This list of problems is not guaranteed to be a complete review. For completeness you must make sure you know how to do all of the homework assigned in the course up to and including that due on October 30, and all the worksheet problems up to and including week 8 (with exclusions as noted below). The emphasis of MT2 is on material from weeks 5–8 but of course you must keep basic skills from weeks 1–4 well honed.

**Exclusions from MT2**: “Electrical engineering” as on worksheet 9984 and two-by-two matrices with complex eigenvalues will NOT be covered on MT2. (But both topics will be in the scope of the final exam, and complex eigenvalues will be needed for solving problems on MT3.)

To study under typical test conditions, here are the ground rules. You may use matrix addition and multiplication on your calculator along with the equivalents of the MATLAB commands `det`, `rref` and `inv`, but you must indicate clearly where and how you used your calculator—otherwise your answers would be considered unjustified and get no credit. Higher-level calculator functions, e.g., those for finding eigenvalues and eigenvectors, are not allowed to justify your work. As usual all algebra and calculus work not involving matrices must be performed by hand and shown clearly.

There will not be any formula sheets with MT2.

**Problem 1**

Solve the initial value problem $y'' + 49y = 0$, $y(0) = 2$, $y'(0) = -10$, expressing the answer in the form $A \cos(\omega t - \delta)$ where $A$, $\omega$ and $\delta$ are positive numbers and furthermore $\delta < 2\pi$. Give numerical approximations for $A$, $\omega$ and $\delta$ accurate to four decimal places.

**Answer.**

$y = C_1 \cos(7t) + C_2 \sin(7t)$ (general solution of harmonic oscillator)

$y(0) = C_1 = 2$

$y'(0) = 7C_2 = -10 \Rightarrow C_2 = -10/7$

$y = 2 \cos(7t) - \frac{10}{7} \sin(7t)$. Note: $(2, -\frac{10}{7})$ is a point in quadrant IV.

$A = \sqrt{2^2 + (-\frac{10}{7})^2} = 2.4578$ and $\delta = \arctan(-\frac{10/7}{2}) + 2\pi = 5.6629$.

$y = 2.4578 \cos(7t - 5.6629)$

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>$\frac{1}{2}\pi$</td>
</tr>
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</table>

← handy “quadrant correction table”

**Date:** October 24, 2018.
**Problem 2**

A mass of 5 kg is suspended from a high ceiling by a spring. When hanging motionless, the mass stretches the spring 2 meters beyond its natural length. Let $y$ denote the height of the mass above the equilibrium point. Initially the mass is pulled down $1/2$ meter below the equilibrium point and released with a velocity of 1 m/sec upward. The mass encounters air resistance of $1.5 \text{ nt/(m/sec)}$. A force of $7 \cos(2t)$ nt is applied to the weight. Write out the initial value problem determining the motion of the mass. Is this system underdamped, overdamped or critically damped? Use $10 \text{ m/sec}^2$ as the acceleration of gravity. You do not have to solve the initial value problem.

**Answer.** The template is $my'' + by' + ky = F(t), y(0) = y_0, y'(0) = v_0$. We are given $m = 5 \text{ kg}$, $b = 1.5 \text{ nt/(m/sec)}$, $F(t) = 7 \cos(2t)$ nt, $y_0 = -0.5 \text{ m}$ and $v_0 = 1 \text{ m/sec}$. The weight of the mass is $(5 \text{ kg}) \cdot (10 \text{ m/sec}^2) = 50 \text{ nt}$, so the value of the spring constant is $k = (50 \text{ nt})/(2 \text{ m}) = 25 \text{ nt/m}$. Thus the initial value problem is

$$5y'' + 1.5y' + 25y = 7 \cos(2t), y(0) = -0.5, y'(0) = 1.$$  

Because $b^2 - 4km = 1.5^2 - 4 \cdot 5 \cdot 25 = -497.75 < 0$, the system is **underdamped**.

