This list of problems is not guaranteed to be a complete review. For a complete review make sure that you know how to do all the homework problems assigned in the course up to and including that due on Tuesday, Oct. 1, and all the worksheets up to and including all those of Week 4. Strive to solve all the problems on this review sheet by hand, except perhaps for the Xenia, Yolanda and Zephyr problem which involves too many decimal places. In general you are expected to solve problems on an exam by hand unless specific instructions to the contrary are supplied on the exam. On the first midterm exam we will permit use of the calculator only sparingly (just for scientific calculator functions like +, −, ×, ÷, √, ln, exp) but on later exams we will ease up and permit calculator use for row reduction and determinants in many cases. However, on all exams, for algebra and calculus work not involving matrices, we insist that you perform and show all the steps needed to justify your final answer by hand calculation.

An answer key for this review sheet will be posted on the course web page

www.math.umn.edu/~gwanders/Math2373

by 5pm on Friday, Sept. 27. There is a link to this URL on the Canvas page.

Remark This review sheet is meant to be both a study aid for the first midterm and later a helpful review for the final exam.

Problem 1

Standing in line at the supermarket I see Xenia, Yolanda and Zephyr ahead of me in the express check-out lane. Xenia buys 2 bags of chips, 3 diet sodas, 1 bag of circus peanuts and spends $5.54. Yolanda buys 3 bags of chips, 1 diet soda, 3 bags of circus peanuts and spends $7.13. Zephyr buys 1 bag of chips, 1 diet soda, 1 bag of circus peanuts and spends $2.97. Find the price for a bag of chips, for a diet soda and for a bag of circus peanuts. You may use your calculator for the numerical work. Use the “$\vec{x} = A^{-1}\vec{b}$” method of solution. Identify $A$, $\vec{x}$ and $\vec{b}$ clearly. Also show the numerical value of $A^{-1}$.

Solution. We have to solve $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \text{price of bag of chips} \\ \text{price of can of diet soda} \\ \text{price of bag of circus peanuts} \end{bmatrix},$$

$$\vec{b} = \begin{bmatrix} \text{Xenia’s bill} \\ \text{Yolanda’s bill} \\ \text{Zephyr’s bill} \end{bmatrix} = \begin{bmatrix} 5.54 \\ 7.13 \\ 2.97 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 1 & -4 \\ 0 & -1/2 & 3/2 \\ -1 & -1/2 & 7/2 \end{bmatrix}.$$
The final answer is
\[
\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\bar{y} = \begin{bmatrix} 1 & 1 & -4 \\ 0 & -1/2 & 3/2 \\ -1 & -1/2 & 7/2 \end{bmatrix} \begin{bmatrix} 5.54 \\ 2.97 \end{bmatrix} = \begin{bmatrix} .79 \\ .89 \end{bmatrix}.
\]

**Problem 2**

Solve the system of equations
\[
\begin{align*}
-2w - x + 5z &= 1 \\
2w + y + z &= 3
\end{align*}
\]

using the “standard method” taught in class, which means: write out the augmented matrix for the system, apply row operations to the augmented matrix to get to reduced row echelon form, and present the final answer in the format of Example 7 on pp. 139-140 (see the next-to-last displayed line in the example). Do your work entirely by hand. Explain each row operation using notation like \( R_2^* = R_2 - 3R_3 \), as in the textbook.

**Solution.** The augmented matrix of the system is
\[
\begin{bmatrix}
-2 & -1 & 0 & 5 & 1 \\
2 & 0 & 1 & 1 & 3
\end{bmatrix}.
\]

Applying \( R_1^* = -R_1/2 \), we get
\[
\begin{bmatrix}
1 & 1/2 & 0 & -5/2 & -1/2 \\
2 & 0 & 1 & 1 & 3
\end{bmatrix}.
\]

Applying \( R_2^* = R_2 - 2R_1 \), we get
\[
\begin{bmatrix}
1 & 1/2 & 0 & -5/2 & -1/2 \\
0 & -1 & 1 & 6 & 4
\end{bmatrix},
\]

Applying \( R_2^* = -R_2 \), we get
\[
\begin{bmatrix}
1 & 1/2 & 0 & -5/2 & -1/2 \\
0 & 1 & -1 & -6 & -4
\end{bmatrix},
\]

which is in row echelon form. Finally, applying \( R_1^* = R_1 - R_2/2 \), we get
\[
\begin{bmatrix}
1 & 0 & 1/2 & 1/2 & 3/2 \\
0 & 1 & -1 & -6 & -4
\end{bmatrix},
\]

which is in reduced row echelon form.

