PRESERVATION OF ALGEBRAICITY IN FREE PROBABILITY

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Abstract. We show that any matrix-polynomial combination of free noncommutative random variables each having an algebraic law has again an algebraic law. Our result answers a question raised in a recent paper of Shlyakhtenko and Skoufranis.

Contents

1. Introduction and statement of the main result

Our main result is as follows:

Theorem 1. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. Let*

$$x_1, \ldots, x_q \in \mathcal{A}$$

be freely independent noncommutative random variables. Let

$$X \in \mathrm{Mat}_p(\mathbb{C}\langle x_1, \ldots, x_q\rangle) \subset \mathrm{Mat}_p(\mathcal{A})$$

be a matrix. If the laws of $x_1, \ldots, x_q$ are algebraic, then so is the law of X.

This paper is devoted to a proof of Theorem 1. We say for short that X is a *free matrix-polynomial combination* of $x_1, \ldots, x_q$. See §2 below for all notation, terminology, and definitions needed to clarify Theorem 1.

Algebraicity of the law of a free matrix-polynomial combination of semicircular variables is a result of Shlyakhtenko and Skoufranis [24, Thm. 5.4]. This was proved as a complement to their main result concerning atoms in the law of a free matrix-polynomial combination of noncommutative random variables each having a nonatomic law. Our main result answers a question raised in [24].

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The phenomenon studied in general here has been verified in many special cases, often merely in passing. For example, it is implicit in the theory of the R-transform that free convolution preserves algebraicity. We note that this fact has been exploited in a practical way in [10]. For another example, the theory of commutators of free random variables worked out explicitly in [22] has implicitly the corollary that formation of free commutators preserves algebraicity of laws. This list of examples could easily be extended, because apparently almost every explicit calculation of a law in free probability theory involves algebraicity.

The papers [8] and [13] are important influences, especially the former. These papers do not treat algebraicity, but they do reveal a rich algebraic and analytic structure. In particular, the equations stated in [8, Thm. 2.2] are in principle the ones we have to analyze to prove Theorem 1. However, it is an important technical point in the cited papers that even though the goal is to clarify analytical properties of a function of one complex variable, namely the Stieltjes transform of a free matrix-polynomial combination of noncommutative random variables, one is forced first to study a many-variable generalization of the Stieltjes transform defined on the matrix upper half-plane; one then finally returns to the one-variable situation of main interest by taking a limit. Such an approach is well-suited to its purpose of computing laws as measures but is too difficult for us to adapt to the proof of Theorem 1. So we have devised an alternative purely algebraic approach to the same set of equations allowing us to avoid the limit step.

The powerful linearization trick introduced in [11] and [12] plays of course a crucial role here, in the self-adjointness-preserving variant introduced in [1] and exploited so successfully in [8]. However, ironically enough, the self-adjointness-preserving aspect of the trick is irrelevant here—rather, it is the extremely simple form of that trick, based on Schur complements, that gives us traction.

While Theorem 1 is a contribution to free probability theory, the underlying form of the problem it solves is of a type long studied in classical probability theory. For example, in [5] algebraicity of the Green function of random walk of a rather general type on a finitely generated free group was proved algebraic by lengthy explicit calculation. For another example, in [27] algebraicity of the Green function of a finitely supported random walk on a group with a finitely generated free subgroup of finite index was proved by a concise method based on formal language theory. With but slight modification, the method of either author yields a proof that a polynomial in free unitary variables has an algebraic law. Note that the latter fact can be recovered as a corollary to Theorem 1 simply by noting that a unitary variable factors as the product of two free Bernoulli variables.

Our method involves study of objects of Green function type, but we do not systematically investigate their algebraicity properties here. For simplicity we focus on algebraicity of laws. We plan to discuss Green functions on another occasion.

For any law μ consider the (formal) *zeta-function* $\exp\left(\sum_{n=1}^{\infty} \mu(\mathbf{X}^n)\frac{t^n}{n}\right)$. In the paper [14], building on the earlier paper [15], a statement is proved which (translated to free probability language) says that the zeta-function of the law of a free matrix-polynomial combination of unitary variables with integer coefficients is an algebraic power series with integer coefficients. Neither [14] nor [15] have any overt connection with free probability. But the results of these papers fit interestingly into the framework of free probability and suggest problems about zeta-functions perhaps susceptible to attack by methods cultivated here.

The overall approach to algebraicity used here is a relatively simple one adapted from the literature of random walk on infinite trees. From this literature, besides [5] and [27], we cite as examples [16], [17] and [21]. This short list could be greatly extended because the idea we are adapting is a fundamental trope. The trope has two main components. Firstly, one gets recursions by exploiting finiteness of cone types, or some related principle of self-similarity. We apply such a strategy here by putting the Boltzmann-Fock space model of free random variables familiar from Voiculescu's theory of the R-transform into suitably "arboreal" form so as to make the needed recursions visible in terms of matrix patterns. Secondly, there are various criteria available for recognizing when the recursions have algebraic solutions. We wish to highlight [16, Prop. 5.1] as an especially clear, simple and general criterion. We develop this criterion here in a suitable direction. The formal language approach to algebraicity of [27] is also quite elegant but we do not pursue it here because it is not clear how to generalize it in the needed direction.

We remark that relations between algebraicity and positivity are highly developed in the random walk literature, e.g., in the previously cited examples [16], [17], and [21], leading to local limit theorems. We do not touch those ideas here but we believe they could be fruitfully developed in the free probability context.

We also remark that when writing the paper [3] wherein was presented a criterion similar to if rather more baroque than [16, Prop. 5.1], the authors were unfortunately unaware of [16]. We acknowledge the priority now.

Our purely algebraic setup bypasses combinatorics, complex analysis and operator theory of the usual sort over the complex numbers. We make no positivity assumptions; moment sequences of variables can be arbitrarily prescribed sequences of complex numbers. We work over the field $\mathbb{C}((1/z))$ of formal Laurent series, using simple ideas about Banach algebras over complete valued fields. Our setup is designed to be as simple as possible to prove Theorem 1. To achieve minimality we sacrifice much, especially control of positivity and branch points. It is an open problem to regain the sacrificed control. Perhaps this is only a matter of unifying features of the several theories mentioned above, but in our opinion some further ingredients from algebraic geometry will be needed. The soliton theory literature, e.g., [20], might provide guidance.

Here is an outline of the paper. In §2, after filling in background in leisurely fashion, and in particular writing down a simple algebraicity criterion similar to [16, Prop. 5.1], namely Proposition 2.4.2 below, we reformulate Theorem 1 as the conjunction of two propositions both of which concern the generalized Schwinger-Dyson equation. In §3 we introduce the formal algebraic variant of operator theory exploited in this paper. In §4 we introduce a suitable model of free random variables and deploy the self-adjoint linearization trick. In §5 we exhibit the solutions of the generalized Schwinger-Dyson equation needed for the proof of Theorem 1. In the remainder of the paper we switch to the viewpoint of algebraic geometry and commutative algebra. In §6 we review topics connected with singularities of plane algebraic curves. In §7 we apply the Weierstrass Preparation Theorem in a perhaps unexpected way. Finally, in §8 we complete the proof of Theorem 1 by checking hypotheses in Proposition 2.4.2.

2. Background for the main result and a reduction of the proof

After recalling principal definitions, fixing notation, and filling in background for Theorem 1, we reduce Theorem 1 to two propositions each treating some aspect of the generalized Schwinger-Dyson equation.

2.1. **Noncommutative probability spaces and free independence.**

We present a brief review to fix notation. See, e.g., [4], [23], or [25] for background.

2.1.1. *Algebras.* All algebras in this paper are unital, associative, and have a scalar field containing $\mathbb{C}$. The unit of an algebra $\mathcal{A}$ is denoted by $1_\mathcal{A}$; other notation, e.g., simply 1, may be used when context permits. Given elements $x_1, \ldots, x_q \in \mathcal{A}$ of an algebra, let $\mathbb{C}\langle x_1, \ldots, x_q\rangle \subset \mathcal{A}$ denote the subring of $\mathcal{A}$ generated by forming all finite $\mathbb{C}$-linear combinations of monomials in the given elements $x_1, \ldots, x_q$, including the "empty monomial" $1_\mathcal{A}$. (But if $\mathcal{A}$ is commutative, instead of $\mathbb{C}\langle x_1, \ldots, x_q\rangle$, we prefer as usual in commutative algebra to write $\mathbb{C}[x_1, \ldots, x_q]$.)

2.1.2. *Noncommutative probability spaces.* A *state* ϕ on an algebra $\mathcal{A}$ is simply a $\mathbb{C}$-linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1_\mathcal{A}) = 1$. In our formal algebraic setup no positivity constraints are imposed. A *noncommutative probability space* is a pair $(\mathcal{A}, \phi)$ consisting of an algebra $\mathcal{A}$ and a state ϕ on that algebra. Given such a pair $(\mathcal{A}, \phi)$, elements of $\mathcal{A}$ are called *noncommutative random variables.*

2.1.3. *Matrices with algebra entries.* Given an algebra $\mathcal{A}$ and a positive integer n, let $\mathrm{Mat}_n(\mathcal{A})$ denote the algebra of n-by-n matrices with entries in $\mathcal{A}$ and let $1 = I_n = I_n \otimes 1_\mathcal{A} \in \mathrm{Mat}_n(\mathcal{A})$ denote the identity matrix, as context permits. More generally let $\mathrm{Mat}_{k\times\ell}(\mathcal{A})$ denote the space of k-by-ℓ matrices with entries in $\mathcal{A}$. For $A \in \mathrm{Mat}_{k\times\ell}(\mathbb{C})$ and $a \in \mathcal{A}$ we define $A\otimes a \in \mathrm{Mat}_{k\times\ell}(\mathcal{A})$ by $(A\otimes a)(i, j) = A(i, j)a$. Let $\mathbf{e}_{ij} \in \mathrm{Mat}_{k\times\ell}(\mathbb{C})$ denote the elementary matrix with 1 in position (i, j) and 0 in every other position. Let $\mathrm{GL}_n(\mathcal{A})$ denote the group of invertible elements of $\mathrm{Mat}_n(\mathcal{A})$. Given a noncommutative probability space $(\mathcal{A}, \phi)$, we regard each matrix $A \in \mathrm{Mat}_n(\mathcal{A})$ as a noncommutative random variable with respect to the state $\phi_n : \mathrm{Mat}_n(\mathcal{A}) \to \mathbb{C}$ given by the formula $\phi_n(A) = \frac{1}{n}\sum_{i=1}^n \phi(A(i, i))$.

2.1.4. *Free independence.* Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_q \subset \mathcal{A}$ be subalgebras such that $1_\mathcal{A} \in \cap_{i=1}^q \mathcal{A}_i$. One says that $\mathcal{A}_1, \ldots, \mathcal{A}_q$ are *freely independent* if for every positive integer k, sequence $i_1, \ldots, i_k \in \{1, \ldots, q\}$ such that $i_1 \neq i_2$, $i_2 \neq i_3$, $\ldots$, $i_{k-1} \neq i_k$ and sequence $x_1 \in \mathcal{A}_{i_1}$, $\ldots$, $x_k \in \mathcal{A}_{i_k}$ such that $\phi(x_1) = \cdots = \phi(x_k) = 0$, one has $\phi(x_1 \cdots x_k) = 0$. As a special case of the preceding general definition, one says that noncommutative random variables $x_1, \ldots, x_q \in \mathcal{A}$ are *freely independent* if the subalgebras $\mathbb{C}\langle x_1\rangle, \ldots, \mathbb{C}\langle x_q\rangle \subset \mathcal{A}$ are freely independent.

2.1.5. *Univariate laws.* Let $\mathbf{X}$ be a variable. A *univariate law* (or, context permitting, simply a *law*) is by definition a state $\mu : \mathbb{C}\langle \mathbf{X}\rangle \to \mathbb{C}$ on the one-variable polynomial algebra $\mathbb{C}\langle \mathbf{X}\rangle$. The value $\mu(\mathbf{X}^n) \in \mathbb{C}$ is called the n^{th} *moment* of μ. In our formal algebraic setup, to give a law is the same as to give a sequence of complex numbers. Given a noncommutative probability space $(\mathcal{A}, \phi)$ and a noncommutative random variable $x \in \mathcal{A}$, the *law* of x is by definition the linear functional $\mu_x : \mathbb{C}\langle \mathbf{X}\rangle \to \mathbb{C}$ determined by the formula $\mu_x(\mathbf{X}^n) = \phi(x^n)$ for integers $n \geq 0$.

2.1.6. *Noncommutative joint laws.* Let $\mathbf{X}_1, \ldots, \mathbf{X}_q$ be independent noncommuting algebraic variables and let $\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle$ be the noncommutative polynomial ring generated by these variables. A *q-variable noncommutative law* or, context permitting, simply a *law*, is a state on the algebra $\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle$. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and let $x_1, \ldots, x_q \in \mathcal{A}$ be noncommutative random variables. The *joint law* $\mu_{x_1,\ldots,x_q} : \mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle \to \mathbb{C}$ of the q-tuple $(x_1, \ldots, x_q)$ is by definition the linear functional defined by the rule $\mu_{x_1,\ldots,x_q}(f(\mathbf{X}_1, \ldots, \mathbf{X}_q)) = \phi(f(x_1, \ldots, x_q))$ for $f(\mathbf{X}_1, \ldots, \mathbf{X}_q) \in \mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle$. The laws $\mu_{x_1}, \ldots, \mu_{x_q}$ of the individual variables (by analogy with classical probabilistic usage) are called the *marginal* laws for the joint law $\mu_{x_1,\ldots,x_q}$. A point worth emphasizing is that if $x_1, \ldots, x_q$ are freely independent, then the joint law $\mu_{x_1,\ldots,x_q}$ is uniquely determined by the marginal laws $\mu_{x_1}, \ldots, \mu_{x_q}$.