**Problem 3**

Solve the following IVP’s:

1. $y'' + 5y' + 6y = 0, y(0) = -2, y'(0) = 11$
2. $y'' + 6y' + 9y = 0, y(0) = 4, y'(0) = -23$
3. $y'' + 4y' + 29y = 0, y(0) = -2, y'(0) = 39$

**Answer.**

**First equation.**

$r^2 + 5r + 6 = (r + 2)(r + 3) = 0 \Rightarrow r = -2, -3$

$y = C_1 e^{-2t} + C_2 e^{-3t}$

$y(0) = C_1 + C_2 = -2, \quad y'(0) = -2C_1 - 3C_2 = 11$

$rref \begin{bmatrix} 1 & 1 & -2 \\ -2 & -3 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \end{bmatrix}$

$y = 5e^{-2t} - 7e^{-3t}$

**Second equation.**

$r^2 + 6r + 9 = (r + 3)^2 = 0, r = -3, -3$ (double root)

$y = C_1 e^{-3t} + C_2 te^{-3t}$

$y(0) = C_1 = 4, \quad y'(0) = -3C_1 + C_2 = -23$

$rref \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & -23 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -11 \end{bmatrix}$

$y = 4e^{-3t} - 11te^{-3t}$
**Third equation.**
\[ r^2 + 4r + 29 = (r + 2)^2 + 5^2 = 0, \quad r = -2 \pm 5i. \]
\[ y = C_1 e^{-2t} \cos(5t) + C_2 e^{-2t} \sin(5t) \]
\[ y(0) = C_1 = -2, \quad y'(0) = -2C_1 + 5C_2 = 39 \]
\[ \text{rref } \begin{bmatrix} 1 & 0 & -2 \\ -2 & 5 & 39 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 7 \end{bmatrix} \]
\[ y = -2e^{-2t} \cos(5t) + 7e^{-2t} \sin(5t) \]

**Problem 4**

A crash-dummy weighs 224 lbs (and thus has mass \( \frac{224}{32} = 7 \) slugs). The crash-dummy encounters air resistance of \( 1.4 \text{ lbs/ft/sec} \). The crash-dummy is tossed off a building 200 ft high, initially with a velocity of 21 ft/sec upward. Let \( y \) denote the height of the crash-dummy above the ground. Set up and solve the initial value problem for the vertical motion of the crash dummy before it hits the ground. Use the method of undetermined coefficients to solve the initial value problem. You can take the acceleration of gravity to be 32 ft/sec downward.

**Answer.**

setup: \((7 \text{ slugs })(y'' \text{ ft/sec}^2) = -(1.4 \text{ lbs/ft/sec})(y' \text{ ft/sec}) - 224 \text{ lbs} \)
\[ y(0) = 200 \text{ ft}, \quad y'(0) = 21 \text{ ft/sec}. \]

after simplification: \( y'' + 0.2y' = -32, \quad y(0) = 200, \quad y'(0) = 21. \)
\[ r^2 + 0.2r = 0, \quad r = 0, -0.2. \]
\[ y_h = C_1 + C_2 e^{-0.2t} \]
\[ y_p = At \] (appropriate guess, because of “trouble”)
\[ y'' + 0.2y'_p = 0.2A = -32 \Rightarrow A = -160. \]
\[ y = -160t + C_1 + C_2 e^{-0.2t} \]
\[ y(0) = C_1 + C_2 = 200 \]
\[ y'(0) = -160 - 0.2C_2 = 21 \]
\[ \text{rref } \begin{bmatrix} 1 & 1 & 200 \\ 0 & -0.2 & 21 + 160 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1105 \\ 0 & 1 & -905 \end{bmatrix} \]
\[ y = -160t + 1105 - 905e^{-0.2t} \]

**Problem 5**

Find particular solutions by the method of undetermined coefficients for each of the following equations:
\[ y'' + 3y' + 2y = te^{3t} \]
\[ y'' + 3y' + 2y = e^{-2t} \]
\[ y'' + 4y = 3 \cos(2t) \]

