We want to solve every initial value problem of the form
\[ x' = Ax, \ x(0) = x_0, \ \text{tr}(A) = 0, \]
in a special way that makes it obvious what the trajectory looks like. When (1) is written out with all individual matrix entries visible the problem looks like this:
\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \]

Notice that we are doing this under the special assumption that the sum of the diagonal elements of the coefficient matrix (the trace of the coefficient matrix) equals zero. The special assumption of trace zero turns out not to be a barrier to applications when the trace zero assumption does not hold for reasons I mentioned in class.

For later reference we mention that under the special assumption \(\text{tr}(A) = 0\):
\[ \text{(3) The eigenvalues of the coefficient matrix } A \text{ are } \pm \sqrt{-\det(A)} = \pm \sqrt{a^2 + bc}. \]

Now let
\[ W = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \] and hence \(WA = \begin{bmatrix} -c & a \\ a & b \end{bmatrix}\).

Our solution procedure is based on the observation that the solution \(x\) satisfies
\[ \frac{d}{dt} x^T W A x = \frac{d}{dt} (-cx^2 + 2axy + by^2) = 0. \]

Proof:
\[
\frac{d}{dt} x^T W A x \\
= \left( \frac{dx}{dt} \right)^T W A x + x^T W A \frac{dx}{dt} \\
= x^T \left( (A^T W A + W A) x \right) \\
= x^T \left( \begin{bmatrix} a & c \\ b & -a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right) x \\
= x^T \left( \begin{bmatrix} -c & a \\ a & b \end{bmatrix} + \begin{bmatrix} 0 & -a^2 - bc \\ a^2 + bc & 0 \end{bmatrix} \right) x = 0. \quad \square
\]

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Thus the solution of the initial value problem (1) stays on the curve
\[ x^T W A x = x_0^T W A x_0 \]
which can also be written in non-matrix form as
\[ -cx^2 + 2axy + by^2 = -cx(0)^2 + 2ax(0)y(0) + by(0)^2. \]
This curve is an ellipse if \( a^2 + bc < 0 \) and a hyperbola if \( a^2 + bc > 0 \). The case \( a^2 + bc = 0 \) is also interesting but we omit discussion of it here.

There is one more important general point we would like to make. The vector
\[ x'(0) = Ax_0 = \begin{bmatrix} ax(0) + by(0) \\ cx(0) - ay(0) \end{bmatrix} \]
points in the direction of travel of the solution of (1) on the curve (6). It is an important arrow to include in any drawing you might make. The magnitude is maybe not so important but the direction is crucial.

Hereafter we will stop writing so many letters and focus on numerical examples. We first consider the IVP
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.
\]
Plugging into (3) or calculating anew for yourself you can see that the eigenvalues of the coefficient matrix are \( \pm \sqrt{26} i = \pm 5.0990 i \). The trajectory \( \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \) travels along the curve
\[ 7x^2 + 6xy + 5y^2 = 37. \]
The latter is the equation of an ellipse. (In general ellipses are associated with complex eigenvalues and hyperbolas with real eigenvalues.) We got this equation by plugging into the general formula (6) above. Note that the initial point \( (x(0), y(0)) = (2, -3) \) is on the curve (9), as it should be. Note that
\[
\begin{bmatrix} x'(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \end{bmatrix}
\]
is a vector establishing the direction of travel around the ellipse. This we got by plugging into (7).

Scroll down for a quick write-up of the graphing process for the ellipse. In order not to interrupt the flow at this point we just take for granted now that the four most important points on the ellipse (9) are
\[ p = \begin{bmatrix} 2.1113 \\ -2.9293 \end{bmatrix}, \quad q = \begin{bmatrix} -1.6302 \\ -1.1750 \end{bmatrix} \]
along with their doppelgängers \( -p \) and \( -q \). The vectors \( \pm p \) and \( \pm q \) point in the directions of the principal axes of the ellipse. There is one subtlety here. You need to choose \( q \) so that it is the next point on the ellipse that you hit as you travel in the direction of the solution of (8) starting from \( p \). That’s how we chose \( q \) here.

Now let
\[ \omega = \sqrt{26} = 5.0990 = \text{absolute value of the eigenvalues of } \begin{bmatrix} 3 & 5 \\ -7 & -3 \end{bmatrix} \]
and let’s enjoy a nice surprise. We have
\[
\begin{bmatrix}
3 & 5 \\
-7 & -3
\end{bmatrix}
\begin{bmatrix}
2.1113 & -1.6302 \\
-2.9293 & -1.1750
\end{bmatrix}
= \begin{bmatrix}
2.1113 & -1.6302 \\
-2.9293 & -1.1750
\end{bmatrix}
\begin{bmatrix}
0 & -5.0990 \\
5.0990 & 0
\end{bmatrix},
\]
i.e., (with letters put back to show a pattern) we have
\[
\begin{bmatrix}
3 & 5 \\
-7 & -3
\end{bmatrix}
[\mathbf{p} \hspace{1em} \mathbf{q}]
= \begin{bmatrix}
0 & -\omega \\
\omega & 0
\end{bmatrix}.
\]
Actually the “nice surprise” is the goal of the process for selecting \(\mathbf{p}\) and \(\mathbf{q}\) in the first place.

