A discrete initial value problem

Solve the following second order linear homogeneous difference equation, taking care to satisfy the indicated initial conditions:

\[ y_{n+2} + 7y_{n+1} + 12y_n = -40, \quad y_0 = 2, \quad y_1 = 1. \]

**Solution.** The guess \( y^{(p)}_n = A \) (no “trouble”) for a “particular solution” satisfies the given difference equation if we take \( A = -2 \). The equation

\[ r^2 + 7r + 12 = (r + 3)(r + 4) = 0 \]

has roots \( r = -3, -4 \). Therefore the “complementary solution” is

\[ y^{(c)}_n = C_1(-3)^n + C_2(-4)^n \]

and the “general solution” takes the form

\[ y_n = -2 + C_1(-3)^n + C_2(-4)^n. \]

We have

\[ y_0 = C_1 + C_2 - 2 = 2, \quad y_1 = -3C_1 - 4C_2 - 2 = 1, \]

and hence

\[
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
-3 & -4
\end{bmatrix}^{-1}
\begin{bmatrix}
4 \\
3
\end{bmatrix}
= \begin{bmatrix}
19 \\
-15
\end{bmatrix}.
\]

The final answer is the closed form formula

\[ y_n = -2 + 19(-3)^n - 15(-4)^n. \]

Brine tank

A very large tank contains 50 lbs of salt dissolved in 400 gallons of water. Brine that contains \( 4/5 \) lbs of salt per gallon of water starts entering the tank at time \( t = 0 \) at the rate of 5 gal/min. The mixture leaves the tank at the lower rate of 3 gal/min. Find an expression for the amount of salt in the tank at time \( t \).
Answer.

\[ y' = \frac{4 \text{ lbs}}{5 \text{ gal}} \times 5 \frac{\text{gal}}{\text{min}} - \frac{y}{400 + (5 - 3)t} \cdot \frac{\text{lbs}}{\text{gal}} \times 3 \frac{\text{gal}}{\text{min}}, \quad y(0) = 50. \]

\[ y' + \frac{3}{2} \frac{y}{t + 200} = 4, \quad y(0) = 50. \]

The integrating factor is \( \exp\left(\frac{3}{2} \int \frac{dt}{t + 200}\right) = (t + 200)^{3/2} \).

\[ \frac{d}{dt}(t + 200)^{3/2}y = 4(t + 200)^{3/2} \Rightarrow (t + 200)^{3/2}y = \frac{8}{5}(t + 200)^{5/2} + C \]

\[ 200^{3/2} \cdot 50 = \frac{8}{5} \cdot 200^{5/2} + C \Rightarrow C = 200^{3/2} \cdot 50 - \frac{8}{5} \cdot 200^{5/2}. \]

Final answer: \( y = \frac{8}{5}(t + 200) + C/(t + 200)^{3/2} \) (with \( C \) as immediately above).

Newton’s Law of Heating and Cooling. A steel ball is heated to a temperature of 200°C and at time \( t = 0 \) is placed in water maintained at 20°C. At \( t = 5 \) minutes the temperature of the ball is 110°C. (i) Find the temperature \( y \) of the ball at time \( t \). Start from the differential equation. (ii) Also find the time at which the temperature of the steel ball equals 50°C, reporting your answer to 4 decimal places of accuracy.

Answer. (i) The differential equation is \( dy/dt = k(20 - y) \). Just to be different let us find the general solution using integrating factors. We have

\[ y' + ky = 20k \Rightarrow \frac{d}{dt}e^{kt}y = 20ke^{kt} \Rightarrow e^{kt}y = 20e^{kt} + C \Rightarrow y = 20 + Ce^{-kt}. \]

Using the information \( y(0) = 200 \) we get the equation \( 200 = 20 + C \) implying \( C = 180 \). Using the information \( y(5) = 110 \) we get \( 110 = 20 + 180e^{-5k} \) implying \( \frac{90}{180} = \frac{1}{2} = e^{-5k} \) and hence

\[ y(t) = 20 + 180 \left(\frac{1}{2}\right)^{t/5}. \]

If you solve all the way for \( k \) then the answer takes instead the equivalent form

\[ y = 20 + 180e^{-0.138629t}. \]

Either form of the answer is okay. You do not have to give both. (ii) The solution of the equation \( y(t) = 50 \) is \( t = \frac{5 \ln 6}{\ln 2} = 12.9248. \)
Convolution

Find the convolution of \( e^t \) and

\[
f(t) = \begin{cases} 
0 & \text{for } 0 \leq t < 1, \\
1 & \text{for } 1 \leq t < 2, \\
0 & \text{for } t \geq 2
\end{cases}
\]

Solution.

\[
\mathcal{L}\{e^t\} = \frac{1}{s-1}, \quad f(t) = u(t-1) - u(t-2), \quad \mathcal{L}\{f(t)\} = \frac{e^{-s} - e^{-2s}}{s},
\]

\[
\mathcal{L}\{e^t \ast f(t)\} = \mathcal{L}\{e^t\}\mathcal{L}\{f(t)\} = \frac{e^{-s} - e^{-2s}}{s(s-1)},
\]

\[
\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s}\right\} = e^t - 1
\]

Another way to find the inverse Laplace transform on the line above is this:

\[
\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \int_0^t e^\tau d\tau = e^t - 1.
\]

Final answer:

\[
e^t \ast f(t) = u(t-1)(e^{t-1} - 1) - u(t-2)(e^{t-2} - 1)
\]