**Answer.**

**First equation.**
\[ r^2 + 3r + 2 = (r + 1)(r + 2) = 0 \Rightarrow r = -1, -2 \]
\[ y_h = C_1 e^{-t} + C_2 e^{-2t} \]
\[ y_p = (At + B)e^{3t} \] (no trouble here)
\[ y'_p = Ae^{3t} + (At + B)(3e^{3t}) = (3At + A + 3B)e^{3t} \]
\[ y''_p = 3Ae^{3t} + (3At + A + 3B)(3e^{3t}) = (9At + 6A + 9B)e^{3t} \]
\[ y''_p + 3y'_p + 2y_p = (20At + 9A + 20B)e^{3t} \]
\[
\text{rref } \begin{bmatrix} 20 & 0 & 1 \\ 9 & 20 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/20 \\ 0 & 1 & -9/400 \end{bmatrix}
\]

\[y_p = (t/20 - 9/400) e^{3t}\]

Second equation.

\[y_h = C_1 e^{-2t} + C_2 e^{-t} \text{ (same as before)}\]

\[y_p = A t e^{-2t} \text{ (trouble), } y'_p = (-2At + A)e^{-2t}, \ y''_p = (4At - 4A)e^{-2t}\]

\[y''_p + 3y'_p + 2y_p = -A e^{-2t}, \ A = -1\]

\[y_p = -te^{-2t}\]

Third equation.

\[y_h = C_1 \cos(2t) + C_2 \sin(2t) \text{ (general solution of harmonic oscillator)}\]

\[y_p = At \cos(2t) + Bt \sin(2t) \text{ (trouble, so multiply by } t)\]

\[y''_p + 4y_p = 4B \cos(2t) - 4A \sin(2t) = 3 \cos(2t)\]

\[A = 0 \text{ and } 4B = 3 \Rightarrow B = 3/4\]

\[y_p = \frac{3t}{4} \sin(2t)\]

Problem 6

Find the eigenvalues and eigenvectors for

\[
\begin{bmatrix} 11 & 15/2 \\ -10 & -9 \end{bmatrix}
\]

Given that the eigenvalues are \(\lambda = 1, 3, 5\), find the eigenvectors for

\[
\begin{bmatrix} 2 & 0 & 1 \\ 3 & -4 & 4 \\ 7 & -15 & 11 \end{bmatrix}
\]

Then use the eigenvectors and eigenvalues you have found to diagonalize the matrices.

Answer.

Two-by-two matrix.

\[
\begin{vmatrix} 11 - t & 15/2 \\ -10 & -9 - t \end{vmatrix} = t^2 - 2t + 24 = (t - 6)(t + 4) \Rightarrow \lambda = 6, -4.
\]

Since eigenvalues for the two-by-two matrix are distinct, diagonalization is guaranteed to be possible. We just need to find some eigenvector for each eigenvalue.

\[
\begin{bmatrix} 11 - 6 & 15/2 \\ -10 & -9 - 6 \end{bmatrix} = \begin{bmatrix} 5 & 15/2 \\ -10 & -15 \end{bmatrix} \Rightarrow \frac{2}{5} \begin{bmatrix} 15/2 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \leftrightarrow 6.
\]

\[
\begin{bmatrix} 11 - (-4) & 15/2 \\ -10 & -9 - (-4) \end{bmatrix} = \begin{bmatrix} 15 & 15/2 \\ -10 & -5 \end{bmatrix} \Rightarrow \frac{2}{15} \begin{bmatrix} 15/2 \\ -15 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \leftrightarrow 4.
\]

We did not bother to hit the \text{rref} button because in each case the two equations are really just one equation we can solve by inspection ("the switcharoo"). Here is a diagonalization:

\[
\begin{bmatrix} 11 & 15/2 \\ -10 & -9 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}
\]
Three-by-three matrix. Since eigenvalues are distinct diagonalization is guaranteed to be possible. We just need to find some eigenvector for each eigenvalue.