2.2. **The Laurent series field $\mathbb{C}((1/z))$ and related notions.** We recall several definitions together providing a framework in which to discuss algebraicity. See the text [6] by Artin for background on valued fields and algebraic functions.

2.2.1. *Definition of $\mathbb{C}((1/z))$ and related objects.* Let $\mathbb{C}((1/z))$ denote the set of *formal Laurent series* in z of the form

$$f = \sum_{i \in \mathbb{Z}} c_i z^i \quad (c_i \in \mathbb{C} \text{ and } c_i = 0 \text{ for } i \gg 0). \tag{1}$$

(The coefficients c_i are not subject to any majorization.) Equipped with addition and multiplication in evident fashion, the set $\mathbb{C}((1/z))$ becomes a field. Note that we have inclusions

$$\mathbb{C}[z] \subset \mathbb{C}(z) \subset \mathbb{C}((1/z)) \text{ and } \mathbb{C}[[1/z]] \subset \mathbb{C}((1/z))$$

where $\mathbb{C}[z]$ is the ring of polynomials in z, $\mathbb{C}[[1/z]]$ is the ring of formal power series in $1/z$, and $\mathbb{C}(z)$ is the field of rational functions of z, all with coefficients in $\mathbb{C}$. Note also that we have an additive direct sum decomposition

$$\mathbb{C}((1/z)) = \mathbb{C}[z] \oplus (1/z)\mathbb{C}[[1/z]].$$

In our algebraic setup the formal variable z corresponds to the classical parameter z in the upper half-plane.

2.2.2. *Algebraic elements of $\mathbb{C}((1/z))$ and their irreducible equations.* Let $\mathbb{C}[x, y]$ be the polynomial ring over $\mathbb{C}$ in two independent (commuting) variables x and y. We say that $f \in \mathbb{C}((1/z))$ is *algebraic* if one and hence all three of the following equivalent conditions hold:

- There exists some $0 \neq P(x, y) \in \mathbb{C}[x, y]$ such that $P(z, f) = 0$.
- There exists some $0 \neq Q(x, y) \in \mathbb{C}[x, y]$ such that $Q(1/z, f) = 0$.
- The field $\mathbb{C}(z, f)$ generated over $\mathbb{C}(z)$ by f is a vector space of finite dimension over $\mathbb{C}(z)$.

The algebraic elements form a subfield of $\mathbb{C}((1/z))$ containing $\mathbb{C}(z)$. For each algebraic element $f \in \mathbb{C}((1/z))$ there exists an irreducible two-variable polynomial $F_f(x, y) \in \mathbb{C}[x, y]$ such that $F_f(1/z, f) = 0$, called an *irreducible equation* for f. Since every nonzero nonmaximal prime ideal of $\mathbb{C}[x, y]$ is principal, in fact $F_f(x, y)$ is unique up to a nonzero factor in $\mathbb{C}$. Thus, with but slight abuse of language, we speak of *the* irreducible equation of f. Note that the degree in y of $F_f(x, y)$ equals the dimension of $\mathbb{C}(z, f)$ over $\mathbb{C}(z)$.

2.2.3. *Algebraic elements of* $\mathbb{C}[[t]]$. From time to time we will find it convenient to change coordinates by the rule $t = 1/z$, e.g., in Proposition 7.1.5 below. Thus it is natural to say that a power series $f(t) \in \mathbb{C}[[t]]$ is *algebraic* if the corresponding series $f(1/z) \in \mathbb{C}[[1/z]] \subset \mathbb{C}((1/z))$ is algebraic in the sense defined in §2.2.2.

2.2.4. *Valuations.* For $f \in \mathbb{C}((1/z))$ expanded as on line (1) we define

$$\operatorname{val} f = \sup\{i \in \mathbb{Z} \mid c_i \neq 0\} = (\text{the } \textit{valuation} \text{ of } f) \in \mathbb{Z} \cup \{-\infty\}.$$

Note that

$$\operatorname{val} f = -\infty \quad \Leftrightarrow \quad f = 0, \tag{2}$$

$$\operatorname{val}(f_1 f_2) \quad = \quad \operatorname{val} f_1 + \operatorname{val} f_2, \tag{3}$$

$$\operatorname{val}(f_1 + f_2) \quad \leq \quad \max(\operatorname{val} f_1, \operatorname{val} f_2) \ \text{ with equality if } \operatorname{val} f_1 \neq \operatorname{val} f_2. \tag{4}$$

Thus val is (the logarithm of) a nonarchimedean valuation in the sense of [6]. Thus it becomes possible to use metric space ideas to reason about $\mathbb{C}((1/z))$, as in [6], and we will do so throughout this paper. We may speak for example of completeness. It is easy to see that $\mathbb{C}((1/z))$ is complete with respect to val.

2.2.5. *Banach algebra structure for* $\mathrm{Mat}_n(\mathbb{C}((1/z)))$. We now extend the metric space ideas a bit farther. We equip the matrix algebra $\mathrm{Mat}_n(\mathbb{C}((1/z)))$ with a valuation by the rule $\operatorname{val} A = \max_{i,j=1}^n \operatorname{val} A(i,j)$. Then (2) and (4) hold for matrices, (3) holds for multiplication of a matrix by a scalar, and (3) holds for multiplication of two matrices provided that "=" is relaxed to "$\leq$." Thus $\mathrm{Mat}_n(\mathbb{C}((1/z)))$ becomes a Banach algebra over $\mathbb{C}((1/z))$. Later, in §3, a certain infinite-dimensional Banach algebra over $\mathbb{C}((1/z))$ will be introduced.

2.2.6. *Composition of Laurent series.* The composition $f \circ g \in \mathbb{C}((1/z))$ of $f, g \in \mathbb{C}((1/z))$ is defined provided that $\operatorname{val} g > 0$. The set $z + \mathbb{C}[[1/z]]$ forms a group under composition. Indeed, it is isomorphic to the familiar group $t + t^2\mathbb{C}[[t]]$ under the map $f(z) \mapsto 1/f(1/t)$. The group $z + \mathbb{C}[[1/z]]$ acts on the right side of $\mathbb{C}((1/z))$ by $\mathbb{C}$-linear field automorphisms.

Lemma 2.2.7. *If $f, g \in z + \mathbb{C}[[1/z]]$ satisfy $f \circ g = z$ and f is algebraic, then g is also algebraic.*

Proof. For some $0 \neq P(x,y) \in \mathbb{C}[x,y]$ we have $0 = P(z,f) \circ g = P(g,z)$ and hence g is algebraic. $\square$

2.3. **Algebraicity of univariate laws.** We recall how to attach to each univariate law a (formal) Stieltjes transform and a (modified formal) R-transform *à la* Voiculescu. Then we recall how in terms of these transforms one can characterize algebraicity of a law.

2.3.1. *Formal Stieltjes transforms.* For a law $\mu : \mathbb{C}\langle \mathbf{X} \rangle \to \mathbb{C}$, the formal sum

$$S_\mu(z) = \sum_{n=0}^{\infty} \mu(\mathbf{X}^n)/z^{n+1} \in \mathbb{C}((1/z))$$

is by definition the *formal Stieltjes transform* of μ. Hereafter we drop the adjective "formal" since no other kind of Stieltjes transform will be considered in this paper.

2.3.2. *Algebraicity of univariate laws.* A law μ will be called *algebraic* if its Stieltjes transform $S_\mu(z) \in \mathbb{C}((1/z))$ is algebraic.

2.3.3. *Free cumulants and R-transforms.* Given a law $\mu : \mathbb{C}\langle\mathbf{X}\rangle \to \mathbb{C}$ one defines in free probability theory for each positive integer n the n^{th} *free cumulant* $\kappa_n(\mu) \in \mathbb{C}$. This can be done various ways, e.g., with generating functions or combinatorially using noncrossing partitions. See, e.g., [4], [23], or [25] for background. The generating function

$$R_\mu(t) = \sum_{n=1}^{\infty} \kappa_n(\mu) t^{n-1} \in \mathbb{C}[[t]]$$

for the free cumulants is the formal version of the *R-transform* of Voiculescu. Hereafter we drop the adjective "formal" since no other kind of R-transform will be considered in this paper.

2.3.4. *Modified R-transforms.* To define and make calculations with free cumulants, we will use the generating function method. Consider the *modified* R-transform

$$\tilde{R}_\mu(z) = z + R_\mu(1/z) = z + \sum_{n=1}^{\infty} \kappa_n(\mu) z^{1-n} \in z + \mathbb{C}[[1/z]],$$

which we will find slightly more convenient. It is known (see [4], [23], or [25]) that $\tilde{R}_\mu(z)$ is the unique solution of the equation

$$\left(\frac{1}{S_\mu(z)}\right) \circ \widetilde{R}_\mu(z) = z. \quad \left(\text{Equivalently: } \widetilde{R}_\mu(z) \circ \left(\frac{1}{S_\mu(z)}\right) = z.\right) \tag{5}$$

Since $z + \mathbb{C}[[1/z]]$ is a group under composition, the modified R-transform $\widetilde{R}_\mu(z)$ is well-defined for every law μ, hence the sequence $\{\kappa_n(\mu)\}_{n=1}^{\infty}$ of free cumulants is defined, and it uniquely determines μ. Furthermore, in our setup the free cumulants of a law can be arbitrarily prescribed.

The next lemma expresses algebraicity in terms of free cumulants.

Lemma 2.3.5. *Let $\mu : \mathbb{C}\langle\mathbf{X}\rangle \to \mathbb{C}$ be a law. Then the following statements are equivalent:*

(I) *μ is algebraic.*
(II) *$S_\mu(z)$ is algebraic.*
(III) *$\widetilde{R}_\mu(z)$ is algebraic.*

Proof. The equivalence (I)$\Leftrightarrow$(II) holds by definition. The equivalence (II)$\Leftrightarrow$(III) holds by Lemma 2.2.7 and (5). $\square$

2.4. **An algebraicity criteria.** We present a simple and general algebraicity criterion in which ultimately we will check hypotheses to prove Theorem 1.

2.4.1. *Setup for the criterion.* Let K/K_0 be an extension of fields. Although we have the extension $\mathbb{C}((1/z))/\mathbb{C}(z)$ uppermost in mind, for the moment we work with a general extension of fields. Let $x = (x_1, \ldots, x_n)$ be an n-tuple of independent (commuting) variables and let $K_0[x]$ denote the polynomial ring generated over K_0 by these variables. Let $f = (f_1, \ldots, f_n) = f(x) \in K_0[x]^n$ be an n-tuple of polynomials. Let $J(x) = \det \frac{\partial f}{\partial x} \in K_0[x]$ be the determinant of the Jacobian matrix of f. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in K^n$ be an n-tuple such that $f(\alpha) = 0$ but $J(\alpha) \neq 0$.

Proposition 2.4.2. *Notation and assumptions are as above. Every entry of the vector α is algebraic over K_0.*

This is a mildly simplified version of [16, Prop. 5.1], with a somewhat different proof. The basic idea is the same.

Proof. We may assume without loss of generality that K_0 is algebraically closed. Let $P \subset K_0[x]$ be the kernel of evaluation at α, which is a prime ideal. Since $J(\alpha) \neq 0$, we have $J \notin P$. Thus by Hilbert's Nullstellensatz there exists a point $a = (a_1, \ldots, a_n) \in K_0^n$ at which every polynomial belonging to P vanishes, in particular $f(a) = 0$, but $J(a) \neq 0$. After a linear change of coordinates we may assume that $a = 0$. Let $M = (x_1, \ldots, x_n) \subset K_0[x_1, \ldots, x_n]$. By construction we have $P \subset M$ but $J \notin M$. It suffices now to prove that $M = P$. In any case, because $J(0) \neq 0$, every polynomial vanishing at the origin can be approximated to arbitrarily high order at the origin by a polynomial without constant term in the given polynomials $f_1, \ldots, f_n$. More precisely, given $g \in M$ and a positive integer k, there exists $h = h(x_1, \ldots, x_n) \in M$ such that $g - h(f_1, \ldots, f_n) \in M^k$. Moreover, clearly, we have $h(f_1, \ldots, f_n) \in P$. Thus $M \subset \bigcap_{k=1}^{\infty}(P + M^k) \subset M$ and hence $M = \bigcap_{k=1}^{\infty}(P + M^k)$. Now a well-known theorem of Krull asserts that for any noetherian domain A and nonunit ideal $I \subset A$ one has $\bigcap_{k=1}^{\infty} I^k = (0)$. (See, e.g., [7, Cor. 10.18, p. 110], [19, Cor. 3, Chap. 4, p. 70], or [18, Thm. 5.6, Chap. X, §5, p. 429].) Krull's theorem applied to the noetherian domain $K_0[x]/P$ and nonunit ideal M/P yields the relation $P = \bigcap_{k=1}^{\infty}(P + M^k)$. Thus we indeed have $M = P$. □

2.4.3. *Remark.* The text [18] by Lang does not give exactly the required version of Krull's result but one can easily obtain it using the materials presented there. In fact all the commutative algebra needed to prove Proposition 2.4.2 is developed in Lang. No material beyond that level is required here.

2.4.4. *Remark.* Proposition 2.4.2 gives a much simpler approach to algebraicity than that used in [3], as would not be surprising in light of our remarks in the introduction.

2.5. **Large-scale organization of the proof of Theorem 1.** We recall the generalized Schwinger-Dyson equation and then we state two technical results about it which together imply Theorem 1.