The last matrix above corresponds to the pair of equations
\[
\begin{align*}
w + y/2 + z/2 &= 3/2 \\
x - y - 6z &= -4
\end{align*}
\]

The variables \( y \) and \( z \) are free since they appear in columns without pivots in the reduced row echelon matrix. Let
\[
y = s \quad \text{and} \quad z = t.
\]

Then, solving for the remaining variables \( w \) and \( x \) in terms of the parameters \( s \) and \( t \), we get
\[
w = -s/2 - t/2 + 3/2 \quad \text{and} \quad x = s + 6t - 4.
\]
The final answer written in the officially accepted manner is
\[
\begin{bmatrix}
w \\
x \\
y \\
z \\
\end{bmatrix} = s \begin{bmatrix} -1/2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/2 \\ 6 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3/2 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]
where \( s \) and \( t \) may take any real values.

**Problem 3**

The following matrix is not quite in row echelon form. (At most two more steps are required.) Clearly identify which matrix is supposed to be in row echelon form. (ii) Then reduce the matrix all the way to reduced row echelon form. The last matrix you write down should be exactly in reduced row echelon form. Furthermore, every step of row-reduction should be explained by using notation like \( R_2^* = R_2 - 3R_4 \), as in the textbook. (We just want to see row operations; no equations are to be solved.) Do the work entirely by hand.

**Solution.**

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
(given matrix)

\[
\begin{align*}
R_4^* &= R_4 - R_3, & R_4^* &= -R_4 \\
R_3^* &= R_3 - R_4, & R_2^* &= R_2 - R_4, & R_1^* &= R_1 - R_4 \\
R_2^* &= R_2 - R_3, & R_1^* &= R_1 - R_3 \\
\end{align*}
\]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
(row echelon form)

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
(reduced row echelon form)
Problem 4

Expand the following determinant along the third row. You need not evaluate the resulting 3 by 3 determinants, just get all the numbers in the right places, and get all the signs right. Then in similar fashion expand along the second column.

\[
\begin{vmatrix}
2 & -1 & 6 & -7 \\
-3 & 4 & 1 & 0 \\
12 & -6 & 0 & 8 \\
5 & 11 & -4 & 10 \\
\end{vmatrix}
\]

column 2

↓

Solution. checkerboard:

+ − + −
− + − +
+ − + − ← row 3
− + − +

expansion along row 3:

\[
\begin{vmatrix}
-1 & 6 & -7 \\
4 & 1 & 0 \\
11 & -4 & 10 \\
\end{vmatrix}
- (-6)
\]
\[
\begin{vmatrix}
2 & 6 & -7 \\
-3 & 1 & 0 \\
5 & -4 & 10 \\
\end{vmatrix}
+ 0 - 8
\]
\[
\begin{vmatrix}
2 & -1 & 6 \\
-3 & 4 & 1 \\
5 & 11 & -4 \\
\end{vmatrix}
\]

expansion along column 2:

\[
\begin{vmatrix}
-1 & 6 & -7 \\
12 & 0 & 8 \\
5 & -4 & 10 \\
\end{vmatrix}
+ 4
\]
\[
\begin{vmatrix}
2 & 6 & -7 \\
12 & 0 & 8 \\
5 & -4 & 10 \\
\end{vmatrix}
- (-6)
\]
\[
\begin{vmatrix}
2 & 6 & -7 \\
-3 & 1 & 0 \\
5 & -4 & 10 \\
\end{vmatrix}
+ 11
\]
\[
\begin{vmatrix}
2 & -1 & 6 \\
-3 & 4 & 1 \\
12 & 0 & 8 \\
\end{vmatrix}
\]

Problem 5

Solve the following initial value problems:

\[
dy \over dt = 4y + t^5 \sin t, \quad y(1) = 2.
\]

\[
dy \over dx = x/y, \quad y(2) = 3
\]

Solution of first IVP. The equation is first order linear so we use integrating factors.