\[
\begin{bmatrix}
1 & 0 & -1 \\
7 & -15 & 10 \\
-1 & 0 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1/5 \\
0 & 0 & 0 \\
1 & 0 & -1/3 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
-5 \\
1 \\
3 \\
\end{bmatrix}
\leftrightarrow 1,
\]

\[
\begin{bmatrix}
3 & -7 & 4 \\
7 & -15 & 8 \\
-3 & 0 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & -1/3 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 \\
3 \\
5 \\
\end{bmatrix}
\leftrightarrow 3,
\]

\[
\begin{bmatrix}
3 & -9 & 4 \\
7 & -15 & 6 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -5/9 \\
0 & 0 & 0 \\
1 & 0 & -2/3 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
5 \\
9 \\
5 \\
\end{bmatrix}
\leftrightarrow 5.
\]

Here is a diagonalization:

\[
\begin{bmatrix}
2 & 0 & 1 \\
3 & -4 & 4 \\
7 & -15 & 11 \\
\end{bmatrix}
\begin{bmatrix}
-5 & 1 & 3 \\
1 & 1 & 5 \\
5 & 1 & 9 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 5 \\
\end{bmatrix}
\]

Problem 7

Consider the matrix:

\[A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}\]

Diagonalize the matrix \(A\), and find the matrix power \(A^n\). Multiply out the matrix so that each element has a concrete expression.

Answer.

Diagonalization:

\[AP = PD, \quad P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad P^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\]

(The details of finding the diagonalization are similar to those in the previous problem and therefore omitted.)

Matrix power:

\[A^n = PD^nP^{-1} = \frac{1}{5} \begin{bmatrix} 2 \cdot 3^n + 3(-2)^n & 3 \cdot 3^n - 3(-2)^n \\ 2 \cdot 3^n - 2(-2)^n & 3 \cdot 3^n + 2(-2)^n \end{bmatrix}.
\]

Problem 8

Find the eigenvalues and all eigenvectors for the following matrices:

\[
\begin{bmatrix}
8 & -2 & -4 \\
3 & 1 & -2 \\
6 & -2 & -2 \\
\end{bmatrix}, \quad \begin{bmatrix}
-7 & -8 \\
18 & 17 \\
\end{bmatrix}
\]

(We leave you to find the eigenvalues of the two-by-two matrix, but we tell you that the eigenvalues of the three-by-three matrix are 2, 2, 3, i.e., 2 is a double
eigenvalue.) Then use the eigenvalues and eigenvectors to diagonalize the given matrices if possible. If not possible, explain why not.

**Answer.**

**Three-by-three matrix.** Since

$$ \begin{bmatrix} 6 & -2 & -4 \\ 3 & -1 & -2 \\ 6 & -2 & -4 \end{bmatrix} \text{ rref } \begin{bmatrix} 1 & -1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(note two nonpivot columns) the eigenvectors for $\lambda = 2$ are all vectors of the form

$$ s \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

where not both $s$ and $t$ are zero. (The eigenspace is two-dimensional.) Since

$$ \begin{bmatrix} 5 & -2 & -4 \\ 3 & -2 & -2 \\ 6 & -2 & -5 \end{bmatrix} \text{ rref } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

(note just one nonpivot column) the eigenvectors for $\lambda = 3$ are all vectors

$$ t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

for $t \neq 0$. We have enough eigenvectors because the total number of nonpivot columns we found in the course of analyzing all the eigenvalues was three, matching the size of the matrix we are diagonalizing, which is three by three. So we get a diagonalization:

$$ \begin{bmatrix} 8 & -2 & -4 \\ 3 & 1 & -2 \\ 6 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 3 \neq 0$$

**Two-by-two matrix.** Eigenvalues are 5, 5, i.e., we have a double root. Since

$$ \begin{bmatrix} -12 & -8 \\ 18 & 12 \end{bmatrix} \text{ rref } \begin{bmatrix} 1 & 2/3 \\ 0 & 0 \end{bmatrix}$$

and there is just one nonpivot column. The eigenvectors for the eigenvalue 5 are

$$ t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

for $t \neq 0$. The eigenspace is only one-dimensional. The total number of nonpivots discovered in the course of trying to find eigenvectors, namely one, does not match the size of the matrix, which is two-by-two. We can’t diagonalize.

