

\[
-12 & 6 & -7 \\
-3 & 1 & 0 \\
5 & -4 & 10 \\
\end{vmatrix}
\]

expansion along column 2:

\[
\begin{vmatrix}
-3 & 1 & 0 \\
12 & 0 & 8 \\
5 & -4 & 10 \\
\end{vmatrix}
\]

\[
-(-1) & 2 & 6 & -7 \\
12 & 0 & 8 & -6 \\
5 & -4 & 10 & 12 \\
\end{vmatrix}
\]

**Problem 5**

Solve the initial value problem

\[
\frac{dy}{dt} = 4y + t^5 \sin t \quad \text{and} \quad y(1) = 2.
\]

**Solution**

\[
\frac{dy}{dt} = \frac{4}{t} \quad \text{(We first put the equation in standard form.)}
\]

\[
\exp \left( - \int \frac{4}{t} \, dt \right) = \exp (-4 \log t) = t^{-4} \quad \text{(Here's the integrating factor.)}
\]

\[
\frac{dt}{dt} \quad t^{-4} \quad \text{sin} \quad t \quad \text{(t^5 sin t)}
\]

\[
2 = y(1) = - \cos(1) + C, \quad C = 2 + \cos(1),
\]

\[
t^{-4} = - \cos t + 2 + \cos(1), \quad y = -t^4 \cos t + (2 + \cos(1))t^4.
\]
Problem 6

Find the inverse of the following matrix:

\[
\begin{bmatrix}
4 & -1 \\
-17 & 5
\end{bmatrix}
\]

Express all the entries of the inverse matrix as fractions, not as decimals. Solve this problem entirely by hand. Show all your work.

Solution

\[
\begin{bmatrix}
4 & -1 \\
-17 & 5
\end{bmatrix}^{-1} = \begin{bmatrix}
5/3 & 1/3 \\
4 & -1 \\
-17/3 & 4/3
\end{bmatrix}.
\]

Problem 7

Find \( k \) such that the following system of homogeneous equations has infinitely many solutions:

\[
\begin{bmatrix}
2 & 3 & -5 \\
-3 & 4 & 7 \\
1 & k & -2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

Solution

It is the same thing to solve the equation

\[
\begin{vmatrix}
2 & 3 & -5 \\
-3 & 4 & 7 \\
1 & k & -2
\end{vmatrix} = 0
\]

because in general, for a square matrix \( A \), the system of equations \( A\vec{x} = 0 \) has infinitely many solutions if and only if \( \det A = 0 \). Using “basketweave” to evaluate the left side we get the equation

\[
2 \cdot 4 \cdot (-2) + 3 \cdot 7 \cdot 1 + (-5) \cdot (-3) \cdot k - (-5) \cdot 4 \cdot 1 - 3 \cdot (-3) \cdot (-2) - 2 \cdot 7 \cdot k = 0
\]

which simplifies to

\[
7 + k = 0.
\]

Thus the solution is \( k = -7 \).

Another way to solve the problem is to expand the determinant along the third row:

\[
(1) \begin{vmatrix}
3 & -5 \\
4 & 7 \\
-3 & 7
\end{vmatrix} - k \begin{vmatrix}
2 & -5 \\
-3 & 7 \\
2 & 3
\end{vmatrix} = 41 + k - 34 = 7 + k = 0
\]

and then solve for \( k \) as before.
Problem 8

Find \( a, b, x \) and \( y \).

\[
\begin{bmatrix}
-24 & 19 & -6 & 1 \\
-25 & 19 & -6 & 1 \\
a & 18 & -6 & 1 \\
b & 22 & -7 & 1
\end{bmatrix}
-1
=\begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
x & 1 & -3 & -1 \\
y & 1 & 1 & -6
\end{bmatrix}
\]

Solution. You have 32 equations to use of which you only need four.