\[
dy \over dt - \frac{4}{t} y = t^4 \sin t \quad \text{(We first put the equation in standard form.)}
\]

\[
\exp \left( -\int \frac{4}{t} dt \right) = \exp (-4 \ln t) = t^{-4} \quad \text{(Here's the integrating factor.)}
\]

\[
\frac{d}{dt}(t^{-4}y) = \sin t, \quad t^{-4}y = -\cos t + C,
\]

\[
2 = y(1) = -\cos(1) + C, \quad C = 2 + \cos(1), \quad t^{-4}y = -\cos t + 2 + \cos(1)
\]

Final answer: \( y = -t^4 \cos t + (2 + \cos(1))t^4 \).
**Solution of second IVP.** The equation is variables separable.

\[ y \, dy = x \, dx \Rightarrow y^2/2 = x^2/2 + C \]

\[ y^2/2 = 2^2/2 + C \Rightarrow 3^2/2 = 2^2/2 + C \Rightarrow C = 5/2 \Rightarrow y^2/2 = x^2/2 + 5/2 \]

\[ \Rightarrow y = \pm \sqrt{x^2 + 5} \]

We take the plus sign so that the initial condition is indeed satisfied.

Final answer: \( y = \sqrt{x^2 + 5} \)

**Problem 6**

Find the inverse of the following matrix:

\[
\begin{bmatrix}
4 & -1 \\
-17 & 5 \\
\end{bmatrix}
\]

Express all the entries of the inverse matrix as fractions, not as decimals. Use the result to solve the system of equations

\[ 4x - y = 17, \quad -17x + 5y = 19. \]

Solve this problem entirely by hand. Show all your work.

**Solution.**

\[
\begin{bmatrix}
4 & -1 \\
-17 & 5 \\
\end{bmatrix}
\]

\[ = \begin{bmatrix}
5/3 & 1/3 \\
17/3 & 4/3 \\
\end{bmatrix}
\]

\[ x = \begin{bmatrix}
5/3 & 1/3 \\
17/3 & 4/3 \\
\end{bmatrix} \begin{bmatrix}
17 \\
19 \\
\end{bmatrix} = \begin{bmatrix}
104/3 \\
365/3 \\
\end{bmatrix}
\]

**Problem 7**

Find \( a, b, x \) and \( y \).

\[
\begin{bmatrix}
-24 & 19 & -6 & 1 \\
-25 & 19 & -6 & 1 \\
\end{bmatrix}
\]

\[ a \begin{bmatrix}
18 \\
22 \\
\end{bmatrix} \]

\[ b \begin{bmatrix}
-6 & 1 \\
-7 & 1 \\
\end{bmatrix}
\]

Solve the problem entirely by hand. There's no need to multiply two four-by-four matrices. Much less work than that is required.

**Solution.** You have 32 equations to use of which you only need four.

\[
\begin{bmatrix}
-24 & 19 & -6 & 1 \\
-25 & 19 & -6 & 1 \\
\end{bmatrix}
\]

\[ a \begin{bmatrix}
18 \\
22 \\
\end{bmatrix} \]

\[ b \begin{bmatrix}
-6 & 1 \\
-7 & 1 \\
\end{bmatrix}
\]

\[ = \begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
\end{bmatrix}
\]

\[ = \begin{bmatrix}
10 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[ x \begin{bmatrix}
1 & -3 & -1 \\
1 & 1 & -6 \\
\end{bmatrix} \begin{bmatrix}
-24 & 19 & -6 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
\end{bmatrix}
\]

\[ x \begin{bmatrix}
1 & -3 & -1 \\
1 & 1 & -6 \\
\end{bmatrix} \begin{bmatrix}
-24 & 19 & -6 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
\end{bmatrix}
\]

\[ x \begin{bmatrix}
1 & -3 & -1 \\
1 & 1 & -6 \\
\end{bmatrix} \begin{bmatrix}
-24 & 19 & -6 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
\end{bmatrix}
\]

\[ x \begin{bmatrix}
1 & -3 & -1 \\
1 & 1 & -6 \\
\end{bmatrix} \begin{bmatrix}
-24 & 19 & -6 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
\end{bmatrix}
\]

\[ x \begin{bmatrix}
1 & -3 & -1 \\
1 & 1 & -6 \\
\end{bmatrix} \begin{bmatrix}
-24 & 19 & -6 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
\end{bmatrix}
\]
0 = \begin{pmatrix} a & 18 & -6 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = -a - 23 \implies a = -23

0 = \begin{pmatrix} b & 22 & -7 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = -b - 28 \implies b = -28.

