REVIEW PROBLEMS FOR MIDTERM I
MATH 2373, SPRING 2020
UNIVERSITY OF MINNESOTA

This list of problems is not guaranteed to be a complete review. For a complete review make sure that you know how to do all the homework problems assigned in the course up to and including those due on Tuesday, February 18, and all the worksheets up to and including all those of Week 4. Strive to solve all the problems on this review sheet by hand, except perhaps for the Xenia, Yolanda and Zephyr problem which involves too many decimal places. An answer key for this review sheet will be posted on the course web page

www.math.umn.edu/~gwanders/Math2373

by 5pm on Thursday, Feb. 13. There is a link to this URL on the Canvas page.

[Ground rules for calculator use on the first midterm:] We do permit graphing calculators (but no laptops or devices with wireless capabilities) and we do permit use of the calculator for scientific calculator functions like

\[ +, -, \times, \div, \sqrt{}, \ln, \exp \]

and so on. But on the first midterm you are expected to perform all matrix work (row reduction, matrix multiplication, inverses, determinants, etc.) by hand with all work clearly shown because otherwise your work would not be considered justified and would not receive credit. Also we may ask you to use notation like

\[ R_2^3 = R_2 - 3R_1 \]

as in the textbook to explain each row operation you perform.

Remark: This review sheet is meant to be both a study aid for the first midterm and later a helpful review for the final exam.

PROBLEM 1

Standing in line at the supermarket I see Xenia, Yolanda and Zephyr ahead of me in the express check-out lane. Xenia buys 2 bags of chips, 3 diet sodas, 1 bag of circus peanuts and spends $5.54. Yolanda buys 3 bags of chips, 1 diet soda, 3 bags of circus peanuts and spends $7.13. Zephyr buys 1 bag of chips, 1 diet soda, 1 bag of circus peanuts and spends $2.97. Find the price for a bag of chips, for a diet soda and for a bag of circus peanuts. You may use your calculator for the numerical work. Use the \[ \vec{x} = A^{-1}\vec{b} \] method of solution. Identify \( A \), \( \vec{x} \) and \( \vec{b} \) clearly. Also show the numerical value of \( A^{-1} \).

PROBLEM 2

Solve the system of equations

\[
\begin{align*}
    x_1 + 2x_2 + 3x_4 + x_5 &= 4 \\
    2x_1 + 4x_2 + 6x_4 + 3x_5 &= 9 \\
    4x_1 + 8x_2 + x_3 + 20x_4 + 8x_5 &= 22
\end{align*}
\]

using the “standard method” taught in class, which means: write out the augmented matrix for the system, apply row operations to the augmented matrix to get first to row echelon form, then to reduced row echelon form, and present the final answer in the format of Example 7 on pp. 139-140 (see the next-to-last displayed line in the example). Also say which matrix at roughly the halfway point is in row echelon form. Do your work entirely by hand. Explain each row operation using notation like $R_2^* = R_2 - 3R_3$, etc., as in the textbook.

**Problem 3**

The following matrix is not quite in row echelon form.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(i) Reduce the matrix to row echelon form. (At most two more steps are required.) Clearly identify which matrix is supposed to be in row echelon form. (ii) Then reduce the matrix all the way to reduced row echelon form. The last matrix you write down should be exactly in reduced row echelon form. Furthermore, every step of row-reduction should be explained by using notation like $R_2^* = R_2 - 3R_1$, as in the textbook. (We just want to see row operations; no equations are to be solved.) Do the work entirely by hand.

**Problem 4**

Expand the following determinant along the third row. You need not evaluate the resulting 3 by 3 determinants, just get all the numbers in the right places, and get all the signs right. Then in similar fashion expand along the second column.

\[
\begin{vmatrix}
2 & -1 & 6 & -7 \\
-3 & 4 & 1 & 0 \\
12 & -6 & 0 & 8 \\
5 & 11 & -4 & 10 \\
\end{vmatrix}
\]

**Problem 5**

Solve the following initial value problems:

\[
t \frac{dy}{dt} = 4y + t^5 \sin t, \quad y(1) = 2.
\]

\[
\frac{dy}{dx} = x/y, \quad y(2) = 3
\]