2.5.1. *The generalized Schwinger-Dyson equation.* In the two technical propositions to be formulated below we consider an instance

$$I_n + a^{(0)}g + \sum_{\theta=1}^{q}\sum_{j=2}^{\infty} \kappa_j^{(\theta)}(a^{(\theta)}g)^j = 0 \tag{6}$$

of the *generalized Schwinger-Dyson equation* for which the data are

$$\left\{\begin{array}{l} \text{positive integers } q \text{ and } n, \\ \text{a matrix } a^{(0)} \in \mathrm{Mat}_n(\mathbb{C}((1/z))), \\ \text{matrices } a^{(1)}, \ldots, a^{(q)} \in \mathrm{Mat}_n(\mathbb{C}), \\ \text{a matrix } g \in \mathrm{Mat}_n(\mathbb{C}((1/z))), \text{ and} \\ \text{a family } \left\{\left\{\kappa_j^{(\theta)}\right\}_{j=2}^{\infty}\right\}_{\theta=1}^{q} \text{ of complex numbers.} \end{array}\right. \tag{7}$$

We assume that

$$\lim_{j\to\infty} \mathrm{val}\,(a^{(\theta)}g)^j = -\infty \ \text{ for } \theta = 1, \ldots, q \tag{8}$$

in order that the left side of (6) have a well-defined value in $\mathrm{Mat}_n(\mathbb{C}((1/z)))$. Furthermore, we impose the following nondegeneracy condition which is perhaps not so often studied:

(9) The linear map

$$\left(h \mapsto a^{(0)}h + \sum_{\theta=1}^{q}\sum_{j=2}^{\infty}\sum_{\nu=0}^{j-1} \kappa_j^{(\theta)}(a^{(\theta)}g)^{\nu}(a^{(\theta)}h)(a^{(\theta)}g)^{j-1-\nu}\right)$$

$: \mathrm{Mat}_n(\mathbb{C}((1/z))) \to \mathrm{Mat}_n(\mathbb{C}((1/z)))$ is invertible.

Note that the map in (9) is at least well-defined by (8).

2.5.2. *Remark.* Note that the linear map considered in (9) has the form of a derivative. This observation suggests correctly that (9) is the key to checking the nondegeneracy hypothesis in Proposition 2.4.2.

2.5.3. *Remark.* The generalized Schwinger-Dyson equation is especially familiar in the case that $\kappa_j^{(\theta)} = 0$ for $j > 2$. In that case (6) arises naturally in the study of free matrix-polynomial combinations of semicircular variables. See, e.g., [1], [4], [8], [12], [13], [23] and [25].

We will prove the following results.

Proposition 2.5.4. *Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. Let*

$$x_1, \dots, x_q \in \mathcal{A}$$

be freely independent noncommutative random variables. Let

$$X \in \mathrm{Mat}_p(\mathbb{C}\langle x_1, \dots, x_q\rangle) \subset \mathrm{Mat}_p(\mathcal{A})$$

be a matrix. (To this point we merely repeat the setup for Theorem 1.) For indices $\theta = 1, \dots, q$ and $j = 2, 3, 4, \dots$ let $\kappa_j^{(\theta)}$ denote the j^{th} free cumulant of the law of the noncommutative random variable x_θ. Then the family $\left\{\left\{\kappa_j^{(\theta)}\right\}_{\theta=1}^{q}\right\}_{j=2}^{\infty}$ of complex numbers for some integer $n > p$ can be completed to a family

$$\left(q, n, a^{(0)}, \left\{a^{(\theta)}\right\}_{\theta=1}^{q}, g, \left\{\left\{\kappa_j^{(\theta)}\right\}_{j=2}^{\infty}\right\}_{\theta=1}^{q}\right)$$

of the form (7) satisfying (6), (8), and (9) along with the further conditions

$$a_0 \in \mathrm{Mat}_n(\mathbb{C}[z]) \;\; s.t. \;\; \frac{d^2}{dz^2}a_0 = 0 \;\; and \tag{10}$$

$$S_{\mu_X} = -\frac{1}{p}\sum_{i=1}^{p} g(i,i). \tag{11}$$

We will prove this proposition §4 and in §5 below by using the self-adjoint linearization trick along with ideas related to random walk on infinite trees.

Proposition 2.5.5. *Let data of the form (7) satisfy (6), (8), and (9). Assume furthermore that*

$$a_0 \in \mathrm{Mat}_n(\mathbb{C}(z)) \;\; and \tag{12}$$

$$\textstyle\sum_{j=2}^{\infty} \kappa_{j+1}^{(\theta)} z^{-j} \in \mathbb{C}((1/z)) \;\; is\ algebraic\ for\ \theta = 1, \dots, q. \tag{13}$$

Then every entry of the matrix g is algebraic.

Proposition 2.5.5 will be proved by using technical ideas emerging from the theory of random walk on infinite trees along with tools from algebraic geometry and commutative algebra. Its proof takes up the remainder of the paper from §6 onward.

In view of Lemma 2.3.5, it is clear that Propositions 2.5.4 and 2.5.5 together imply Theorem 1.

2.5.6. *Remark.* The viewpoint on free matrix-polynomial combinations of noncommutative random variables expressed in Propositions 2.5.4 and 2.5.5 is that of operator-valued free probability theory, much the same as in, e.g., [8]. The motivating ideas for the proofs of Propositions 2.5.4 and 2.5.5 are drawn from the literature of random walk on free groups and infinite trees, typical examples from which being [16], [17] and [21]. Several further tools taken "off the shelf" will be needed to overcome obstructions coming up along the way, e.g., Newton polygons and the Weierstrass Preparation Theorem. Ultimately the novelty in the proof of Theorem 1 is in the stitching rather than in the fabric.

3. Hessenberg-Toeplitz matrices and free cumulants

We introduce the formal version of operator theory used in this paper and then as an illustration we revisit a key insight of Voiculescu concerning the free cumulants.

3.1. **The algebras $\mathfrak{M}$ and $\mathfrak{M}((1/z))$.** We introduce two algebras of infinite matrices, the first an algebra over $\mathbb{C}$ and the second a larger algebra over $\mathbb{C}((1/z))$ possessing Banach algebra structure.

3.1.1. *Notation.* Let $\mathbb{N}$ denote the set of **nonnegative** integers.

3.1.2. *The algebra* $\mathfrak{M}$. Let $\mathfrak{M}$ denote the vector space over $\mathbb{C}$ consisting of $\mathbb{N}$-by-$\mathbb{N}$ matrices M such that for each $j \in \mathbb{N}$ there exist only finitely many $i \in \mathbb{N}$ such that $M(i,j) \neq 0$. Every upper-triangular $\mathbb{N}$-by-$\mathbb{N}$ matrix with entries in $\mathbb{C}$ belongs to $\mathfrak{M}$. Informally, $\mathfrak{M}$ consists of the "almost upper-triangular" matrices. It is easy to see that matrix multiplication is well-defined on $\mathfrak{M}$ and moreover associative, thus making $\mathfrak{M}$ into a unital associative algebra with scalar field $\mathbb{C}$. Indeed, it is clear that $\mathfrak{M}$ is a copy of the algebra of linear endomorphisms of a complex vector space of countably infinite dimension. We write $\mathbf{1} = 1_{\mathfrak{M}}$ to abbreviate notation. We equip $\mathfrak{M}$ with the state $\phi(M) = M(0,0)$, thus defining a noncommutative probability space $(\mathfrak{M}, \phi)$.

3.1.3. *The algebra* $\mathfrak{M}((1/z))$. Let $\mathfrak{M}((1/z))$ denote the set of $\mathbb{N}$-by-$\mathbb{N}$ matrices M with entries in $\mathbb{C}((1/z))$ satisfying one and hence both of the following equivalent conditions:

- There exists a *Laurent expansion* $M = \sum_{n\in\mathbb{Z}} M_n z^n$ with coefficients $M_n \in \mathfrak{M}$ such that $M_n = 0$ for $n \gg 0$.
- One has $\lim_{i\to\infty} \operatorname{val} M(i,j) = -\infty$ for each $j \in \mathbb{N}$ (without any requirement of uniformity in j) and furthermore one has $\sup_{i,j\in\mathbb{N}} \operatorname{val} M(i,j) < \infty$.

From the equivalent points of view described above it is clear that $\mathfrak{M}((1/z))$ becomes a unital $\mathbb{C}((1/z))$-algebra with respect to the usual notion of matrix multiplication. For $M \in \mathfrak{M}((1/z))$ we define $\operatorname{val} M = \sup_{i,j\in\mathbb{N}} \operatorname{val} M(i,j)$. With respect to the valuation function val thus extended to $\mathfrak{M}((1/z))$ the latter becomes a unital Banach algebra over $\mathbb{C}((1/z))$. We write $\mathbf{1} = 1_{\mathfrak{M}} = 1_{\mathfrak{M}((1/z))}$.

3.1.4. *Elementary matrices and an abuse of notation.* Let $\mathbf{e}[i,j] \in \mathfrak{M}$ denote the elementary matrix with entries given by the rule

$$\mathbf{e}[i,j](k,\ell) = \delta_{ik}\delta_{j\ell} \text{ for } i,j,k,\ell \in \mathbb{N}.$$

The square-bracket notation is intended to contrast with the notation $\mathbf{e}_{ij} \in \mathrm{Mat}_{k\times\ell}(\mathbb{C})$ previously introduced for elementary matrices with finitely many rows and columns. For $M \in \mathfrak{M}$ supported in a set $S \subset \mathbb{N}\times\mathbb{N}$ we abuse notation by writing

$$M = \sum_{(i,j)\in S} M(i,j)\mathbf{e}[i,j]$$

as a convenient shorthand to indicate placement of entries.

The following simple lemma is a key motivation for the definition of $\mathfrak{M}((1/z))$.

Lemma 3.1.5. *Fix $M \in \mathfrak{M}$ arbitrarily and let μ denote the law of M. Then the matrix*

$$z\mathbf{1} - M \in \mathfrak{M}((1/z))$$

is invertible and

$$(z\mathbf{1} - M)^{-1}(0,0) = S_\mu(z).$$

Proof. One has

$$(z\mathbf{1} - M)^{-1} = \frac{1}{z}\sum_{k=0}^\infty \frac{M^k}{z^k} \in \mathfrak{M}((1/z))$$

by the usual geometric series argument in a Banach algebra, and this noted, it is clear that $(z\mathbf{1} - M)^{-1}(0,0)$ is the Stieltjes transform of the law of M. □

3.2. **Hessenberg-Toeplitz matrices.**

3.2.1. *Basic definitions.* Let $\{\kappa_j\}_{j=1}^\infty$ be any sequence of complex numbers. Consider the infinite matrix

$$C = \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 & \dots & & & & \\ 1 & \kappa_1 & \kappa_2 & \kappa_3 & \dots & & & \\ & 1 & \kappa_1 & \kappa_2 & \kappa_3 & \dots & & \\ & & 1 & \kappa_1 & \kappa_2 & \kappa_3 & \dots & \\ & & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathfrak{M}. \tag{14}$$

Equivalently, in terms of the elementary matrices $\mathbf{e}[i,j] \in \mathfrak{M}$ we have

$$C = \sum_{k\in\mathbb{N}}\left(\mathbf{e}[1+k,k] + \sum_{j\in\mathbb{N}} \kappa_{j+1}\mathbf{e}[k,j+k]\right). \tag{15}$$

The matrix C displays the (upper) *Hessenberg* pattern: $i > j+1 \Rightarrow C(i,j) = 0$ for $i,j \in \mathbb{N}$. The matrix C also displays the *Toeplitz* pattern: $C(i+1,j+1) = C(i,j)$ for $i,j \in \mathbb{N}$. Accordingly we call C a *Hessenberg-Toeplitz matrix.*

The reason for our interest in the matrix C is explained by the next lemma.

Lemma 3.2.2. *Assumptions and notation are as above. Then for every positive integer j the j^{th} free cumulant of C viewed as a noncommutative random variable in the noncommutative probability space $(\mathfrak{M},\phi)$ equals κ_j.*

This fact is well-known—it is a key insight for Voiculescu's theory of the R-transform. It is therefore not necessary to give a proof. But we nevertheless give a proof in §3.4 below to prepare the reader for the more difficult calculations undertaken in §5 below.

3.3. **Inversion of block-decomposed matrices.** We pause to review a method of calculation used repeatedly in the sequel.

3.3.1. *Basic identities.* Let

$$\left[\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{array}\right]$$

be an invertible square matrix (in practice infinite) decomposed into blocks where $\mathbf{a}$ and $\mathbf{d}$ are square and $\mathbf{d}$ is also invertible. Then we have a factorization

$$\left[\begin{array}{cc} \mathbf{a}-\mathbf{b}\mathbf{d}^{-1}\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{d} \end{array}\right] = \left[\begin{array}{cc} \mathbf{1} & -\mathbf{b}\mathbf{d}^{-1} \\ \mathbf{0} & \mathbf{1} \end{array}\right] \left[\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{array}\right] \left[\begin{array}{cc} \mathbf{1} & \mathbf{0} \\ -\mathbf{d}^{-1}\mathbf{c} & \mathbf{1} \end{array}\right] \tag{16}$$

from which in particular we infer that the *Schur complement* $\mathbf{a}-\mathbf{b}\mathbf{d}^{-1}\mathbf{c}$ is invertible. Let

$$\mathbf{g} = (\mathbf{a}-\mathbf{b}\mathbf{d}^{-1}\mathbf{c})^{-1}.$$

From (16) one then straightforwardly derives the inversion formula

$$\left[\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{array}\right]^{-1} = \left[\begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}^{-1} \end{array}\right] + \left[\begin{array}{c} \mathbf{1} \\ -\mathbf{d}^{-1}\mathbf{c} \end{array}\right] \mathbf{g} \left[\begin{array}{cc} \mathbf{1} & -\mathbf{b}\mathbf{d}^{-1} \end{array}\right]. \tag{17}$$

The latter formula also shows that invertibility of $\mathbf{d}$ and $\mathbf{a}-\mathbf{b}\mathbf{d}^{-1}\mathbf{c}$ implies invertibility of $\left[\begin{array}{cc} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{array}\right]$. For convenient application in §5, we restate in abstract form a couple of relations among blocks recorded in formula (17).