0 = \begin{pmatrix} x & 1 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = x - 3 \implies x = 3.

1 = \begin{pmatrix} y & 1 & 1 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = y - 4 \implies y = 5.

**Problem 8**

Solve the system of equations

\[
\begin{align*}
3X - 2Y + 9 & = sX \\
5X + 6Y - 7 & = sY
\end{align*}
\]

where \(s\) is a parameter. Use Cramer’s Rule. Simplify your final answer.

**Solution.** First “pretty print” the system of equations so that each variable is its own column:

\[
\begin{align*}
(s - 3)X + 2Y & = 9 \\
-5X + (s - 6)Y & = -7
\end{align*}
\]

Then we can plug into Cramer’s Rule:

\[
X = \frac{\begin{vmatrix} 9 & 2 \\ -7 & s - 6 \end{vmatrix}}{\begin{vmatrix} s - 3 & 2 \\ -5 & s - 6 \end{vmatrix}} = \frac{9s - 54 + 14}{s^2 - 9s + 18 + 10} = \frac{9s - 40}{s^2 - 9s + 28}
\]

\[
Y = \frac{\begin{vmatrix} s - 3 & 9 \\ -5 & -7 \end{vmatrix}}{\begin{vmatrix} s - 3 & 2 \\ -5 & s - 6 \end{vmatrix}} = \frac{-7s + 21 + 45}{s^2 - 9s + 28} = \frac{-7s + 66}{s^2 - 9s + 28}
\]
Problem 9

(i) Find the general solution of the autonomous differential equation

\[ y' = (1 + y)(5 - y). \]

(ii) Also find the solution which satisfies the initial condition \( y(0) = 7 \).

(iii) Find the equilibrium solutions. (iv) Determine the stability of each equilibrium solution. (v) Sketch the direction field for the differential equation well enough to justify your determination of stability.

Solution. First find the general solution. Start by separating variables:

\[
\frac{dy}{(1 + y)(5 - y)} = dt
\]

Do a partial fraction expansion:

\[
\frac{1}{(1 + y)(5 - y)} = \frac{A}{y + 1} + \frac{B}{y - 5}
\]

Multiply on both sides by \((y + 1)(y - 5)\):

\[-1 = A(y - 5) + B(y + 1)\]

Plug in \(y = -1\) to get \(A = 1/6\); plug in \(y = 5\) to get \(B = -1/6\). We can rewrite the separated differential equation as

\[
\frac{1}{6} \frac{dy}{y + 1} - \frac{1}{6} \frac{dy}{y - 5} = dt
\]

Clear denominators:

\[
\frac{dy}{y + 1} - \frac{dy}{y - 5} = 6 dt
\]

Integrate on both sides:

\[
\ln \left| \frac{y + 1}{y - 5} \right| = 6t + C_1
\]

\[(*) \quad \frac{y + 1}{y - 5} = Ce^{6t} \quad (C = \pm e^{C_1})\]

Solve for \(y\):

\[\frac{7 + 1}{7 - 5} = 4 = C.\]

Thus

\[y = \frac{20e^{6t} + 1}{4e^{6t} - 1}\]

is the solution satisfying \(y(0) = 7\). Equilibrium solutions are \(y = -1\) and \(y = 5\). (The numbers \(-1\) and \(5\) are just the roots of \((1 + y)(y - 5) = 0\).) Looking at Figure 1 (which is a dfIELD plot) we can see that \(y = 5\) is stable and \(y = -1\) is unstable. Using dfIELD here was overkill. Any legible sketch of the direction field which establishes where the slopes are positive and where the slopes are negative is good enough to determine stability.
Problem 10

Find the least squares line \( y = k + mx \) for the following data points.

\((1, 4), (3, 8), (4, 9), (6, 12)\).

Do so by first expressing the two equations for the unknowns \( k \) and \( m \) in matrix form. You can do the numerical work with your calculator.