\[
\begin{bmatrix}
-24 & 19 & -6 & 1 \\
-25 & 19 & -6 & 1 \\
a & 18 & -6 & 1 \\
b & 22 & -7 & 1
\end{bmatrix}
=\begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
x & 1 & -3 & -1 \\
y & 1 & 1 & -6
\end{bmatrix}
=\begin{bmatrix}
1 & 0 & 0 & 0 \\
-24 & 19 & -6 & 1 \\
-25 & 19 & -6 & 1 \\
a & 18 & -6 & 1 \\
b & 22 & -7 & 1
\end{bmatrix}
\]

\[
0 = \begin{bmatrix}
a & 18 & -6 & 1
\end{bmatrix}
= -a - 23 \Rightarrow a = -23
\]

\[
0 = \begin{bmatrix}
b & 22 & -7 & 1
\end{bmatrix}
= -b - 28 \Rightarrow b = -28.
\]

\[
0 = \begin{bmatrix}
x & 1 & -3 & -1
\end{bmatrix}
= x - 3 \Rightarrow x = 3.
\]

\[
1 = \begin{bmatrix}
y & 1 & 1 & -6
\end{bmatrix}
= y - 4 \Rightarrow y = 5.
\]

With so many equations available obviously this is not the only possible solution. But possibly it is the laziest. :)

Problem 9

(i) Find the general solution of the autonomous differential equation

\[
y' = (1 + y)(5 - y).
\]

(ii) Also find the solution which satisfies the initial condition \( y(0) = 7 \).

(iii) Find the equilibrium solutions. (iv) Determine the stability of each equilibrium solution. (v) Sketch the direction field for the differential equation well enough to justify your determination of stability.
Solution

First find the general solution. Start by separating variables

\[ \frac{dy}{(1 + y)(5 - y)} = dt \]

Partial fractions:

\[ \frac{1}{(1 + y)(5 - y)} = \frac{A}{y + 1} + \frac{B}{y - 5} \]

Multiply on both sides by \((y + 1)(y - 5)\) to get

\[-1 = A(y - 5) + B(y + 1)\]

Plug in \(y = -1\) to get \(A = 1/6\). Plug in \(y = 5\) to get \(B = -1/6\). We can rewrite the separated differential equation as

\[ \frac{1}{6} \frac{dy}{y + 1} - \frac{1}{6} \frac{dy}{y - 5} = dt \]

Clear denominators:

\[ \frac{dy}{y + 1} - \frac{dy}{y - 5} = 6 dt \]

Integrate on both sides and use the old math professor’s trick:

\[ \log(y + 1) - \log(y - 5) = 6t + \ln C \]

(\(*\) \[ \frac{y + 1}{y - 5} = Ce^{6t} \]

Solve for \(y\):

\[ y + 1 = yCe^{6t} - 5Ce^{6t}, \quad y(1 - Ce^{6t}) = -5Ce^{6t} - 1, \quad y = \frac{5Ce^{6t} + 1}{Ce^{6t} - 1} \]

Next we solve the indicated initial value problem. Plug \(t = 0\) in earlier line marked with \((*)\) to get

\[ \frac{7 + 1}{7 - 5} = 4 = C. \]

Thus

\[ y = \frac{20e^{6t} + 1}{4e^{6t} - 1} \]

is the solution satisfying \(y(0) = 7\). Equilibrium solutions are \(y = -1\) and \(y = 5\). (The numbers \(-1\) and \(5\) are just the roots of \((1 + y)(y - 5) = 0\).) Looking at Figure 1 (which is a dfield plot) we can see that \(y = 5\) is stable and \(y = -1\) is unstable. Using dfield here was overkill. Any legible sketch of the direction field which establishes where the slopes are positive and where the slopes are negative is good enough to determine stability.

Problem 10

‘‘frozen out’’ by the weather emergency of Week 2---ignore for now

Find the least squares line \(y = k + mx\) for the following data points.

\((1, 4), (3, 8), (4, 9), (6, 12)\).

Do so by first expressing the two equations for the unknowns \(k\) and \(m\) in matrix form. You can do the numerical work with your calculator.
Solution. The inconsistent system of equations we wish that we could solve exactly (but cannot) is the following:

\[
\begin{bmatrix}
1 & 1 \\
1 & 3 \\
1 & 4 \\
1 & 6
\end{bmatrix}
\begin{bmatrix}
k \\
m
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
8 \\
9 \\
12
\end{bmatrix}
\]

In order to do the best we can by the method of least squares, we use the trick to “left-multiply both sides by the transpose.”