Lemma 3.3.2. *Let $\mathcal{A}$ be a unital associative algebra (perhaps not commutative). Let $\pi, \sigma \in \mathcal{A}$ satisfy $\pi^2 = \pi \neq 0$, $\sigma^2 = \sigma \neq 0$, $\pi\sigma = \sigma\pi = 0$ and $1 = 1_{\mathcal{A}} = \pi + \sigma$. Let $A \in \mathcal{A}$ be invertible. Assume furthermore that $\sigma A \sigma$ is invertible in the algebra $\sigma\mathcal{A}\sigma$ and let A_σ^{-1} denote the inverse. Then we have*

$$\sigma A^{-1}\pi = -A_\sigma^{-1} A \pi A^{-1} \pi \quad \textit{and} \tag{18}$$

$$A^{-1}\sigma = (1_{\mathcal{A}} - A^{-1}\pi A \sigma) A_\sigma^{-1}. \tag{19}$$

Proof. We have

$$\sigma A \sigma A^{-1} \pi = -\sigma A A^{-1} \pi + \sigma A \sigma A^{-1} \pi = -\sigma A \pi A^{-1} \pi.$$

Now left-multiply extreme terms by A_σ^{-1} to recover (18). Similarly, we have

$$\sigma = \sigma A A_\sigma^{-1} = (A - \pi A \sigma) A_\sigma^{-1}.$$

Now left-multiply extreme terms by A^{-1} to recover (19). □

3.4. **Proof of Lemma 3.2.2.** Consider the Laurent series

$$f = f(z) = z + \sum_{j=1}^{\infty} \kappa_j z^{j-1} \in z + \mathbb{C}[[1/z]].$$

We must prove that f is equal to the modified R-transform of the law of C. Consider also the Stieltjes transform

$$g = g(z) = S_{\mu_C}(z) \in (1/z) + (1/z^2)\mathbb{C}[[1/z]]$$

of the law of C. Since $z + \mathbb{C}[[1/z]]$ forms a group under composition, it will suffice to show that

$$z = f \circ \frac{1}{g},$$

or equivalently

$$z = g^{-1} + \sum_{j=1}^{\infty} \kappa_j g^{j-1},$$

or equivalently

$$1 = (z - \kappa_1)g - \sum_{j=2}^{\infty} \kappa_j g^j.$$

Let

$$A = z\mathbf{1} - C \in \mathfrak{M}((1/z)).$$

By Lemma 3.1.5 the inverse

$$G = A^{-1} \in \mathfrak{M}((1/z))$$

exists and furthermore

$$g = G(0, 0).$$

In view of the relation

$$1 = \sum_{k \in \mathbb{N}} A(0, k)G(k, 0)$$

holding because $G = A^{-1}$, it will be enough simply to prove that

$$G(i, 0) = g^{i+1} \text{ for } i \in \mathbb{N}. \tag{20}$$

Now with an eye toward applying (17) above, consider the block decomposition

$$A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$$

where

$$\mathbf{a} = z - \kappa_1, \;\; \mathbf{b} = -\begin{bmatrix} \kappa_2 & \kappa_3 & \dots \end{bmatrix}, \;\; \mathbf{c} = -\begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}, \text{ and } \mathbf{d} = A.$$

By (17) we have

$$\begin{bmatrix} G(1,0) \\ G(2,0) \\ \vdots \end{bmatrix} = -\mathbf{d}^{-1}\mathbf{c}\mathbf{g} = \begin{bmatrix} G(0,0) \\ G(1,0) \\ \vdots \end{bmatrix} g,$$

whence (20). The proof is complete. □

4. The linearization step

In this section we apply the the self-adjoint linearization trick to a suitable model of free noncommutative random variables with prescribed free cumulants, thus advancing the proof of Proposition 2.5.4.

4.1. **Stars and diamonds.**

We build a model for the free unital associative monoid on q generators for which $\mathbb{N}$ is the underlying set. Using this monoid structure we will be able to construct and manipulate usefully patterned matrices in $\mathfrak{M}((1/z))$.

4.1.1. *Improper representations to the base q.* Suppose at first that $q > 1$. In grade school one learns to represent nonnegative integers to the base q using place-notation and digits selected from the set $\{0, \ldots, q-1\}$. It is not hard to see that using instead digits selected from the set $\{1, \ldots, q\}$ one still gets a unique representation for every member of $\mathbb{N}$, it being understood that 0 is represented by the empty digit string $\emptyset$. A representation to the base q of a nonnegative integer using digits $\{1, \ldots, q\}$ will be called *improper*. Improper representations to the base q make sense also for $q = 1$. In the latter extreme case each $x \in \mathbb{N}$ is represented by a string of 1's of length x.

4.1.2. *Example: counting improperly to the base* 3.

$$\emptyset, 1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, \ldots$$

4.1.3. *The degree function* $\deg_q x$. Let $\deg = \deg_q : \mathbb{N} \to \mathbb{N}$ be the function characterized by the inequality

$$\frac{q^{\deg x} - 1}{q - 1} \leq x \leq q \cdot \left(\frac{q^{\deg x} - 1}{q - 1} \right).$$

In the extreme case $q = 1$ it is understood that one should evaluate the bounds by L'Hôpital's Rule. Informally, $\deg x$ is simply the number of digits in the improper expansion of x to the base q.

4.1.4. *The binary operation* $\star_q$. We define the binary operation

$$\star = \star_q : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

by the rule

$$x \star y = x q^{\deg y} + y.$$

Informally, in terms of improper representations to the base q, the operation $\star$ is simply that of concatenating digit strings. The operation $\star$ makes $\mathbb{N}$ into a free associative monoid freely generated by the digits $1, \ldots, q$ with 0 as the identity element.

Lemma 4.1.5. $\mathbb{N} \setminus \{0\}$ *is the disjoint union of the sets* $\mathbb{N} \star \theta$ *for* $\theta = 1, \ldots, q$.

There is nothing to prove. We record this for convenient reference.

4.1.6. *The binary operation* $\diamond_q$. We define the binary operation

$$\diamond = \diamond_q : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

by the formula

$$x^{\diamond y} = \underbrace{x \star \cdots \star x}_{y}.$$

We use exponential-style notation to emphasize the analogy with exponentiation.

Lemma 4.1.7. *For $\theta = 1, \ldots, q$, every $x \in \mathbb{N}$ has a unique factorization $x = \theta^{\diamond i} \star k$ where $i \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \theta \star \mathbb{N}$.*

Again, there is nothing to prove. We record this for convenient reference.

4.1.8. *Remark.* Consider the graph $\Gamma = \Gamma_q$ with vertex set $\mathbb{N}$ and an edge connecting x to $\theta \star x$ for all pairs $(\theta, x) \in \{1, \ldots, q\} \times \mathbb{N}$. With $0 \in \mathbb{N}$ designated as the root, the resulting graph Γ is an infinite rooted planar tree in which every vertex has a "birth-ordered" set of q children, i.e., a q-ary rooted tree. We do not explicitly use the q-ary tree in this paper because we rely on the monoid $(\mathbb{N}, \star)$ directly. Nonetheless the q-ary tree is an important source of intuition.

4.2. **Kronecker products and the isomorphism $\natural$.** We introduce notation which is tedious to define but convenient to calculate with.

4.2.1. *Classical Kronecker products.* Recall that for matrices of finite size the *Kronecker product*

$$A^{(1)} \otimes A^{(2)} \in \mathrm{Mat}_{k_1k_2 \times \ell_1\ell_2}(\mathbb{C}) \quad \left(A^{(\alpha)} \in \mathrm{Mat}_{k_\alpha \times \ell_\alpha}(\mathbb{C}) \ \text{ for } \alpha = 1, 2\right)$$

is defined by the rule

$$A^{(1)} \otimes A^{(2)} = \begin{bmatrix} A^{(1)}(1,1)A^{(2)} & \ldots & A^{(1)}(1,\ell_1)A^{(2)} \\ \vdots & & \vdots \\ A^{(1)}(k_1,1)A^{(2)} & \ldots & A^{(1)}(k_1,\ell_1)A^{(2)} \end{bmatrix}.$$

4.2.2. *Kronecker products involving infinite matrices.* In the mixed infinite/finite case we define the *Kronecker product*

$$x \otimes a \in \mathfrak{M}((1/z)) \quad (x \in \mathfrak{M}((1/z)) \ \text{ and } \ a \in \mathrm{Mat}_n(\mathbb{C}((1/z))))$$

by the rule

$$x \otimes a = \begin{bmatrix} x(0,0)a & x(0,1)a & \ldots \\ x(1,0)a & x(1,1)a & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

or equivalently and more explicitly

$$(x \otimes a)(i_1n + i_2 - 1, j_1n + j_2 - 1) = x(i_1, j_1)a(i_2, j_2) \qquad \text{for } i_1, j_1 \in \mathbb{N} \text{ and } i_2, j_2 = 1, \ldots, n.$$

We also define

$$a \otimes x \in \mathfrak{M}((1/z)) \quad (x \in \mathfrak{M}((1/z)) \ \text{ and } \ a \in \mathrm{Mat}_n(\mathbb{C}((1/z))))$$

by the somewhat ungainly iterated index formula

$$((a \otimes x)(i_1, j_1))(i_2, j_2) = a(i_1, j_1)x(i_2, j_2)) \qquad \text{for } i_1, j_1 \in \mathbb{N} \text{ and } i_2, j_2 = 1, \ldots, n.$$

4.2.3. *The operation* $\natural$. For $M \in \mathrm{Mat}_n(\mathfrak{M}((1/z)))$ we define $M^\natural \in \mathfrak{M}((1/z))$ by the formula

$$M^\natural = \sum_{i_1,j_1\in\mathbb{N}} \sum_{i_2,j_2=1}^{n} M(i_2,j_2)(i_1,j_1)\mathbf{e}[ni_1+i_2-1, nj_2+j_2-1]$$

thus defining an isometric isomorphism

$$(M \mapsto M^\natural) : \mathrm{Mat}_n(\mathfrak{M}((1/z))) \to \mathfrak{M}((1/z))$$

of Banach algebras over $\mathbb{C}((1/z))$, where the source algebra is given Banach structure by declaring that $\mathrm{val}\, A = \max_{i,j=1}^n \mathrm{val}\, A(i,j)$ for $A \in \mathrm{Mat}_n(\mathfrak{M}((1/z)))$. Finally, note that

$$(a \otimes x)^\natural = x \otimes a \quad \text{for } a \in \mathrm{Mat}_n(\mathbb{C}((1/z))) \text{ and } x \in \mathfrak{M}((1/z)). \tag{21}$$

Thus the operation $\natural$ has a natural interpretation as exchange of tensor factors.

4.3. **Digital linearization.** Here is the main result in this section.

Proposition 4.3.1. *Let* $(\mathcal{A}, \phi)$ *be a noncommutative probability space. Let*

$$x_1, \dots, x_q \in \mathcal{A}$$

be freely independent noncommutative random variables. Fix a matrix

$$X \in \mathrm{Mat}_p(\mathbb{C}\langle x_1, \dots, x_q\rangle) \subset \mathrm{Mat}_p(\mathcal{A}).$$

Let

$$\kappa_j^{(\theta)} = \kappa_j(\mu_{x_\theta}) \ \text{ for } j = 1, 2, \dots \text{ and } \theta = 1, \dots, q.$$

Then for some integer $N > 0$ *there exist matrices*

$$L_0, L_1, \dots, L_q \in \mathrm{Mat}_{p+N}(\mathbb{C})$$

all of which vanish identically in the upper left p*-by-*p *block such that*

$$\begin{aligned} L = \mathbf{1} \otimes \left(L_0 + \begin{bmatrix} zI_p & 0 \\ 0 & 0 \end{bmatrix}\right) + \sum_{\theta=1}^{q} \sum_{k\in\mathbb{N}} \mathbf{e}[\theta \star k, k] \otimes L_\theta \\ + \sum_{\theta=1}^{q} \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} \kappa_{j+1}^{(\theta)} \mathbf{e}[k, \theta^{\diamond j} \star k] \otimes L_\theta \in \mathfrak{M}((1/z)) \ \text{ is invertible and} \end{aligned} \tag{22}$$

$$S_{\mu_X}(z) = \frac{1}{p} \sum_{i=0}^{p-1} L^{-1}(i,i). \tag{23}$$

We call L a *digital linearization* of X. It is worth remarking that the linearization is thoroughgoing in the sense that not only do the variables $x_1, \dots, x_q$ appear linearly—so also does the variable z. This proposition has in many respects a similarity in form to [8, Thm. 4.1] although the setups are vastly different. The proof will be completed in §4.6 below.