**Solution.** The inconsistent system of equations we wish that we could solve exactly (but cannot) is the following:

\[
\begin{bmatrix}
1 & 1 \\
1 & 3 \\
1 & 4 \\
1 & 6 \\
\end{bmatrix}
\begin{bmatrix}
k \\
m \\
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
8 \\
9 \\
12 \\
\end{bmatrix}
\]

In order to do the best we can by the method of least squares, we use the trick to left-multiply both sides by the transpose of the coefficient matrix.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 4 & 6 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 3 \\
1 & 4 \\
1 & 6 \\
\end{bmatrix}
\begin{bmatrix}
k \\
m \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 4 & 6 \\
\end{bmatrix}
\begin{bmatrix}
4 \\
8 \\
9 \\
12 \\
\end{bmatrix}
\]
After multiplying out using MATLAB or a graphing calculator, we get

\[
\begin{bmatrix}
4 & 14 \\
14 & 62
\end{bmatrix}
\begin{bmatrix}
k \\
m
\end{bmatrix}
= 
\begin{bmatrix}
33 \\
136
\end{bmatrix}.
\]

Finally, using the \( A^{-1} \vec{x} = \vec{b} \) method we get

\[
\begin{bmatrix}
k \\
m
\end{bmatrix}
= 
\begin{bmatrix}
4 & 14 \\
14 & 62
\end{bmatrix}^{-1}
\begin{bmatrix}
33 \\
136
\end{bmatrix}
= 
\begin{bmatrix}
2.7308 \\
1.5769
\end{bmatrix}.
\]

(You could also use Cramer’s Rule.) Thus the least squares line is

\[ y = 2.7308 + 1.5769x. \]

Figure 2 is a plot of the data points and the least squares line that fits these data points. We included this figure only as an illustration—it is not an official part of the solution of this line-fitting problem.

**Problem 11**

The rate at which the amount of a radioactive substance changes is proportional to the amount present. At a certain time a sample of a certain radioactive substance contains 40 mg. The same sample only contains 25 mg of the substance 50 years later. (i) How much of the substance will remain after 120 years? (ii) What is the half-life of the substance? Start with the differential equation.

**Solution.** Let \( y \) denote the amount of the substance in milligrams and let \( t \) denote the time in years measured from the start of observations of the sample. We need to solve

\[
\frac{dy}{dt} = ky, \quad y(0) = 40, \quad y(50) = 25.
\]

Solving for \( k \) is part of the problem. The general solution of the differential equation is \( y = Ce^{kt} \). We have \( C = 40 \) because \( y(0) = 40 \). We have \( 40e^{50k} = 25 \) because \( y(50) = 25 \). We then have

\[
e^{50k} = \frac{25}{40} \Rightarrow k = \frac{\ln(25/40)}{50} = -0.0094001 \text{ reciprocal years}
\]

hence

\[
y = 40e^{-0.0094001t}
\]

is the amount of substance at time \( t \), hence

\[
y = 40e^{-0.0094001t} \bigg|_{t=120} = 40e^{-0.0094001\cdot120} = 12.947 \text{ mg}
\]

is the answer to part (i). The half-life is the length of time it takes for the amount of substance to reduce by \( 1/2 \), i.e., the solution \( T \) of the equation \( \frac{1}{2} = e^{kT} \). Then

\[
T = \frac{\ln(1/2)}{k} = \frac{\ln(1/2)}{-0.0094001} = 73.738 \text{ years}
\]

is the answer to part (ii).

**Remark** An exact form of the solution is \( y = 40(25/40)^{\,t/50} \). In this form it is easy to check in your head that \( y(0) = 40 \) and \( y(50) = 25 \).
Initially an 800 gallon tank contains 400 gallons of water in which is dissolved 43 lbs of salt. Brine containing 4 lbs of salt per gallon enters the tank at the rate of 7 gallons per minute. The well mixed brine leaves the tank at the slower rate of 2 gallons per minute. (i) Find an expression for the number of pounds of salt in
the tank at time $t$ which is valid until the tank overflows. (ii) When does the tank overflow?

**Solution.** Let $y = y(t)$ be the amount (lbs) of salt in the tank at time $t$ (min). We have to solve the initial value problem

$$\frac{dy}{dt} = 4 \frac{\text{lbs}}{\text{gal}} \cdot 7 \frac{\text{gal}}{\text{min}} - \frac{y}{400 + (7 - 2)t} \frac{\text{lbs}}{\text{gal}} \cdot 2 \frac{\text{gal}}{\text{min}}, \quad y(0) = 43.$$ 

After simplifying and rewriting the differential equation in the standard form required by the integrating factor method we get

$$\frac{dy}{dt} + \frac{2y}{400 + 5t} = 28, \quad y(0) = 43.$$ 

Next we calculate the integrating factor:

$$\exp\left(\int \frac{2}{400 + 5t} dt\right) = \exp\left(\frac{2}{5} \ln(t + 80)\right) = (t + 80)^{2/5}.$$ 