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 4 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 4 & 6
\end{bmatrix}
\begin{bmatrix}
k \\
m
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
8 \\
9 \\
12
\end{bmatrix}
\]

After multiplying out using MATLAB or hand calculator, we get

\[
\begin{bmatrix}
4 & 14 & 14 \\
14 & 62 & 33
\end{bmatrix}
\begin{bmatrix}
k \\
m
\end{bmatrix}
= 
\begin{bmatrix}
33 \\
136
\end{bmatrix}.
\]

Finally,

\[
\begin{bmatrix}
k \\
m
\end{bmatrix}
= 
\begin{bmatrix}
4 & 14 & 14 \\
14 & 62 & 33
\end{bmatrix}^{-1}
\begin{bmatrix}
33 \\
136
\end{bmatrix}
= 
\begin{bmatrix}
2.7308 \\
1.5769
\end{bmatrix}.
\]

Thus the least squares line is

\[y = 2.7308 + 1.5769x.\]

Figure 2 is a plot of the data points and the least squares line that fits these data points. We included this figure only as an illustration—it is not an official part of the solution of this line-fitting problem.
Problem 11

The rate at which the amount of a radioactive substance changes is proportional to the amount present. At a certain time, a sample of a certain radioactive substance contains 40 mg. The same sample only contains 25 mg of the substance 50 years later. (i) How much of the substance will remain after 120 years? (ii) What is the half-life of the substance? Start with the differential equation.

Solution

Let $y$ denote the amount of the substance in milligrams and let $t$ denote the time in years measured from the time observations of the substance began. We are given that

$$\frac{dy}{dt} = ky, \quad y(0) = 40, \quad y(50) = 25.$$  

Here $k$ is a negative constant we need to determine. The general solution of the differential equation is $y = Ce^{kt}$. We have $C = 40$ because $y(0) = 40$. We have $40e^{50k} = 25$ because $y(50) = 25$. Thus

$$e^{50k} = \frac{25}{40} = \frac{5}{8} \quad \text{and hence} \quad y = 40\left(\frac{5}{8}\right)^{t/50}.$$
We then have

\[ y(120) = 40(5/8)^{120/50} = 12.947 \text{mg} \]

which answers part (i) in several different formats any of one of which is acceptable but with the last being preferred since easiest to read. To answer part (ii) of the question we have to calculate the half-life \( T \) of the substance. This is determined by either of the following two equations:

\[ 40(5/8)^{T/50} = 20 \quad \text{and} \quad 40e^{-0.009400T} = 20 \]

Solving the former we have

\[ T = 50 \frac{\ln(1/2)}{\ln(5/8)} = 73.738 \text{ years} \]

Solving the latter we get

\[ T = \frac{\ln(1/2)}{-0.0094} = 73.738 \text{ years}, \]

which (no surprise) is the same answer.

**ANOTHER SOLUTION BY WAY OF SOLVING FOR \( k \)**

Some students may prefer to solve for \( k \), so we present that route as well. From, say, the equation \( e^{50k} = 5/8 \) above we get

\[ k = \frac{\ln(5/8)}{50} = -0.009400 \]

and hence

\[ y = 40e^{-0.009400t}, \]

which is an equivalent expression for the amount of substance at time \( t \). If you do solve for \( k \) you have to keep at least four significant digits in order to achieve reasonable accuracy.

**Problem 12**

Initially an 800 gallon tank contains 400 gallons of water in which is dissolved 43 lbs of salt. Brine containing 4 lbs of salt per gallon enters the tank at the rate of 7 gallons per minute. The well mixed brine leaves the tank at the slower rate of 2 gallons per minute. (i) Find an expression for the number of pounds of salt in the tank at time \( t \) which is valid until the tank overflows. (ii) When does the tank overflow?