4.3.2. *Remark.* Picking up again on the idea mentioned in §4.1.8, and adopting the absurd point of view that probabilities can be square matrices with arbitrary complex number entries, the matrix L describes a random walk on the q-ary tree Γ_q such that from a given vertex $x \in \mathbb{N}$, one may (i) step one unit back toward the root (if not already at the root), (ii) stay in place, or (iii) move away from the root arbitrarily far along along a geodesic $\{\theta^{\diamond i} \star x \mid i \in \mathbb{N}\}$ for some $\theta \in \{1, \dots, q\}$. Whether or not this interpretation of L is absurd, it is does make random walk intuition available to analyze L. Guided by this intuition we will prove in §5 below that the generalized Schwinger-Dyson equation holds upper left corners of matrices of the form L, as well as for more general infinite matrices.

4.4. **Free random variables with prescribed free cumulants.** We exhibit a model for q free noncommutative random variables with prescribed free cumulants. The model is described by Proposition 4.4.3 below.

4.4.1. *Self-embeddings of* $\mathfrak{M}$. For $\theta = 1, \dots q$ and $A \in \mathfrak{M}$ we define

$$A^{(\theta)} = \sum_{k \in \mathbb{N} \setminus \theta \star \mathbb{N}} \sum_{i,j \in \mathbb{N}} A(i,j) \mathbf{e}[\theta^{\diamond i} \star k, \theta^{\diamond j} \star k]. \tag{24}$$

By Lemma 4.1.7 the matrix $A^{(\theta)}$ is block-diagonal with copies of A indexed by $\mathbb{N} \setminus \theta \star \mathbb{N}$ repeated along the diagonal. Thus the map $(A \mapsto A^{(\theta)}) : \mathfrak{M} \to \mathfrak{M}$ is a unital one-to-one homomorphism of algebras. Note that $A^{(\theta)}(0,0) = A(0,0)$ and hence the map $A \mapsto A^{(\theta)}$ is law-preserving. Let $\mathfrak{M}^{(\theta)}$ denote the embedded image of $\mathfrak{M}$ under the map $A \mapsto A^{(\theta)}$.

Lemma 4.4.2. *The subalgebras* $\mathfrak{M}^{(1)}, \dots, \mathfrak{M}^{(q)} \subset \mathfrak{M}$ *are freely independent.*

Proof. Fix $\theta_1, \dots, \theta_k \in \{1, \dots, q\}$ such that

$$\theta_1 \neq \theta_2, \;\; \theta_2 \neq \theta_3 \;\; , \dots , \;\; \theta_{k-1} \neq \theta_k.$$

Fix $A_1, \dots, A_k \in \mathfrak{M}$ such that

$$A_1(0,0) = \cdots = A_k(0,0) = 0.$$

Our task is to verify that

$$(A_1^{(\theta_1)} \cdots A_k^{(\theta_k)})(0,0) = 0. \tag{25}$$

Now by definition, for any matrix $A \in \mathfrak{M}$ such that $A(0,0) = 0$ and $\theta = 1, \dots, q$, the matrix entry $A^{(\theta)}(i,j)$ vanishes for $i, j \in \mathbb{N} \setminus \theta \star \mathbb{N}$. Using this observation, one can then verify inductively for $\ell = 1, \dots, k$ that the top row of $A^{(\theta_1)} \cdots A^{(\theta_\ell)}$ is supported in columns indexed by $\theta_\ell \star \mathbb{N}$. For $\ell = k$ this proves (25). □

Proposition 4.4.3. *Let* $\{\{\kappa_j^{(\theta)}\}_{j=1}^\infty\}_{\theta=1}^q$ *be any family of complex numbers. Then the family*

$$\sum_{k \in \mathbb{N}} \mathbf{e}[\theta \star k, k] + \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \kappa_{j+1}^{(\theta)} \mathbf{e}[k, \theta^{\diamond j} \star k] \in \mathfrak{M} \;\; \textit{for } \theta = 1, \dots, q \tag{26}$$

of noncommutative random variables is freely independent and moreover the j^{th} *free cumulant of the* θ^{th} *noncommutative random variable equals* $\kappa_j^{(\theta)}$.

Proof. Let C_θ be a copy of the matrix C defined in (14) and equivalently in (15), with κ_j specialized to $\kappa_j^{(\theta)}$. By Lemma 3.2.2 we know already that the j^{th} free cumulant of the law of C_θ equals $\kappa_j^{(\theta)}$. Substituting directly into the definition (24) we have

$$C_\theta^{(\theta)} = \sum_{i\in\mathbb{N}} \sum_{k\in\mathbb{N}\backslash\theta\star\mathbb{N}} \mathbf{e}[\theta^{\diamond(i+1)}\star k, \theta^{\diamond i}\star k] + \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}\backslash\theta\star\mathbb{N}} \kappa_{j+1}^{(\theta)}\mathbf{e}[\theta^{\diamond i}\star k, \theta^{\diamond(i+j)}\star k].$$

The result follows now via Lemmas 4.1.7 and 4.4.2. □

4.4.4. *Remark.* Voiculescu introduced the Boltzmann-Fock space model of free random variables using lowering and raising operators for his striking proof of additivity of the R-transform for addition of free random variables. See, e.g., [4, Cor. 5.3.23] and its proof for a brief review of this background. Proposition 4.4.3 is merely an alternative description of the Boltzmann-Fock space model in terms emphasizing "arboreal" aspects, thus making recursions easily accessible. In our setup the matrices

$$\sum_{k\in\mathbb{N}} \mathbf{e}[\theta\star k, k] \in \mathfrak{M} \text{ and } \sum_{k\in\mathbb{N}} \mathbf{e}[k, \theta\star k] \in \mathfrak{M} \text{ for } \theta = 1,\dots,q$$

are the lowering and raising operators, respectively.

4.5. **Technical lemmas.** The next lemma recalls what we need of the self-adjoint linearization trick. For background see [1], [2] or [8].

Lemma 4.5.1. *For each $f \in \mathrm{Mat}_p(\mathbb{C}\langle \mathbf{X}_1,\dots,\mathbf{X}_q\rangle)$ there exists a factorization $f = bd^{-1}c$ (called a* linearization *of f) where*

$$b \in \mathrm{Mat}_{p\times N}(\mathbb{C}\langle \mathbf{X}_1,\dots,\mathbf{X}_q\rangle), c \in \mathrm{Mat}_{N\times p}(\mathbb{C}\langle \mathbf{X}_1,\dots,\mathbf{X}_q\rangle, d \in \mathrm{GL}_N(\mathbb{C}\langle \mathbf{X}_1,\dots,\mathbf{X}_q\rangle),$$

and each entry of each of these matrices belongs to the $\mathbb{C}$-linear span of $1, \mathbf{X}_1,\dots,\mathbf{X}_q$.

Proof. If every entry of f belongs to the $\mathbb{C}$-linear span of $1, \mathbf{X}_1,\dots,\mathbf{X}_q$, then, say, $f = fI_p^{-1}I_p$ is a linearization. Thus it will be enough to demonstrate that given linearizable $f_1, f_2 \in \mathrm{Mat}_p(\mathbb{C}\langle \mathbf{X}_1,\dots,\mathbf{X}_q\rangle)$, again f_1+f_2 and f_1f_2 are linearizable. So suppose that $f_i = b_id_i^{-1}c_i$ for $i = 1, 2$ are factorizations of the desired form. We then have

$$f_1 + f_2 = \left[\begin{array}{cc} b_1 & b_2 \end{array}\right] \left[\begin{array}{cc} d_1 & 0 \\ 0 & d_2 \end{array}\right]^{-1} \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] \text{ and}$$

$$f_1f_2 = \left[\begin{array}{ccc} b_1 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc} 0 & 0 & d_2 \\ 0 & 1 & b_2 \\ d_1 & c_1 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} c_2 \\ 0 \\ 0 \end{array}\right].$$

To assist the reader in checking the second formula, we note that

$$\left[\begin{array}{ccc} 0 & 0 & d_2 \\ 0 & 1 & b_2 \\ d_1 & c_1 & 0 \end{array}\right]^{-1} = \left[\begin{array}{ccc} d_1^{-1}c_1b_2d_2^{-1} & -d_1^{-1}c_1 & d_1^{-1} \\ -b_2d_2^{-1} & 1 & 0 \\ d_2^{-1} & 0 & 0 \end{array}\right].$$

Thus $f_1 + f_2$ and f_1f_2 have linearizations and consequently the lemma holds. □

The following statement slightly refines Lemma 3.1.5.

Lemma 4.5.2. *Fix $A \in \mathrm{Mat}_n(\mathfrak{M})$. Then the following statements hold:*

$zI_n \otimes \mathbf{1} - A \in \mathrm{Mat}_n(\mathfrak{M}((1/z)))$ *and* $z\mathbf{1} - A^\natural \in \mathfrak{M}((1/z))$ *are invertible.*

$\left((zI_n \otimes \mathbf{1} - A)^{-1}\right)^\natural = (z\mathbf{1} - A^\natural)^{-1}$.

$$S_{\mu_A}(z) = \frac{1}{n}\sum_{i=0}^{n-1}(z - A^\natural)^{-1}(i,i) = \left(\frac{1}{n}\sum_{i=1}^{n}(zI_n \otimes \mathbf{1} - A)^{-1}(i,i)\right)(0,0).$$

This statement supplements Lemma 3.1.5 only by some minor bookkeeping details. We therefore omit proof.

4.6. **Proof of Proposition 4.3.1.** Without loss of generality we may assume that $(\mathcal{A}, \phi)$ is the noncommutative probability space $(\mathfrak{M}, \phi)$ and we may take $\{x_\theta\}_{\theta=1}^q$ to be the family constructed in Proposition 4.4.3. Fix

$$f = f(\mathbf{X}_1, \ldots, \mathbf{X}_q) \in \mathrm{Mat}_p(\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q\rangle)$$

such that $f(x_1, \ldots, x_q) = X$, write $f = bd^{-1}c$ as in Lemma 4.5.1, and then write

$$\begin{aligned} \left[\begin{array}{cc} 0 & b \\ c & d \end{array}\right] &= L_0 \otimes 1_{\mathbb{C}\langle \mathbf{X}_1,\ldots,\mathbf{X}_q\rangle} + L_1 \otimes \mathbf{X}_1 + \cdots + L_q \otimes \mathbf{X}_q \\ &\in \mathrm{Mat}_{p+N}(\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q\rangle) \end{aligned}$$

in the unique possible way. Finally, let

$$L = \mathbf{1} \otimes \left(L_0 + \left[\begin{array}{cc} zI_p & 0 \\ 0 & 0 \end{array}\right]\right) + x_1 \otimes L_1 + \cdots + x_q \otimes L_q \in \mathfrak{M}((1/z))$$

noting that this expression when expanded in terms of elementary matrices takes by (26) the desired form (22). Let

$$B \in \mathrm{Mat}_{p\times N}(\mathfrak{M}), \quad C = \mathrm{Mat}_{N\times p}(\mathfrak{M}), \quad D \in \mathrm{GL}_N(\mathfrak{M}), \quad F \in \mathrm{Mat}_p(\mathfrak{M})$$

be the evaluations of b, c, d, f respectively at $\mathbf{X}_i = x_i$ for $i = 1, \ldots, q$. Then the matrix

$$\left[\begin{array}{cc} zI_p \otimes \mathbf{1} & B \\ C & D \end{array}\right] \in \mathrm{Mat}_{p+N}(\mathfrak{M}((1/z)))$$

is invertible since $zI_p \otimes \mathbf{1} - F$ is invertible by Lemma 4.5.2 and D, since the image of an invertible matrix under a unital algebra homomorphism, is again invertible. More precisely, we have

$$\begin{aligned} &\left[\begin{array}{cc} zI_p \otimes \mathbf{1} & B \\ C & D \end{array}\right]^{-1} - \left[\begin{array}{cc} 0 & 0 \\ 0 & D^{-1} \end{array}\right] \\ &= \left[\begin{array}{c} I_p \otimes \mathbf{1} \\ -D^{-1}C \end{array}\right](zI_p \otimes \mathbf{1} - F)^{-1}\left[\begin{array}{cc} I_p \otimes \mathbf{1} & -BD^{-1} \end{array}\right] \end{aligned} \tag{27}$$

by the inversion formula (17). In turn, by (21) we have

$$L = \left[\begin{array}{cc} zI_p \otimes \mathbf{1} & B \\ C & D \end{array}\right]^\natural,$$

hence L is invertible and moreover (23) holds by Lemma 4.5.2. The proof of Proposition 4.3.1 is complete. □

5. Solving the generalized Schwinger-Dyson equation

We finish the proof of Proposition 2.5.4 by constructing an adequate supply of solutions of the generalized Schwinger-Dyson equation.

5.1. **Statement of the construction.** Here is our main result in this section.

Proposition 5.1.1. *Fix data of the form* (7). *Consider the matrix*

$$A = -\mathbf{1}\otimes a^{(0)} - \sum_{\theta=1}^{q}\sum_{k\in\mathbb{N}}\left(\mathbf{e}[\theta\star k,k]+\sum_{j=1}^{\infty}\kappa_{j+1}^{(\theta)}\mathbf{e}[k,\theta^{\diamond j}\star k]\right)\otimes a^{(\theta)} \in \mathfrak{M}((1/z)) \tag{28}$$

constructed by using these data. Assume that

$$G = A^{-1} \in \mathfrak{M}((1/z)) \text{ exists, and} \tag{29}$$

$$g(i,j) = G(i-1,j-1) \text{ for } i,j=1,\dots,n. \tag{30}$$

Then (6), (8), *and* (9) *hold, i.e., the data* (7) *constitute a solution of the generalized Schwinger-Dyson equation.*

We complete the proof below in §5.2 after deducing Proposition 2.5.4 from Proposition 5.1.1 and proving a lemma excusing us from having to verify (9) when proving Proposition 5.1.1.