For more information on what happens when diagonalization is not possible, proceed to an advanced linear algebra course to learn about Jordan canonical form.
Problem 9

Solve the following initial value problem:

\[ y'' + 6y' + 34y = 663 \cos(7t), \quad y(0) = 0, \quad y'(0) = 63 \]

Identify the steady-state and transient parts of the solution clearly. Express the steady-state part of the solution in the form \( A \cos(\omega t - \delta) \), where \( A > 0, \omega > 0, \delta \geq 0 \) and \( \delta < 2\pi \). Numerical approximations to four decimal places of accuracy are requested for \( A, \omega \) and \( \delta \).

\[
\begin{align*}
 \text{Answer.} \\
r^2 + 6r + 34 &= (r + 3)^2 + 5^2 = 0 \Rightarrow r = -3 \pm 5i. \\
y_h &= C_1 e^{-3t} \cos(5t) + C_2 e^{-3t} \sin(5t) \\
y_p &= A \cos(7t) + B \sin(7t) \\
y''_p + 6y'_p + 34y_p &= (-15A + 42B) \cos(7t) + (-42A - 15B) \sin(7t)
\end{align*}
\]

\[
\text{rref} = \begin{bmatrix} -15 & 42 & 663 \\ -42 & -15 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 14 \end{bmatrix}
\]

\[
y = y_p + y_h = -5 \cos(7t) + 14 \sin(7t) + C_1 e^{-3t} \cos(5t) + C_2 e^{-3t} \sin(5t) \\
y(0) = -5 + C_1 = 0 \Rightarrow C_1 = 5 \\
y'(0) = 7 \cdot 14 - 3C_1 + 5C_2 = 63 \Rightarrow 83 + 5C_2 = 63 \Rightarrow 5C_2 = -20 \Rightarrow C_2 = -4. \\
y = -5 \cos(7t) + 14 \sin(7t) + 5e^{-3t} \cos(5t) - 4e^{-3t} \sin(5t)
\]

\[
\begin{array}{c c}
\text{steady state} & \text{transient} \\
A &= \sqrt{(-5)^2 + 14^2} = 14.8660, \delta = \arctan(\frac{14}{5}) + \pi = 1.9138 \\
\text{(note that the point } (-5, 14) \text{ is in quadrant II)}
\end{array}
\]

\[
\boxed{y = 14.8660 \cos(7t - 1.9138) + 5e^{-3t} \cos(5t) - 4e^{-3t} \sin(5t)}
\]

Problem 10

Solve the initial value problem

\[ y'' - 4y = 24e^{2t}, \quad y(0) = 4, \quad y'(0) = 10. \]

Solution.

\[
r^2 - 4 = (r - 2)(r + 2) = 0 \Rightarrow r = \pm 2 \Rightarrow y_h = C_1 e^{-2t} + C_2 e^{2t} \\
y_p = Ate^{2t} \text{ (trouble so multiply by } t) \\
y''_p - 4y_p = A(4e^{2t} + 4te^{2t}) - 4Ate^{2t} = 4Ae^{2t} \\
\text{(We used the double Leibniz formula } (fg)'' = f''g + 2f'g' + fg'') \\
4A = 24 \text{ and hence } A = 6, \text{ hence } y_p = 6te^{2t} \\
y = 6te^{-2t} + C_1 e^{-2t} + C_2 e^{2t} \\
y(0) = C_1 + C_2 = 4, \quad y'(0) = 6 - 2C_1 + 2C_2 = 10
\]