**Solution**

Let \( y = y(t) \) be the amount (lbs) of salt in the tank at time \( t \) (min). We have to solve the initial value problem

\[ \frac{dy}{dt} = 4 \frac{\text{lbs}}{\text{gal}} \cdot \frac{7 \text{ gal}}{\text{min}} - \frac{y}{400 + (7 - 2)t} \frac{\text{lbs}}{\text{gal}} \cdot \frac{2 \text{ gal}}{\text{min}} \quad y(0) = 43. \]

After “cleaning up” we have

\[ \frac{dy}{dt} + \frac{2y}{400 + 5t} = 28, \quad y(0) = 43. \]
We now use the integrating factor method.

\[
\exp\left(\int \frac{2 \, dt}{400 + 5t}\right) = \exp\left(\frac{2}{5} \int \frac{dt}{80 + t}\right) = \exp\left(\frac{2}{5} \log(t + 80)\right) = (t + 80)^{2/5}
\]

\[
\frac{d}{dt}(t + 80)^{2/5}y = 28(t + 80)^{2/5}
\]

\[
(t + 80)^{2/5}y = 28 \cdot \frac{5}{t} \cdot (t + 80)^{7/5} + C = 20(t + 80)^{7/5} + C
\]

\[
80^{2/5} \cdot 43 = 20 \cdot 80^{7/5} + C \Rightarrow C = 80^{2/5} \cdot 43 - 20 \cdot 80^{7/5} (= -8985)
\]

\[
y = 20(t + 80) + C/(t + 80)^{2/5} \quad \text{with } C \text{ as immediately above.}
\]

The tank overflows when 400 + 5t = 800, i.e., at time \( t = \frac{800-400}{5} = 80 \) minutes.

**Problem 13**

A steel ball with an initial temperature of 60°F is placed in an oven which is maintained at 420°F. After 4 minutes the temperature of the ball is 180°F. (i) Find an expression for the temperature of the ball at time \( t \). (ii) Find the surface temperature of the ball after 12 minutes. (iii) Find the time at which the ball will have a temperature of 350°F. Start with the differential equation.

**Solution**

Let \( y = y(t) \) be the temperature in degrees Fahrenheit of the ball at time \( t \) in minutes after being placed in the oven. We know that

\[
\frac{dy}{dt} = k(420 - y)
\]

by Newton’s law of heating, where \( k \) is a constant we have to determine using the information given in the problem. We are also given that

\[
y(0) = 60 \quad \text{and} \quad y(4) = 180.
\]

We solve using the integrating factor method. We start by rewriting the differential equation as

\[
\frac{dy}{dt} + ky = 420k.
\]

The integrating factor is \( \exp(\int k \, dt) = e^{kt} \). We then have

\[
\frac{d}{dt}e^{kt}y = 420ke^{kt} \Rightarrow e^{kt}y = 420e^{kt} + C \Rightarrow y = 420 + Ce^{-kt}.
\]

To get the constant \( C \) we use the initial value information:

\[
y(0) = 60 = 420 + C \Rightarrow C = -360.
\]

Thus we have

\[
y = 420 - 360e^{-kt}.
\]

To answer part (i) of the question, we are going to replace \( e^{-kt} \) by an expression in which the only variable is \( t \). One way to do that is to solve for \( k \) and another way is to use algebra to get such an expression. We will show both ways, and we will show below how to pursue both ways to answer parts (ii) and (iii) of the question as well. We have

\[
180 = y(4) = 420 - 360e^{-4k} \Rightarrow e^{-4k} = \frac{240}{360} = \frac{2}{3} \Rightarrow e^{-kt} = \left(\frac{2}{3}\right)^{t/4}.
\]
It follows that

\[ y = 420 - 360 \left( \frac{2}{3} \right)^{t/4}, \]

which answers part (i) of the question. We can also solve the equation \( e^{-4k} = \frac{2}{3} \) for \( k \), obtaining

\[ k = -\frac{1}{4} \ln \left( \frac{2}{3} \right) = 0.10137. \]

Then we get another (equivalent) answer for part (i), namely

\[ y = 420 - 360 e^{-0.10137t}. \]

We then have

\[ y(12) = 420 - 360 \left( \frac{2}{3} \right)^{12/4} = 420 - 360 \exp(-0.10137 \cdot 12) = 313.3. \]

which answers part (ii) of the question both ways. Finally, to find the time \( T \) asked for in part (iii) we have to solve the equation

\[ y(T) = 420 - 360 \left( \frac{2}{3} \right)^{T/4} = 420 - 360 \exp(-0.10137T) = 350. \]

This can be rewritten

\[ \frac{70}{360} = \left( \frac{2}{3} \right)^{T/4} = \exp(-0.10137T). \]

Solving by the usual precalculus methods, we get

\[ T = 4 \frac{\ln(7/36)}{\ln(2/3)} = \frac{\ln(7/36)}{-0.10137} = 16.155 \text{ minutes}, \]

which answers part (iii) of the question (in both ways).