Now let \(\alpha\) be any constant. Consider the vector function
\[
\mathbf{x}(t) = [\mathbf{p} \hspace{1em} \mathbf{q}]
\begin{bmatrix}
\cos(\omega t + \alpha) \\
\sin(\omega t + \alpha)
\end{bmatrix}
\]
Then
\[
\mathbf{x}'(t) = [\mathbf{p} \hspace{1em} \mathbf{q}]
\begin{bmatrix}
0 & -\omega \\
\omega & 0
\end{bmatrix}
\begin{bmatrix}
\cos(\omega t + \alpha) \\
\sin(\omega t + \alpha)
\end{bmatrix}
= \begin{bmatrix}
3 & 5 \\
-7 & -3
\end{bmatrix} \mathbf{x}(t),
\]
i.e., \(\mathbf{x}(t)\) satisfies the differential equation in (8), and we hope we can find \(\alpha\) to satisfy the initial condition. To find \(\alpha\) we calculate as follows:
\[
\begin{bmatrix}
\cos \alpha \\
\sin \alpha
\end{bmatrix}
= [\mathbf{p} \hspace{1em} \mathbf{q}]^{-1}
\begin{bmatrix}
2 \\
-3
\end{bmatrix}
= \begin{bmatrix}
0.9979 \\
0.0655
\end{bmatrix}
= \begin{bmatrix}
\cos(0.656) \\
\sin(0.656)
\end{bmatrix}
\Rightarrow \alpha = 0.656.
\]
Fortunately the point
\[
\begin{bmatrix}
0.9979 \\
0.0655
\end{bmatrix}
\]
turned out to be on the unit circle. Actually the process for finding \(\mathbf{p}\) and \(\mathbf{q}\) guarantees that this is always going to happen in the elliptical case.

Here’s the FINAL ANSWER in its full numerical glory:
\[
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix}
= \cos(5.0990t + 0.656)
\begin{bmatrix}
2.1113 \\
-2.9293
\end{bmatrix}
+ \sin(5.0990t + 0.656)
\begin{bmatrix}
-1.6302 \\
-1.1750
\end{bmatrix}.
\]
Scroll down to see a graph. To check I plugged exactly this formula into MAPLE and checked that up to very tiny errors due to round-off it indeed satisfies (8).

As promised we now quickly run the procedure for graphing the ellipse (8). We write the equation of the ellipse as
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}^T
\begin{bmatrix}
7 & 3 \\
3 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} = 37.
\]
The eigenvalues of \(\begin{bmatrix}
7 & 3 \\
3 & 5
\end{bmatrix}\) are the roots of \(\lambda^2 - 12\lambda + 26 = 0\) which are
\[
\lambda = 6 \pm \sqrt{10} = 2.8377, 9.1622.
\]
Here are some eigenvectors for the two eigenvalues, respectively:
\[
\begin{bmatrix}
-3 \\
7 - 2.8377
\end{bmatrix}
= \begin{bmatrix}
-3 \\
4.1623
\end{bmatrix},
\begin{bmatrix}
-3 \\
7 - 9.1622
\end{bmatrix}
= \begin{bmatrix}
-3 \\
-2.1623
\end{bmatrix}
\]
We need to rescale these vectors to points on the ellipse, which is to say, we first have to solve the equation
\[
e^2
\begin{bmatrix}
-3 \\
4.1623
\end{bmatrix}^T
\begin{bmatrix}
7 & 3 \\
3 & 5
\end{bmatrix}
\begin{bmatrix}
-3 \\
4.1623
\end{bmatrix}
= 74.7018e^2 = 37.
getting \( c = \pm 0.7038 \) and then apply the scale factor \( c \) getting

\[
p = -0.7038 \begin{bmatrix} -3 \\ 4.1623 \end{bmatrix} = \begin{bmatrix} 2.1113 \\ -2.9293 \end{bmatrix}.
\]

(We were allowed to take the minus sign here and decided we felt like using it.)

Similarly we have to solve the equation

\[
c^2 \begin{bmatrix} -3 & -3 \\ -2.1623 & -2.1623 \end{bmatrix}^T \begin{bmatrix} 7 \\ 3 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -3 \\ -2.1623 \end{bmatrix} = 125.2982c^2 = 37
\]

getting \( c = \pm 0.5434 \) and then apply the scalar factor getting

\[
q = 0.5434 \begin{bmatrix} -3 \\ -2.1623 \end{bmatrix} = \begin{bmatrix} -1.6302 \\ -1.1750 \end{bmatrix}.
\]

Finally we have to wonder for a minute having already picked \( p \) whether we have to change the choice of \( q \) to \(-q\) or not. As it happens, our choice of \( q \) is good.

If you pick the wrong \( q \) you get

\[
\begin{bmatrix} 3 & 5 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}
\]

instead of

\[
\begin{bmatrix} 3 & 5 \\ -7 & -3 \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}
\]

in which case you just replace \( q \) by \(-q\) and you are back on track. So it is easy to check if you made the right choice and fix it if you didn’t.
restart; with(linalg): with(plots):

aspect:=-4..4,-4..4:
pbold:=[2.1113,-2.9293]:
qbold:=[-1.6302,-1.1750]:
xnought:=[2,-3]:
xnoughtprime:=[-9,-5]:
xbold:=matadd(pbold,qbold,cos(t),sin(t)):
tangent:=matadd(xnought,xnoughtprime,1,t):
f:=plot([xbold[1],xbold[2],t=0..2*Pi],aspect,color=BLACK):
g:=plot([pbold[1]*t,pbold[2]*t,t=0..1],aspect,color=GREEN):
# gbis:=plot([pbold[1]*t,pbold[2]*t,t=-1..0],aspect,color=YELLOW):
h:=plot([qbold[1]*t,qbold[2]*t,t=0..1],aspect,color=RED):
ell:=plot([tangent[1],tangent[2],t=0..0.3],aspect,color=CYAN):
# hbis:=plot([qbold[1]*t,qbold[2]*t,t=-1..0],aspect,color=MAGENTA):
display({f,g,h,ell});

> # pbold is the green segment pointing into quadrant IV.
> # qbold is the red segment pointing into quadrant III.
> # The cyan segment indicates the direction of travel around the ellipse, which is clockwise.

Figure 1. Elliptical trajectory