**Problem 6**

Find the inverse of the matrix \[
\begin{bmatrix}
4 & -1 \\
-17 & 5 \\
\end{bmatrix}
\]. Express all the entries of the inverse matrix as fractions, not as decimals. Use the result to solve the system of equations

\[
4x - y = 17,
\]

\[
-17x + 5y = 19.
\]

Solve this problem entirely by hand. Show all your work.
Problem 7

Find \( a, b, x, \) and \( y. \)

\[
\begin{bmatrix}
-24 & 19 & -6 & 1 \\
-25 & 19 & -6 & 1 \\
a & 18 & -6 & 1 \\
b & 22 & -7 & 1 \\
\end{bmatrix}
\begin{bmatrix}
-1 \\
2 \\
x \\
y \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & -1 & 0 & 0 \\
2 & -1 & -1 & 0 \\
x & 1 & -3 & -1 \\
y & 1 & 1 & -6 \\
\end{bmatrix}
\]

Solve the problem entirely by hand. There’s no need to multiply two four-by-four matrices. Much less work than that is required.

Problem 8

Solve the system of equations

\[
sX = 3X - 2Y + 9 \\
sY = 5X + 6Y - 7
\]

where \( s \) is a parameter. Use Cramer’s Rule. Simplify your final answer.

Problem 9

(i) Find the general solution of the autonomous differential equation

\[ y' = (1 + y)(5 - y). \]

(ii) Also find the solution which satisfies the initial condition \( y(0) = 7. \)

(iii) Find the equilibrium solutions. (iv) Determine the stability of each equilibrium solution. (v) Sketch the direction field for the differential equation well enough to justify your determination of stability.

Problem 10

Find the least squares line \( y = k + mx \) for the following data points.

\[(1, 4), (3, 8), (4, 9), (6, 12).\]

Do so by first expressing the two equations for the unknowns \( k \) and \( m \) in matrix form. You can do the numerical work with your calculator.

Problem 11

The rate at which the amount of a radioactive substance changes is proportional to the amount present. At a certain time a sample of a certain radioactive substance contains 40 mg. The same sample only contains 25 mg of the substance 50 years later. (i) How much of the substance will remain after 120 years? (ii) What is the half-life of the substance? Start with the differential equation.

Problem 12

Initially an 800 gallon tank contains 400 gallons of water in which is dissolved 43 lbs of salt. Brine containing 4 lbs of salt per gallon enters the tank at the rate of 7 gallons per minute. The well mixed brine leaves the tank at the slower rate of 2 gallons per minute. (i) Find an expression for the number of pounds of salt in the tank at time \( t \) which is valid until the tank overflows. (ii) When does the tank overflow?
Problem 13

A steel ball with an initial temperature of 60°F is placed in an oven which is maintained at 420°F. After 4 minutes the temperature of the ball is 180°F. (i) Find an expression for the temperature of the ball at time $t$. (ii) Find the temperature of the ball after 12 minutes. (iii) Find the time at which the ball will have a temperature of 350°F. Start with the differential equation.

Problem 14

Find the determinant

$$
\begin{vmatrix}
7 & 7 & 7 & -7 \\
1 & 0 & 0 & 1 \\
0 & 3 & 0 & 1 \\
-2 & 1 & -2 & 1 \\
\end{vmatrix}
$$

by hand. Use whatever combination of methods seems the easiest. Explain each step of your calculation by a word or phrase.

Problem 15

Find the inverse of the following matrix.

$$
\begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 1 \\
-2 & -1 & -3 \\
\end{bmatrix}
$$

You must show all work. You must use the textbook notation for row operations to explain each step.

Problem 16

Solve the system of equations

$$
5x_4 - x_3 = 7, \quad 4x_3 - 2x_2 + x_4 = 9, \quad 2x_2 + x_4 - x_3 = 17, \quad x_1 + x_2 + x_3 + x_4 = 14
$$

for $x_3$ using Cramer’s Rule. You are expected first to rewrite the system of equations neatly, and then to express $x_3$ as a ratio of 4-by-4 determinants. You do not have to evaluate any determinants. The emphasis is on demonstrating that you know the pattern of Cramer’s Rule.

Problem 17

Initially $10000$ is deposited in a savings account paying interest of 3%/yr compounded continuously. Money is then steadily withdrawn from the account at a rate of $450/yr$. (i) Write out and solve an IVP for the amount of money in the account at time $t$ after the initial deposit. (ii) Does the money run out? If so when?