5.1.2. *Completion of the proof of Proposition 2.5.4 with Proposition 5.1.1 granted.* We identify X in Proposition 2.5.4 with X in Proposition 4.3.1. We complete the choice of positive integer q and the given family

$$\left\{\left\{\kappa_j^{(\theta)}\right\}_{\theta=1}^{q}\right\}_{j=2}^{\infty}$$

to a family

$$\left(q,n,a^{(0)},\left\{a^{(\theta)}\right\}_{\theta=1}^{q},g,\left\{\left\{\kappa_j^{(\theta)}\right\}_{j=2}^{\infty}\right\}_{\theta=1}^{q}\right) \tag{31}$$

of the form (7) where

$$\begin{array}{rcl}
n & = & p+N>p,\\
a^{(0)} & = & -\left(L_0+\left[\begin{array}{cc} zI_p & 0\\ 0 & 0\end{array}\right]-\sum_{\theta=1}^{q}L_\theta\right)\in \mathrm{Mat}_n(\mathbb{C}[z]),\\
a^{(\theta)} & = & -L^{(\theta)}\in \mathrm{Mat}_n(\mathbb{C}) \text{ for } \theta=1,\dots,q \text{ and}\\
g(i,j) & = & -L^{-1}(i-1,j-1) \text{ for } i,j=1,\dots,n.
\end{array}$$

For the family (31) the hypotheses (29) and (30) of Proposition 5.1.1 are fulfilled by (22) and the definition of g, respectively. Thus (31) is a solution of the generalized Schwinger-Dyson equation, i.e., (6), (8), and (9) hold. Property (10) holds by construction and property (11) holds by (23). The proof of Proposition 2.5.4 is complete modulo the proof of Proposition 5.1.1. □

Lemma 5.1.3. *To prove Proposition 5.1.1 it is necessary only to verify statements* (6) *and* (8) *for data* (7) *satisfying hypotheses* (29) *and* (30).

In the proof below we are recycling elements of the "secondary trick" used in [1] to obtain certain correction terms.

Proof. The weakened version of Proposition 5.1.1 delivering only conclusions (6) and (8) for data (7) satisfying (29) and (30) we will call Proposition 5.1.1−ϵ.

Our task is to derive Proposition 5.1.1 from Proposition 5.1.1$-\epsilon$. To that end fix $b \in \mathrm{Mat}_n(\mathbb{C}((1/z))$ arbitrarily and consider new data consisting of

$$\left\{\begin{array}{l}\text{a positive integer } \hat{n} = 2n \text{ (but } q \text{ the same as before)},\\ \text{a matrix } \hat{a}^{(0)} = \left[\begin{array}{cc} a^{(0)} & b \\ 0 & a^{(0)} \end{array}\right] \in \mathrm{Mat}_{\hat{n}}(\mathbb{C}((1/z))),\\ \text{matrices } \hat{a}^{(\theta)} = \left[\begin{array}{cc} a^{(\theta)} & 0 \\ 0 & a^{(\theta)} \end{array}\right] \in \mathrm{Mat}_{\hat{n}}(\mathbb{C}) \text{ for } \theta = 1, \dots, q,\\ \text{a matrix } \hat{g} = \left[\begin{array}{cc} g & h \\ 0 & g \end{array}\right] \in \mathrm{Mat}_{\hat{n}}(\mathbb{C}((1/z))) \text{ (} h \text{ to be determined), and}\\ \text{a family } \{\{\kappa_j^{(\theta)}\}_{j=2}^{\infty}\}_{\theta=1}^{q} \text{ of complex numbers (same as before).}\end{array}\right. \tag{32}$$

We will apply Proposition 5.1.1$-\epsilon$ to the new data (32) thereby deriving (9) for the old data (7). To apply Proposition 5.1.1$-\epsilon$ we need first to verify invertibility of the matrix

$$\hat{A} = -\hat{a}^{(0)} - \sum_{\theta=1}^{q} \sum_{k \in \mathbb{N}} \left(\mathbf{e}[\theta \star k, k] + \sum_{j=1}^{\infty} \kappa_{j+1}^{(\theta)} \mathbf{e}[k, \theta^{\diamond j} \star k] \right) \otimes \hat{a}^{(\theta)}.$$

Because $\natural$ is an isomorphism, there exists unique $\tilde{A} \in \mathrm{Mat}_n(\mathfrak{M}((1/z)))$ such that $(\tilde{A})^\natural = A$. Using (21) we obtain the relation

$$\hat{A} = \left[\begin{array}{cc} \tilde{A} & -b \otimes \mathbf{1} \\ 0 & \tilde{A} \end{array}\right]^\natural \in \mathfrak{M}((1/z)),$$

and we have explicitly

$$\hat{G} = \left(\left[\begin{array}{cc} \tilde{A} & -b \otimes \mathbf{1} \\ 0 & \tilde{A} \end{array}\right]^{-1} \right)^\natural = \left[\begin{array}{cc} \tilde{A}^{-1} & \tilde{A}^{-1}(b \otimes \mathbf{1})\tilde{A}^{-1} \\ 0 & \tilde{A}^{-1} \end{array}\right]^\natural.$$

Thus the new data (32) satisfy (29), and moreover there is a choice of h we can in principle read off from the last displayed line above so that hypothesis (30) is satisfied. (An explicit formula for h is not needed.) Statement (8) of Proposition 5.1.1$-\epsilon$ applied to the new data asserts that

$$\lim_{j \to \infty} \mathrm{val} \left[\begin{array}{cc} a^{(\theta)} g & a^{(\theta)} h \\ 0 & a^{(\theta)} g \end{array}\right]^j = -\infty \ \text{ for } \theta = 1, \dots, q.$$

This can also be deduced directly from (8) as it pertains to the old data (7). Finally, the key point is that by statement (6) as it pertains to the new data (32) we have

$$I_{\hat{n}} + \left[\begin{array}{cc} a^{(0)} & b \\ 0 & a^{(0)} \end{array}\right] \left[\begin{array}{cc} g & h \\ 0 & g \end{array}\right] + \sum_{\theta=1}^{q} \sum_{j=2}^{\infty} \kappa_j^{(\theta)} \left(\left[\begin{array}{cc} a^{(\theta)} & 0 \\ 0 & a^{(\theta)} \end{array}\right] \left[\begin{array}{cc} g & h \\ 0 & g \end{array}\right] \right)^j = 0.$$

Looking in the upper left corners, we obtain an identity

$$a^{(0)} h + bg + \sum_{\theta=1}^{q} \sum_{j=2}^{\infty} \sum_{\nu=0}^{j-1} \kappa_j^{(\theta)} (a^{(\theta)} g)^\nu (a^{(0)} h)(a^{(\theta)} g)^{j-1-\nu} = 0.$$

The latter equation, because b is arbitrary and g is invertible by (6) as it pertains to the old data (7), proves that (9) holds for the old data. In other words, Proposition 5.1.1$-\epsilon$ does indeed imply Proposition 5.1.1. $\square$

5.2. **Proof of Proposition 5.1.1.** In broad outline the proof is similar to the proof we previously gave for Lemma 3.2.2. But more machinery is needed.

5.2.1. *Block decompositions.* Throughout the proof it will be convenient to work with the block decompositions defined by the formulas

$$A = \sum_{i,j\in\mathbb{N}} \mathbf{e}[i,j] \otimes A\langle i,j\rangle \text{ and } G = \sum_{i,j\in\mathbb{N}} \mathbf{e}[i,j] \otimes G\langle i,j\rangle$$

where

$$A\langle i,j\rangle, G\langle i,j\rangle \in \mathrm{Mat}_n(\mathbb{C}((1/z))).$$

The key point is contained in the following result.

Lemma 5.2.2. *We have*

$$G\langle \theta_1 \star \cdots \star \theta_k, 0\rangle = ga^{(\theta_1)}ga^{(\theta_2)}\cdots ga^{(\theta_k)}g \tag{33}$$

for $k \in \mathbb{N}$ *and* $\theta_1, \dots, \theta_k \in \{1, \dots, q\}$.

Proof. For the proof we will use Lemma 3.3.2 in the case

$$\mathcal{A} = \mathfrak{M}((1/z)), \;\; A \in \mathcal{A} : \text{ as on line (28)},$$
$$\pi = \mathbf{e}[0,0] \otimes I_n \in \mathcal{A} \text{ and } \sigma = \sum_{i=1}^{\infty} \mathbf{e}[i,i] \otimes I_n \in \mathcal{A}.$$

We will also use the matrices

$$R^{(\theta)} = \sum_{k\in\mathbb{N}} \mathbf{e}[k\star\theta, k] \otimes I_n \in \mathfrak{M} \text{ and } \hat{R}^{(\theta)} = \sum_{k\in\mathbb{N}} \mathbf{e}[k, k\star\theta] \otimes I_n \in \mathfrak{M} \quad \text{for } \theta = 1, \dots, q.$$

These matrices satisfy

$$\hat{R}^{(\theta)}R^{(\theta')} = \delta_{\theta\theta'}\mathbf{1} \text{ for } \theta, \theta' = 1, \dots, q, \tag{34}$$

$$\hat{R}^{(\theta)}AR^{(\theta')} = \delta_{\theta\theta'}A \text{ for } \theta, \theta' = 1, \dots, q, \text{ and} \tag{35}$$

$$\sum_{\theta=1}^{q} R^{(\theta)}\hat{R}^{(\theta)} = \sum_{i=1}^{\infty} \mathbf{e}[i,i] \otimes I_n, \tag{36}$$

as can be verified by straightforward calculation. It follows by (35) and (36) that

$$\sigma A \sigma = \left(\sum_{\theta=1}^{q} R^{(\theta)}\hat{R}^{(\theta)}\right) A \left(\sum_{\theta=1}^{q} R^{(\theta')}\hat{R}^{(\theta')}\right) = \sum_{\theta=1}^{q} R^{(\theta)}A\hat{R}^{(\theta)}.$$

It follows in turn that A_σ^{-1} exists and more precisely that

$$A_\sigma^{-1} = \sum_{\theta=1}^{q} R^{(\theta)}G\hat{R}^{(\theta)}, \tag{37}$$

as one verifies using (34). We furthermore have

$$\sigma A \pi A^{-1} \pi = -\sum_{\theta=1}^{q} R^{(\theta)}\mathbf{e}[0,0] \otimes a^{(\theta)}g$$

as one can immediately check. Finally, we have the following chain of equalities:

$$
\begin{aligned}
\sum_{\theta=1}^{q}\sum_{i\in\mathbb{N}}\mathbf{e}[i\star\theta,0]\otimes G\langle i\star\theta,0\rangle &= \sigma A^{-1}\pi \;=\; -A_\sigma^{-1}A\pi A^{-1}\pi \\
&= \sum_{\theta=1}^{q}\sum_{\theta'=1}^{q} R^{(\theta)}G\hat{R}^{(\theta)}R^{(\theta')}\mathbf{e}[0,0]\otimes a^{(\theta')}g \\
&= \sum_{\theta=1}^{q}\sum_{i\in\mathbb{N}}\mathbf{e}[i\star\theta,0]\otimes G\langle i,0\rangle a^{(\theta)}g.
\end{aligned}
$$

At the first step we used Lemma 4.1.5, at the second step equation (18) of Lemma 3.3.2, at the third step (37), and at the last step (34). Thus (33) holds. □

5.2.3. *Proof of* (8). By definition of $\mathfrak{M}((1/z))$ we have

$$\lim_{i\to\infty}\operatorname{val} G\langle i,0\rangle = -\infty.$$

Thus by Lemma 5.2.2 we have for $\theta = 1,\ldots,q$ that

$$\lim_{i\to\infty}\operatorname{val}\,(a^{(\theta)}g)^i = \lim_{i\to\infty}\operatorname{val} a^{(\theta)}G\langle\theta^{\diamond i},0\rangle = 0,$$

which proves statement (8).

5.2.4. *Proof of* (6). Consider the following calculation:

$$
\begin{aligned}
I_n &= \sum_{j\in\mathbb{N}} A\langle 0,j\rangle G\langle j,0\rangle \\
&= -a^{(0)}G\langle 0,0\rangle - \sum_{\theta=1}^{q}\sum_{j=1}^{\infty}\kappa_{j+1}^{(\theta)}a^{(\theta)}G\langle\theta^{\diamond j},0\rangle \\
&= -a^{(0)}g - \sum_{\theta=1}^{q}\sum_{j=1}^{\infty}\kappa_{j+1}^{(\theta)}(a^{(\theta)}g)^j.
\end{aligned}
$$

The first step holds by definition of G, the second by definition of A, and the last by Lemma 5.2.2. This calculation proves statement (6).

5.2.5. *Proof of* (9). By Lemma 5.1.3 it is necessarily the case that statement (9) holds. The proof of Proposition 5.1.1 is complete and in turn the proof of Proposition 2.5.4 is complete. □

5.2.6. *Remark.* In Lemma 5.2.2 and its proof we are exploiting the type of recursion used, for example, in [21], and used more generally in many investigations of random walk on infinite trees.

5.2.7. *Remark.* By exploiting the identities (34), (35), and (36), we are recycling some features of the proof of [1, Prop. 9].

5.2.8. *Remark.* By continuing the line of argument in the proof of Lemma 5.2.2 and using (19) of Lemma 3.3.2, it is not difficult to prove that

$$GR^{(\theta)} = (R^{(\theta)} - G\pi AR^{(\theta)})G \text{ for } \theta = 1, \dots, q. \tag{38}$$

Since statement (38) is not needed for the proof of Theorem 1, we omit its proof. It is easy to see that Lemma 5.2.2 and (38) together allow one to make every block $G\langle i,j\rangle$ explicit in terms of g and A. This possibility to make G completely explicit is worth pointing out because in principle G generalizes the Green function studied in [5] and [27]. These connections remain to be investigated.

