Without discussing too much general theory we just work two numerical examples which ought to give a good overview.

First we graph the curve
\[ E : \quad 7x^2 + 2xy + 3y^2 = 1. \]

We rewrite the equation as
\[
\mathbf{x}^T \mathbf{A} \mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.
\]

We will find the four most important solutions of the equation of the ellipse \( E \). The first step is to diagonalize the symmetric matrix. In this case \( \mathbf{A} = \mathbf{P} \mathbf{D} \mathbf{P}^T \) reads
\[
\begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0.9732 & 0.2298 \\ 0.2298 & 0.9732 \end{bmatrix} \begin{bmatrix} 0.9732 & 0.2298 \\ 0.2298 & 0.9732 \end{bmatrix} = \begin{bmatrix} 7.2361 & 0 \\ 0 & 2.7639 \end{bmatrix}.
\]
The matrix \( \mathbf{P} \) was supplied by MATLAB and as a courtesy that program serves up column vectors that are unit vectors. Note these eigenvectors are perpendicular to each other, as is always the case when diagonalizing a symmetric matrix. Let
\[
\mathbf{p} = \frac{\begin{bmatrix} 0.9732 \\ 0.2298 \end{bmatrix}}{\sqrt{7.2361}} = \begin{bmatrix} 0.3618 \\ 0.0854 \end{bmatrix}, \quad \lambda = 7.2361,
\]
\[
\mathbf{q} = \frac{-\begin{bmatrix} 0.2298 \\ 0.9732 \end{bmatrix}}{\sqrt{2.7639}} = \begin{bmatrix} -0.1382 \\ 0.5854 \end{bmatrix}, \quad \mu = 2.7639.
\]
The vectors \( \mathbf{p} \) and \( \mathbf{q} \) are again eigenvectors and still perpendicular but now they satisfy
\[
\mathbf{p}^T \mathbf{p} = \frac{1}{\lambda}, \quad \mathbf{q}^T \mathbf{q} = \frac{1}{\mu},
\]
and hence
\[
\mathbf{p}^T \mathbf{A} \mathbf{p} = \lambda \mathbf{p}^T \mathbf{p} = 1, \quad \mathbf{q}^T \mathbf{A} \mathbf{q} = \mu \mathbf{q}^T \mathbf{q} = 1
\]
so that they are actually points on the ellipse \( E \) we are trying to graph. The points \( \pm \mathbf{p} \) in quadrants I and III are nearest to the origin. The points \( \pm \mathbf{q} \) in quadrants II and IV are farthest from the origin. The formula
\[
\mathbf{x}(t) = \cos(t) \mathbf{p} + \sin(t) \mathbf{q} \quad (0 \leq t \leq 2\pi)
\]
is a parameterization of the ellipse. I have posted a MAPLE printout separately on the web page to show the graph of the ellipse. You have to scroll down a ways to get finally to the graph of \( E \) plus various decorations.

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To check that $x(t)$ is really running around $E$ we calculate as follows:

\[
x(t)^T A x(t) = (\cos(t)p^T + \sin(t)q^T)A(\cos(t)p + \sin(t)q)
\]

\[
= \cos(t)^2 A_p^T p + \cos(t)\sin(t)\mu p^T q + \sin(t)\cos(t)\lambda q^T p + \sin(t)^2 \mu q^T q
\]

\[
= \cos(t)^2 + \sin(t)^2 = 1.
\]

Secondly we graph the pair of equations

\[
H_\pm : 3x^2 + 10xy + 5y^2 = \pm 1
\]

don the same axes. We start by rewriting the equations as

\[
x^T A x = \begin{bmatrix} x & y \end{bmatrix}^T \begin{bmatrix} 3 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x \\
0 \end{bmatrix} = \pm 1.
\]

We look for the four most important points on the hyperbolas $H_\pm$. To do it we diagonalize thanks again to MATLAB getting $AP = PD$ as follows:

\[
\begin{bmatrix} 3 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 0.6340 & -0.7733 \\ 0.7733 & 0.6340 \end{bmatrix} = \begin{bmatrix} 0.6340 & -0.7733 \\ 0.7733 & 0.6340 \end{bmatrix} \begin{bmatrix} 9.0990 & 0 \\ 0 & -1.0990 \end{bmatrix}.
\]

Let

\[
p = \begin{bmatrix} 0.6340 \\ 0.7733 \end{bmatrix} / \sqrt{9.0990} = \begin{bmatrix} 0.2102 \\ 0.2564 \end{bmatrix}, \quad \lambda = 9.0990,
\]

\[
q = \begin{bmatrix} -0.7733 \\ 0.6340 \end{bmatrix} / \sqrt{1.0990} = \begin{bmatrix} -0.7377 \\ 0.6048 \end{bmatrix}, \quad \mu = -1.0990.
\]

The vectors $p$ and $q$ are again eigenvectors and still perpendicular but now they satisfy

\[
p^T p = \frac{1}{\lambda}, \quad q^T q = \frac{1}{\mu}
\]

(notice the minus sign appearing on the right) and hence

\[
p^T A p = \lambda p^T p = 1, \quad q^T A q = \mu q^T q = -1
\]

(notice the minus sign appearing on the right). Thus the points $\pm p$ in quadrants I and III are points on $H_+$ and the points $\pm q$ in quadrants II and IV are points on $H_-$. These points are the ones closest on their branches to the origin. The two branches of $H_+$ are

\[
\pm(\cosh(t)p + \sinh(t)q) \quad (-\infty < t < \infty).
\]

The two branches of $H_-$ are

\[
\pm(\cosh(t)q + \sinh(t)p) \quad (-\infty < t < \infty).
\]

The vectors $\pm p \pm q$ point in the directions of the two asymptotes. You have to scroll down all the way to the bottom of the web-page post to find the graph.

We can check our work by a calculation similar to the one we did in the case of ellipses. We have

\[
(\cosh(t)p + \sinh(t)q)^T A (\cosh(t)p + \sinh(t)q)
\]

\[
= \cosh(t)^2 A_p^T p + \cosh(t)\sinh(t)\mu p^T q + \sinh(t)\cosh(t)\lambda q^T p + \sinh(t)^2 \mu q^T q
\]

\[
= \cosh(t)^2 - \sinh(t)^2 = 1.
\]

and similarly

\[
(\cosh(t)q + \sinh(t)p)^T A (\cosh(t)q + \sinh(t)p) = -\cosh(t)^2 + \sinh(t)^2 = -1.
\]