**Remark.** Just so there is no misunderstanding: I solved the problem two different ways. When you are solving problems like this you only have to pick one of the ways and then stick with it.

**Problem 14**

Find the determinant

\[
\begin{vmatrix}
1 & 1 & 1 & -1 \\
1 & 0 & 0 & 1 \\
0 & 3 & 0 & 1 \\
-2 & 1 & -2 & 1
\end{vmatrix}
\]

by hand. Use whatever combination of methods seems the easiest. Explain each step of your calculation by a word or phrase.
Solution

\[
\begin{bmatrix}
1 & 1 & 1 & -1 \\
1 & 0 & 0 & 1 \\
0 & 3 & 0 & 1 \\
-2 & 1 & -2 & 1
\end{bmatrix}
\]

(use the row operation \( R_4^* = R_4 + 2R_1 \))

\[
\begin{bmatrix}
1 & 1 & 1 & -2 \\
1 & 0 & 0 & 1 \\
0 & 3 & 0 & 1 \\
0 & 3 & 0 & -1
\end{bmatrix}
\]

(expand along third column)

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 3 & 1 \\
0 & 3 & -1
\end{bmatrix}
\]

(expand along first column)

\[
\begin{bmatrix}
3 & 1 \\
3 & -1
\end{bmatrix}
\]

\(-3 - 3 = -6.\)

(This is not the only possible solution.)

Problem 15

Find the inverse of the following matrix.

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 1 \\
-2 & -1 & -3
\end{bmatrix}
\]

You must show all work. You must use the textbook notation for row operations to explain each step. Check at least three times that you wrote out the correct starting matrix for your calculation.

Solution

We use the “side-by-side” method.

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 0 & 1 \\
-2 & -1 & -3 & 0 & 0 & 1
\end{bmatrix}
\]

\( R_2^* = R_2 - 2R_1, R_3^* = R_3 + 2R_1 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & -2 & 1 & 0 \\
0 & 1 & -1 & 2 & 0 & 1
\end{bmatrix}
\]

\( R_2 \leftrightarrow R_3 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & 2 & 0 & 1 \\
0 & 0 & -1 & -2 & 1 & 0
\end{bmatrix}
\]

\( R_3^* = -R_3 \)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 & -1 & 0
\end{bmatrix}
\]

\( R_2^* = R_2 + R_3, R_1^* = R_1 - R_3 \)

\[
\begin{bmatrix}
1 & 1 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 4 & -1 & 1 \\
0 & 0 & 1 & 2 & -1 & 0
\end{bmatrix}
\]

\( R_1^* = R_1 - R_2 \)
Problem 16

Solve the system of equations
\[ 5x_4 - x_3 = 7, \quad 4x_3 - 2x_2 + x_4 = 9, \quad 2x_2 + x_4 - x_3 = 17, \quad x_1 + x_2 + x_3 + x_4 = 14 \]
for \( x_3 \) using Cramer’s Rule. You are expected first to rewrite the system of equations neatly, and then to express \( x_3 \) as a ratio of 4-by-4 determinants. You do not have to evaluate any determinants. The emphasis is on demonstrating that you know the pattern of Cramer’s Rule.

Solution

Note that the equations were written “sloppily.” You have to begin by rewriting the equations neatly, with each variable placed in its own column.

\[
\begin{align*}
-x_3 + 5x_4 &= 7 \\
-2x_2 + 4x_3 + x_4 &= 9 \\
+2x_2 - x_3 + x_4 &= 17 \\
+x_1 + x_2 + x_3 + x_4 &= 14
\end{align*}
\]

Having written things out neatly, I can read off the required ratio of determinants easily:

\[
x_3 = \frac{\begin{vmatrix} 0 & 0 & 7 & 5 \\ 0 & -2 & 9 & 1 \\ 0 & 2 & 17 & 1 \\ 1 & 1 & 14 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & -1 & 5 \\ 0 & -2 & 4 & 1 \\ 0 & 2 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}}
\]