6. Notes on Newton-Puiseux series

At this point in the paper we switch from the viewpoint of noncommutative algebra and formal operator theory to that of commutative algebra and elementary algebraic geometry. The main result of this section is Proposition 6.2.3 below.

6.1. **Newton-Puiseux series.** We review a basic device for understanding and resolving singularities of plane algebraic curves in characteristic zero.

6.1.1. *The algebraic closure of* $\mathbb{C}((1/z))$. Let

$$\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{C}((1/z^{1/n!})).$$

In other words, $\mathbb{K}$ is the field obtained by adjoining to $\mathbb{C}((1/z))$ roots of z of all orders. When discussing $\mathbb{K}$ below we will sometimes use the abbreviated notation $\mathbb{K}_0 = \mathbb{C}((1/z))$ since it is more apposite. It has long been known that $\mathbb{K}$ is the algebraic closure of $\mathbb{K}_0$. The original insight is due to Newton. See for example [6, Chap. 2, Sec. 5] or [9, Part III, Chap. 8, Sec. 3] for background. The last reference reproduces some of Newton's correspondence on this subject.

6.1.2. *Extension of the valuation function* val *to* $\mathrm{Mat}_n(\mathbb{K})$. Each element $g \in \mathbb{K}$ has by definition a unique *Newton-Puiseux expansion*

$$g = \sum_{q \in \mathbb{Q}} b_q z^q$$

with coefficients $b_q \in \mathbb{C}$ such that for some positive integer $N = N_g$ one has $b_q = 0$ unless $q \leq N$ and $q \in \frac{1}{N}\mathbb{Z}$. For each such $g \in \mathbb{K}$ we define

$$\operatorname{val} g = \sup\{q \in \mathbb{Q} \mid b_q \neq 0\} = (\text{the } \textit{valuation} \text{ of } g) \in \mathbb{Q} \cup \{-\infty\},$$

thus extending to the field $\mathbb{K}$ the valuation val we already defined on $\mathbb{K}_0$. The properties (2), (3) and (4) of the function val on $\mathbb{K}_0$ continue to hold for the extension of val to $\mathbb{K}$. But the field $\mathbb{K}$ is not complete. More generally, we extend val from $\mathrm{Mat}_n(\mathbb{K}_0)$ to $\mathrm{Mat}_n(\mathbb{K})$ by the rule $\operatorname{val} A = \max_{i,j=1}^n \operatorname{val} A(i,j)$. Then $\mathrm{Mat}_n(\mathbb{K})$ satisfies all the axioms of a Banach algebra over $\mathbb{K}$ except of course for completeness.

Lemma 6.1.3. *Let $P(y) \in \mathbb{K}_0[y]$ be a polynomial monic of degree n in a variable y with coefficients in the field $\mathbb{K}_0$. Write*

$$P(y) = \sum_{i=0}^{n} (-1)^i s_i y^{n-i} = \prod_{j=1}^{n} (y - r_j) \quad (s_i \in \mathbb{K}_0 \textit{ and } r_j \in \mathbb{K})$$

in the unique possible way such that

$$\operatorname{val} r_1 \geq \cdots \geq \operatorname{val} r_n.$$

Note that $s_0 = 1$. *Then we have*

$$\operatorname{val} s_i \leq \operatorname{val} r_1 \cdots r_i \text{ for } i = 1, \dots, n, \text{ where} \tag{39}$$

$$\text{equality holds for } i < n \text{ s.t. } \operatorname{val} r_i > \operatorname{val} r_{i+1} \text{ and also for } i = n. \tag{40}$$

Proof. Since s_i for $i > 0$ is the i^{th} symmetric function of the roots $r_1, \dots, r_n$, the result follows immediately from (2), (3) and (4). □

6.1.4. *Newton polygons.* The relations (39) and (40) are traditionally represented graphically by a *Newton polygon*. See, e.g., [6, loc. cit.] or [9, loc. cit.]. In the present setup, the associated Newton polygon can be described as follows. Assume for simplicity that $s_n \neq 0$ and hence $\operatorname{val} r_n > -\infty$. Let $\psi : [0, n] \to \mathbb{R}$ be the unique continuous piecewise linear function such that $\psi(0) = 0$ and such that for $i = 1, \dots, n$ the derivative $\psi'(t)$ evaluates to $\operatorname{val} r_i$ on the interval $(i-1, i)$. By construction $\psi(t)$ is a concave function. Relation (39) says that $\operatorname{val} s_i \leq \psi(i)$ for $i = 0, \dots, n$. Relation (40) says that $\operatorname{val} s_i = \psi(i)$ for $i \in \{0, \dots, n\}$ such that either $i \in \{0, n\}$ or $\psi'(i)$ is not defined. In other words, Lemma 6.1.3 says that the graph of $\psi(t)$ is the upper boundary of the convex hull of the set of points $\{(i, \operatorname{val} s_i) \mid i = 0, \dots, n\} \subset \mathbb{R}^2$.

Lemma 6.1.5. *Continuing in the setup of Lemma 6.1.3, and assuming for simplicity that* $s_n \neq 0$, *we have*

$$\operatorname{val} r_{n-1} \geq 0 > \operatorname{val} r_n \quad \text{if and only if} \quad \max_{i=0}^{n} \operatorname{val} s_i = \operatorname{val} s_{n-1} = \operatorname{val} s_n - \operatorname{val} r_n. \tag{41}$$

Proof. In this case the function $\psi(t)$ considered in the preceding paragraph is non-decreasing for $t \leq n-1$ and strictly decreasing for $t \geq n-1$. □

6.2. **Applications.** We now present two applications of the preceding circle of ideas both of which are crucial for the proof of Theorem 1. Strikingly, neither application refers overtly to the field $\mathbb{K}$.

Lemma 6.2.1. *Let* $A \in \operatorname{Mat}_n(\mathbb{C}((1/z)))$ *be any matrix. Write*

$$\det(I_n + tA) = 1 + \textstyle\sum_{i=1}^n e_i t^i \quad (e_i \in \mathbb{C}((1/z))).$$

The following statements are equivalent:

(I) $e_1, \dots, e_n \in (1/z)\mathbb{C}[[1/z]]$.
(II) $\lim_{k\to\infty} \operatorname{val}(A^k) = -\infty$.

Proof. To the list of statements (I) and (II) we add

(III) Every eigenvalue of A in $\mathbb{K}$ has negative valuation.

Statements (I) and (III) are equivalent by Lemma 6.1.3. It remains only to prove the equivalence (II)⇔(III). It is actually easier to prove more. We will prove the equivalence (II)⇔(III) for $A \in \operatorname{Mat}_n(\mathbb{K})$. Supposing at first that A consists of a single Jordan block, one verifies the equivalence by inspection. In general we can write $A = WJW^{-1}$ where $W \in \operatorname{GL}_n(\mathbb{K})$ and $J \in \operatorname{Mat}_n(\mathbb{K})$ is block-diagonal with diagonal blocks of the Jordan form and we have a bound

$$\left|\operatorname{val} A^k - \operatorname{val} J^k\right| \leq \operatorname{val} W^{-1} + \operatorname{val} W$$

which establishes the equivalence (II)⇔(III) in general. □

6.2.2. *Negative spectral valuation.* Hereafter, we say that $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ has *negative spectral valuation* if invertible and the equivalent conditions (I) and (II) above hold.

The next statement summarizes just enough of the theory of desingularization of plane algebraic curves in characteristic zero for our purposes.

Proposition 6.2.3. *Let*

$$f = \sum_i c_i z^i \in \mathbb{C}((1/z)) \ \ (c_i \in \mathbb{C} \text{ and } c_i = 0 \text{ for } i \gg 0)$$

be algebraic. For integers $N > 0$ let

$$f_N = z^N \sum_{i \leq -2N} c_i z^i = z^N \left(f - \sum_{i > -2N} c_i z^i \right) \in (1/z^N)\mathbb{C}[[1/z]].$$

Let

$$F_N(x, y) = F_{f_N}(x, y) \in \mathbb{C}[x, y]$$

be the irreducible equation of f_N. Then we have

$$F_N(0,0) = \frac{\partial F_N}{\partial x}(0,0) = \cdots = \frac{\partial^{N-1} F_N}{\partial x^{N-1}}(0,0) = 0 \ \text{ and } \ \frac{\partial F_N}{\partial y}(0,0) \neq 0$$

provided that N is sufficiently large depending on f.

Proof. If $f_{N_0} = 0$ for some N_0 then $f_N = 0$ for all $N \geq N_0$ and hence $F_N(x, y) = y$ for all $N \geq N_0$, in which case there is nothing to prove. Thus we may assume without loss of generality that $f_N \neq 0$ for all N.

Let $h_N = f - f_N/z^N$. By the definitions we have

$$\mathrm{val}(f - h_N) \leq -2N \ \text{ and hence } \ \mathrm{val}\,(z^N(f - h_N)) = -N. \tag{42}$$

Let n denote the dimension of $\mathbb{C}(z, f)$ over $\mathbb{C}(z)$. It is clear that $f \in \mathbb{C}((1/z))$ and $f_N \in \mathbb{C}((1/z))$ generate the same extension of $\mathbb{C}(z)$. Thus the polynomials $F(x, y) = F_f(x, y)$ and $F_N(x, y) = F_{f_N}(x, y)$ have the same degree in y, namely n. Now write

$$F_N(x, y) = \sum_{i=0}^{n} p_{i,N}(x) y^{n-i} \ \ (p_{i,N}(x) \in \mathbb{C}[x], \ p_{n,N}(x) \neq 0).$$

For any $p(x) \in \mathbb{C}[x]$ let $\mathrm{ord}_{x=0}\, p(x)$ denote the exponent of the highest power of x dividing $p(x)$. Note that

$$\min_{i=0}^{n} \mathrm{ord}_{x=0}\, p_{i,N}(x) = 0$$

since otherwise x would divide $F_N(x, y)$, which is impossible since $F_N(x, y)$ is irreducible and of positive degree in y. It will be enough to prove that

$$\min_{i=0}^{n} \mathrm{ord}_{x=0}\, p_{i,N}(x) = \mathrm{ord}_{x=0}\, p_{n-1,N}(x) \leq -N + \mathrm{ord}_{x=0}\, p_{n,N}(x). \tag{43}$$

Since for $p(x) \in \mathbb{C}[x]$ we have

$$\mathrm{ord}_{x=0}\, p(x) = -\mathrm{val}\, p(1/z), \tag{44}$$

the natural tool for proving (43) is Lemma 6.1.5.

Let $r_1, \ldots, r_n$ denote the roots in $\mathbb{K}$ of $F(1/z, y) \in \mathbb{K}_0[y]$, numbered so that $r_n = f$. Then for a suitable enumeration $r_{1,N}, \ldots, r_{n,N}$ of the roots in $\mathbb{K}$ of $F_N(1/z, y) \in \mathbb{K}_0[y]$, we have

$$r_{i,N} = z^N(r_i - h_N) \text{ for } i = 1, \ldots, n$$

and in particular

$$r_{n,N} = z^N(r_n - h_N) = z^N(f - h_N) = f_N.$$

Furthermore, the roots $r_1, \ldots, r_n$ are distinct due to irreducibility of $F(x, y)$. Thus via (42) it follows that for some integer $N_0 > 0$ and all integers $N \geq N_0$ we have

$$\min_{i=1}^{n-1} \operatorname{val} r_{i,N} = N + \min_{i=1}^{n-1} \operatorname{val}(r_i - r_n) \geq 0. \tag{45}$$

Hereafter we assume that $N \geq N_0 > 0$, in which case

$$\min_{i=1}^{n-1} \operatorname{val} r_{i,N} \geq 0 > -N \geq \operatorname{val} r_{n,N}. \tag{46}$$

There is no loss of generality in assuming now that $\operatorname{val} r_{i,N}$ is a monotone decreasing function of i. Then relations (44), (45) and (46) via Lemma 6.1.5 imply (43). The proof of Proposition 6.2.3 is complete. $\square$

7. Evaluation of algebraic power series on matrices

In this section we develop a method to evaluate algebraic power series on matrices with entries in $\mathbb{C}((1/z))$ of negative spectral valuation which will make possible a proof of Proposition 2.5.5 by checking hypotheses in Proposition 2.4.2. We follow a line of thought initially suggested to us by the discussion of efficient computation of the matrix exponential in the undergraduate text [26]. The main result of this section is Proposition 7.1.5 below. The Cayley-Hamilton Theorem, the Weierstrass Preparation Theorem, and Proposition 6.2.3 are the main tools used here.

7.1. **Description of the evaluation method.**

7.1.1. *Notation.* Throughout §7 we fix a positive integer n and we work with the family of independent commuting algebraic variables $\{u_i\}_{i=1}^n \cup \{v_i\}_{i=1}^{2n} \cup \{t, x, y\}$. It is understood that these variables are independent of and commute with z. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_{2n})$. We write $\mathbb{C}[u] = \mathbb{C}[u_1, \ldots, u_n]$, $\mathbb{C}[[u]] = \mathbb{C}[[u_1, \ldots, u_n]]$, $\mathbb{C}[u, v] = \mathbb{C}[u_1, \ldots, u_n, v_1, \ldots, v_{2n}]$ and so on. Similar notation for building other rings from these variables is used below without further comment. For any commutative algebra $\mathcal{A}$ and matrix $A \in \operatorname{Mat}_n(\mathcal{A})$ we write

$$\det(1 + tA) = 1 + \sum_{i=1}^n e_i(A)t^i \in \mathcal{A}[t], \quad e(A) = (e_1(A), \ldots, e_n(A)) \in \mathcal{A}^n \text{ and}$$

$$A^\flat = \begin{bmatrix} A(1,1) & \ldots & A(1,n) & \ldots & A(n,1) & \ldots & A(n,n) \end{bmatrix}^{\mathrm{T}} \in \operatorname{Mat}_{n^2 \times 1}.$$

The exact definition of $A^\flat$ is not so important; we just need to have a definite rule for writing a matrix as a column vector. Note that the Cayley-Hamilton Theorem takes the form

$$A^n + \sum_{i=1}^n (-1)^i e_i(A) A^{n-i} = 0 \tag{47}$$

in this setup.