Then we calculate in the usual way to get the answer to part (i):

$$\frac{d}{dt} \left(\frac{(t + 80)^{2/5} y}{t}\right) = 28(t + 80)^{2/5},$$

$$y = 420 + 43 = 20 \cdot 80^{7/5} + C = 80^{2/5} \cdot 43 - 20 \cdot 80^{7/5} = -8985$$

The tank overflows when $400 + 5t = 800$. Thus the answer to question (ii) is $t = \frac{800 - 400}{5} = 80$ minutes.

**Problem 13**

A steel ball with an initial temperature of 60°F is placed in an oven which is maintained at 420°F. After 4 minutes the temperature of the ball is 180°F. (i) Find an expression for the temperature of the ball at time $t$. (ii) Find the temperature of the ball after 12 minutes. (iii) Find the time at which the ball will have a temperature of 350°F. Start with the differential equation.

**Solution.** Let $y = y(t)$ be the temperature in degrees Fahrenheit of the ball at time $t$ in minutes after being placed in the oven. Newton’s law along with the numerical data supplied yield the problem

$$\frac{dy}{dt} = k(420 - y), \quad y(0) = 60, \quad y(4) = 180$$

for us to solve. Finding $k$ is part of the problem. To solve we calculate as follows:

$$\frac{dy}{dt} + ky = 420k,$$

$$\exp\left(\int k dt\right) = e^{kt},$$

$$\frac{d}{dt} \left(e^{kt} y\right) = 420ke^{kt},$$

$$e^{kt} y = 420e^{kt} + C,$$

$$y = 420 + Ce^{-kt}.$$
\[ 60 = y(0) = 420 + C \Rightarrow C = -360. \]

\[ y = 420 - 360e^{-kt} \]

\[ 180 = y(4) = 420 - 360e^{-4k} \Rightarrow e^{-4k} = \frac{240}{360} \Rightarrow e^{-k} = \left( \frac{240}{360} \right)^{1/4} \Rightarrow e^{-kt} = \left( \frac{240}{360} \right)^{t/4} \]

Then

\[ y = 420 - 360 \left( \frac{240}{360} \right)^{t/4} \]

is the answer to part (i). An equivalent acceptable form of the answer is

\[ y = 420 - 360e^{-kt} \text{ where } k = \frac{1}{4} \ln \left( \frac{360}{240} \right) = 0.10137. \]

In turn

\[ y(12) = 420 - 360 \left( \frac{240}{360} \right)^{12/4} = 313.34^o \]

is the answer to part (ii). To write down the answer for part (iii) we have to solve the equation

\[ 420 - 360 \left( \frac{240}{360} \right)^{T/4} = 350 \]

for \( T \) where \( k \) is as above. Then

\[ T = 4 \frac{\ln(70/360)}{\ln(240/360)} = 16.1554 \text{ min} \]

is the answer to part (iii).

**Problem 14**

Find the determinant

\[
\begin{vmatrix}
7 & 7 & 7 & -7 \\
1 & 0 & 0 & 1 \\
0 & 3 & 0 & 1 \\
-2 & 1 & -2 & 1 \\
\end{vmatrix}
\]

by hand. Use whatever combination of methods seems the easiest. Explain each step of your calculation by a word or phrase.
Solution.

\[
\begin{bmatrix}
  7 & 7 & 7 & -7 \\
  1 & 0 & 0 & 1 \\
  0 & 3 & 0 & 1 \\
-2 & 1 & -2 & 1
\end{bmatrix}
\] (use the row operation \( R_1^* = R_1/7 \))

\[
= 7 \begin{bmatrix}
  1 & 1 & 1 & -1 \\
  1 & 0 & 0 & 1 \\
  0 & 3 & 0 & 1 \\
-2 & 1 & -2 & 1
\end{bmatrix}
\] (use the row operation \( R_4^* = R_4 + 2R_1 \))

\[
= 7 \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  0 & 3 & 0 & 1 \\
  0 & 3 & 0 & -1
\end{bmatrix}
\] (expand along third column)

\[
= 7 \begin{bmatrix}
  3 & 1 \\
  1 & 0 \\
  0 & 1
\end{bmatrix} = 7(-3 - 3) = -42.
\]

**Problem 15**

Find the inverse of the following matrix.

\[
\begin{bmatrix}
  1 & 1 & 1 \\
  2 & 2 & 1 \\
-2 & -1 & -3
\end{bmatrix}
\]

You must show all work. You must use the textbook notation for row operations to explain each step.