\[
\text{rref} = \begin{bmatrix} 1 & 1 & 4 \\ -2 & 2 & 10 - 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}
\]

\[
y = 6te^{2t} + e^{-2t} + 3e^{2t}
\]
Problem 11

Express the vector \[
\begin{bmatrix}
-5 \\
3 \\
22 \\
5
\end{bmatrix}
\] as a linear combination of \[
\begin{bmatrix}
1 \\
1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
3 \\
1
\end{bmatrix}, \text{ and } \begin{bmatrix}
-1 \\
1 \\
4 \\
1
\end{bmatrix}
\] if possible. If not possible, explain why. Then same question with \[
\begin{bmatrix}
-5 \\
3 \\
22 \\
5
\end{bmatrix}
\] replaced by \[
\begin{bmatrix}
-5 \\
3 \\
22 \\
4
\end{bmatrix}.
\]

Answer. To answer the first part of the question we have to solve the equations

\[
x \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 22 \\ 5 \end{bmatrix}
\]

Well, we have \( \text{rref} \)

\[
\begin{bmatrix}
1 & 1 & -1 & -5 \\
1 & 0 & 1 & 3 \\
2 & 3 & 4 & 22 \\
1 & 1 & 1 & 5
\end{bmatrix}
\] =

\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] and thus

\[
\begin{bmatrix}
-5 \\
3 \\
22 \\
5
\end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\]

To answer the second part of the question we have to solve the slightly different equations

\[
x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 22 \\ 4 \end{bmatrix}.
\]

Well, we have \( \text{rref} \)

\[
\begin{bmatrix}
1 & 1 & -1 & -5 \\
1 & 0 & 1 & 3 \\
2 & 3 & 4 & 22 \\
1 & 1 & 1 & 4
\end{bmatrix}
\] =

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\] The last equation here is \( 0 = 1 \) which can't be solved, so no linear combination of the requested type can be formed. (Note that it is easy to make a typing mistake and arrive at the conclusion of no solution so be really careful.)
Problem 12

Are the vectors \[
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}, \quad
\begin{bmatrix}
5 \\
6 \\
7 \\
8
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
10 \\
11 \\
12 \\
13
\end{bmatrix}
\] linearly independent? If not, express one of the vectors as a linear combination of the others.

Answer. The equations to be analyzed are

\[
x \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix} + y \begin{bmatrix}
5 \\
6 \\
7 \\
8
\end{bmatrix} + z \begin{bmatrix}
10 \\
11 \\
12 \\
13
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

If these equation have only the solution \((x, y, z) = (0, 0, 0)\) then the answer is “yes, linearly independent” and otherwise the answer is “no, linearly dependent.” Well, we have

\[
\text{rref } \begin{bmatrix}
1 & 5 & 10 & 0 \\
2 & 6 & 11 & 0 \\
3 & 7 & 12 & 0 \\
4 & 8 & 13 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -5/4 & 0 \\
0 & 1 & 9/4 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

So indeed no, the vectors are linearly dependent because there is some solution \((x, y, z)\) of the equations above different from \((0, 0, 0)\). Reading off one of these solutions different from \((0, 0, 0)\) from the reduced row echelon matrix, we find that

\[
\begin{bmatrix}
5/4 \\
3 \\
4
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} \begin{bmatrix}
9/4 \\
6 \\
7
\end{bmatrix} + \begin{bmatrix}
10/4 \\
11 \\
12
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

Finally, we can rearrange the preceding equation to get the equation

\[
\begin{bmatrix}
10 \\
11 \\
12 \\
13
\end{bmatrix} = \begin{bmatrix}
5/4 \\
3 \\
4
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} + \begin{bmatrix}
9/4 \\
6 \\
7
\end{bmatrix},
\]

which expresses the last of the vectors as a linear combination of the preceding two.