**Solution.** We use the row-reduction method.

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 0 & 0 \\
  2 & 2 & 1 & 0 & 1 & 0 \\
-2 & -1 & -3 & 0 & 0 & 1
\end{bmatrix}, R_2^* = R_2 - 2R_1, R_3^* = R_3 + 2R_1
\]

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 0 & 0 \\
  0 & 0 & -1 & -2 & 1 & 0 \\
  0 & 1 & -1 & 2 & 0 & 1
\end{bmatrix}, R_2 \leftrightarrow R_3
\]

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 0 & 0 \\
  0 & 1 & -1 & 2 & 0 & 1 \\
  0 & 0 & -1 & -2 & 1 & 0
\end{bmatrix}, R_3^* = -R_3
\]

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 & 0 & 0 \\
  0 & 1 & -1 & 2 & 0 & 1 \\
  0 & 0 & 1 & 2 & -1 & 0
\end{bmatrix}, R_2^* = R_2 + R_3, R_1^* = R_1 - R_3
\]

\[
\begin{bmatrix}
  1 & 1 & 0 & -1 & 1 & 0 \\
  0 & 1 & 0 & 4 & -1 & 1 \\
  0 & 0 & 1 & 2 & -1 & 0
\end{bmatrix}, R_1^* = R_1 - R_2
\]
Solve the system of equations

\[5x_4 - x_3 = 7, \quad 4x_3 - 2x_2 + x_4 = 9, \quad 2x_2 + x_4 - x_3 = 17, \quad x_1 + x_2 + x_3 + x_4 = 14\]

for \(x_3\) using Cramer’s Rule. You are expected first to rewrite the system of equations neatly, and then to express \(x_3\) as a ratio of 4-by-4 determinants. You do not have to evaluate any determinants. The emphasis is on demonstrating that you know the pattern of Cramer’s Rule.

**Solution.** Note that the equations were written “sloppily.” You have to begin by rewriting the equations neatly, with each variable placed in its own column.

\[
\begin{align*}
-x_3 + 5x_4 &= 7 \\
-2x_2 + 4x_3 + x_4 &= 9 \\
+2x_2 - x_3 + x_4 &= 17 \\
+x_1 + x_2 + x_3 + x_4 &= 14
\end{align*}
\]

Having written things out neatly, I can read off the required ratio of determinants easily:

\[
x_3 = \frac{\begin{vmatrix}
0 & 0 & 7 & 5 \\
0 & -2 & 9 & 1 \\
0 & 2 & 17 & 1 \\
1 & 1 & 14 & 1
\end{vmatrix}}{\begin{vmatrix}
0 & 0 & -1 & 5 \\
0 & -2 & 4 & 1 \\
0 & 2 & -1 & 1 \\
1 & 1 & 1 & 1
\end{vmatrix}}.
\]

Initially \$10000 is deposited in a savings account paying interest of 3%/yr compounded continuously. Money is then steadily withdrawn from the account at a rate of \$450/yr. (i) Write out and solve an IVP for the amount of money in the account at time \(t\) after the initial deposit. (ii) Does the money run out? If so when?

**Solution.** Let \(y = y(t)\) be the number of dollars ($) in the account \(t\) years after the initial deposit. The IVP is

\[
\begin{align*}
\frac{dy}{dt} &= 0.03y - 450, \quad y(0) = 10000, \\
y' - 0.03y &= -450 \\
d\left(e^{-0.03t}y\right)/dt &= -450e^{-0.03t} \\
e^{-0.03t}y &= 15000e^{-0.03t} + C \\
y &= 15000 + Ce^{0.03t}.
\end{align*}
\]

\(10000 = y(0) = 15000 + C \Rightarrow C = -5000.\)

The answer to part (i) is \(y = 15000 - 5000e^{0.03t}.\) We have

\[
y(T) = 15000 - 5000e^{0.03T} = 0 \Rightarrow T = \frac{\ln(15000/5000)}{0.03} = 36.62040963 \text{ yrs}.
\]

The answer to part (ii) is that the money runs out in 36.620 years.