7.1.2. *Pre-widgets.* In anticipation of defining the notion of widget below, we call a $2n$-tuple $P(u,v) \in \mathbb{C}[u,v]^{2n}$ a *pre-widget* if

$$P(0,0) = 0 \text{ and} \tag{48}$$

$$\det_{i,j=1}^{2n} \frac{\partial P_i}{\partial v_j}(0,0) \neq 0. \tag{49}$$

Let $\frac{\partial P}{\partial u}(u,v) \in \mathrm{Mat}_{2n\times n}(\mathbb{C}[u,v])$ denote the matrix with entries $\frac{\partial P_i}{\partial u_j}(u,v)$ and let $\frac{\partial P}{\partial v}(u,v) \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}[u,v])$ denote $2n$-by-$2n$ matrix with entries $\frac{\partial P_i}{\partial v_j}(u,v)$. Hereafter, for example, we write the determinant in (49) in the abbreviated form $\det \frac{\partial P}{\partial v}(0,0)$. Similar matrix notation for derivatives and Jacobian determinants of vector functions will be used below without further comment.

Lemma 7.1.3. *Let $P(u,v) \in \mathbb{C}[u,v]^{2n}$ be a pre-widget. Then for every matrix $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ of negative spectral valuation there exists unique $\gamma \in (1/z)\mathbb{C}[[1/z]]^{2n}$ such that $P(e(A),\gamma) = 0$. Furthermore, the matrix $\frac{\partial P}{\partial v}(e(A),\gamma)$ is invertible in the matrix ring $\mathrm{Mat}_{2n}(\mathbb{C}[[1/z]])$.*

Proof. By Lemma 6.2.1 we have $e(A) \in (1/z)\mathbb{C}[[1/z]]^n$. We temporarily make the change of variable $t = 1/z$. We write $F(t,v) = P(e(A)|_{z=1/t}, v) \in \mathbb{C}[[t,v]]$. Since $F(0,0) = 0$ and $\det \frac{\partial F}{\partial v}(0,0) \neq 0$, the equation $F(t,g(t)) = 0$ has a unique solution $g(t) \in t\mathbb{C}[[t]]^{2n}$ by the Implicit Function Theorem for formal power series. It follows that $\gamma = g(1/z) \in (1/z)\mathbb{C}[[1/z]]^{2n}$ exists and is unique. Furthermore, since $\det \frac{\partial F}{\partial v}(t,g(t))|_{t=0} = \det \frac{\partial P}{\partial v}(0,0) \neq 0$, the matrix $\frac{\partial F}{\partial v}(t,g(t))$ is invertible in $\mathrm{Mat}_{2n}(\mathbb{C}[[t]])$ and hence the matrix $\frac{\partial P}{\partial v}(e(A),\gamma)$ is invertible in $\mathrm{Mat}_{2n}(\mathbb{C}[[1/z]])$. $\square$

7.1.4. *Widgets.* Let $P(u,v) \in \mathbb{C}[u,v]^{2n}$ be a pre-widget. Suppose now further that for every pair (A,γ) considered in Lemma 7.1.3 and integer $N \geq 0$ we have

$$\left[\begin{array}{ccc} (A^N)^\flat & \ldots & (A^{N+2n-1})^\flat \end{array}\right] \frac{\partial P}{\partial v}(e(A),\gamma)^{-1} \frac{\partial P}{\partial u}(e(A),\gamma) = 0. \tag{50}$$

Under the further assumption (50) we call the pre-widget $P(u,v)$ a *widget*. The property (50) is contrived to expedite a later calculation of a Jacobian determinant.

Here is our main result in this section. Recall that $f(t) \in \mathbb{C}[[t]]$ is called *algebraic* if $f(1/z) \in \mathbb{C}[[1/z]] \subset \mathbb{C}((1/z))$ is algebraic in the sense defined in §2.2.2 above.

Proposition 7.1.5. *Fix an algebraic power series*

$$f(t) = \sum_{i=0}^{\infty} c_i t^i \in \mathbb{C}[[t]] \quad (c_i \in \mathbb{C}). \tag{51}$$

Then for each integer $N \gg 0$ there exists a widget $P(u,v) \in \mathbb{C}[u,v]^{2n}$ such that for each $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ of negative spectral valuation the unique $\gamma \in (1/z)\mathbb{C}[[1/z]]^{2n}$ satisfying $P(e(A),\gamma) = 0$ also satisfies

$$\sum_{j=2N}^{\infty} c_j \left[\begin{array}{cc} A & B \\ 0 & A \end{array}\right]^j = \sum_{j=1}^{2n} \gamma_j \left[\begin{array}{cc} A & B \\ 0 & A \end{array}\right]^{N+j-1} \tag{52}$$

for every $B \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$.

The proof takes up the rest of §7.

7.1.6. *Remark.* The single statement (52) is equivalent to the conjunction of the statements

$$\sum_{j=2N}^{\infty} c_j A^j = \sum_{j=1}^{2n} \gamma_j A^{N+j-1} \quad \text{and}$$
$$\sum_{j=2N}^{\infty} \sum_{\nu=0}^{j-1} c_j A^\nu B A^{j-1-\nu} = \sum_{j=1}^{2n} \gamma_j \sum_{\nu=0}^{N+j-2} c_j A^\nu B A^{N+j-2-\nu}.$$

For the proof of Proposition 2.5.5 the latter two properties are what we will actually need to use.

7.2. **I-adic convergence, power series, and Weierstrass division.** We pause to review generalities connected with the Weierstrass Preparation Theorem.

7.2.1. *I-adic convergence.* Given a commutative ring R with unit, an ideal I, and a sequence $\{a\}\cup\{a_i\}_{i=1}^{\infty}$ in R, one says $\lim_{i\to\infty} a_i = a$ holds *I-adically* if for every positive integer k there exists a positive integer $i_0 = i_0(k)$ such that $a - a_i \in I^k$ for all $i \geq i_0$. Similarly, one can speak of I-adic Cauchy sequences and I-adic completeness. Consider, e.g., the ring $\mathbb{C}[[u_1, \ldots, u_n]] = \mathbb{C}[[u]]$ and the maximal ideal $I = (u_1, \ldots, u_n) \subset \mathbb{C}[[u]]$. Then $f_i \in \mathbb{C}[[u]]$ converges I-adically to $f \in \mathbb{C}[[u]]$ if and only if for every n-tuple $(\nu_1, \ldots, \nu_n)$ of nonnegative integers and every sufficiently large index i depending on $(\nu_1, \ldots, \nu_n)$, the Taylor coefficient $\frac{1}{\nu_1!\cdots\nu_n!}\frac{\partial^{\nu_1+\cdots+\nu_n} f_i}{\partial u_1^{\nu_1}\cdots\partial u_n^{\nu_n}}(0)$ equals the Taylor coefficient $\frac{1}{\nu_1!\cdots\nu_n!}\frac{\partial^{\nu_1+\cdots+\nu_n} f}{\partial u_1^{\nu_1}\cdots\partial u_n^{\nu_n}}(0)$. It is easy to see that the ring $\mathbb{C}[[u_1, \ldots, u_n]]$ is I-adically complete.

7.2.2. *Weierstrass division.* We now briefly recall the *Weierstrass Preparation Theorem* from a more active point of view emphasizing the algorithm of Weierstrass division. See, e.g., [28, Thm. 5, p. 139, Chap. VII, §1] for background and proof. The theorem concerns an $(n+1)$-variable power series ring over a field, with one of the variables singled out for special treatment. For definiteness we take the coefficient field to be $\mathbb{C}$. Consider the ring $\mathbb{C}[[u_1, \ldots, u_n, t]] = \mathbb{C}[[u, t]]$, with t distinguished. One is given a *divisand* $F(u,t) \in \mathbb{C}[[u,t]]$ and a *divisor* $D(u,t) \in \mathbb{C}[[u,t]]$. Of the latter it is assumed that there exists a positive integer m (called the *multiplicity* of the divisor) such that $D(0,t) = t^m U(t)$ for some $U(t) \in \mathbb{C}[[t]]$ such that $U(0) \neq 0$. The Weierstrass division process delivers a *quotient* $Q(u,t) \in \mathbb{C}[[u,t]]$ and a *remainder* $R(u,t) \in \mathbb{C}[[u]][t]$. The pair $(Q(u,t), R(u,t))$ is uniquely determined by two requirements. Firstly, the division equation $F(u,t) = Q(u,t)D(u,t) + R(u,t)$ must hold. Secondly, $R(u,t)$ must be a polynomial in t of degree $< m$. It bears emphasis that if $D(u,t) \in \mathbb{C}[[u]][t]$ is monic of degree m such that $D(0,t) = t^m$, and $F(u,t) \in \mathbb{C}[[u]][t]$, then the Euclidean (i.e., high school) and Weierstrass division processes deliver the same quotient and remainder.

Lemma 7.2.3. *We continue in the setting of the preceding paragraph. However, for simplicity we assume now that $D(u,t)$ is monic of degree m such that $D(0,t) = t^m$. Consider the ideal $I = (u_1, \ldots, u_n) \subset \mathbb{C}[[u]][t]$. Let k be a positive integer. If t^k divides $F(u,t)$, then $R(u,t)$ belongs to the ideal $I^{\lfloor k/m \rfloor}$. (Here $\lfloor c \rfloor$ denotes the greatest integer not exceeding c.)*

It follows that formation of Weierstrass remainder upon division by $D(u,t)$ viewed as a function from $\mathbb{C}[[u,t]]$ to $\mathbb{C}[[u]][t]$ is continuous with respect to the (t)-adic topology on the source and the $(u_1, \ldots, u_n)$-adic topology on the target.

Proof. Let $F_0(u,t) = F(u,t)/t^k$. Let $R_0(u,t)$ denote the remainder of $F_0(u,t)$ upon Weierstrass division by $D(u,t)$. Then $R(u,t)$ is the remainder of $t^k R_0(u,t)$ upon high school division by $D(u,t)$. This noted, there is no loss of generality in assuming that $F(u,t) = t^k$. Write $D(u,t) = t^m + \sum_{i=0}^{m-1} a_i t^i$ with coefficients $a_i = a_i(u) \in \mathbb{C}[[u]]$ such that $a_i(0) = 0$. Write $R(u,t) = \sum_{i=0}^{m-1} b_i t^i$ with coefficients $b_i = b_i(u) \in \mathbb{C}[[u]]$. Then we have

$$\left[\begin{array}{cccc} & & & -a_0 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & -a_{m-1} \end{array}\right]^k \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] = \left[\begin{array}{c} b_0 \\ \vdots \\ b_{m-1} \end{array}\right],$$

where the matrix on the left is the so-called *companion matrix* for $D(u,t)$. Clearly every entry of the m^{th} power of the companion matrix belongs to the ideal I, and hence every entry of the k^{th} power belongs to the ideal $I^{\lfloor k/m \rfloor}$. $\square$

7.3. **Construction of gadgets.** We make our first application of the Weierstrass division process.

7.3.1. *Construction.* Let

$$D(u,t) = t^n + \sum_{i=1}^{n} (-1)^i u_i t^{n-i} \in \mathbb{C}[u,t].$$

Note that the left side of (47) equals $D(e(A), A)$. This is the motivation for the definition of $D(u,t)$. Now fix $f(t) \in \mathbb{C}[[t]]$ and perform Weierstrass division by $D(u,t)^2$ to obtain an identity

$$f(t) = \sum_{i=1}^{2n} \varphi_i(u) t^{i-1} + Q_1(u,t) D(u,t)^2, \tag{53}$$

where

$$\varphi(u) = (\varphi_1(u), \ldots, \varphi_{2n}(u)) \in \mathbb{C}[[u]]^{2n} \text{ and } Q_1(u,t) \in \mathbb{C}[[u,t]].$$

By differentiation we immediately deduce that

$$\sum_{i=1}^{2n} \frac{\partial \varphi_i}{\partial u_j}(u) t^{i-1} = -\left(\frac{\partial Q_1}{\partial u_j}(u,t) D(u,t) + 2 Q_1(u,t) \frac{\partial D}{\partial u_j}(u,t)\right) D(u,t) \tag{54}$$

for $j = 1, \ldots, n$. We call $\varphi(u) \in \mathbb{C}[[u]]^{2n}$ the *gadget* associated with $f(t)$.

Lemma 7.3.2. *Let $f(t) \in \mathbb{C}[[t]]$ be expanded as a power series in t as on line* (51) *above. Let $\varphi(u) \in \mathbb{C}[[u]]^{2n}$ be the gadget associated with $f(t)$ according to the rule* (53) *above. Then for every $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ of negative spectral valuation and any $B \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ one has*

$$\sum_{j=0}^{\infty} c_j \left[\begin{array}{cc} A & B \\ 0 & A \end{array}\right]^j = \sum_{j=1}^{2n} \varphi_j(e(A)) \left[\begin{array}{cc} A & B \\ 0 & A \end{array}\right]^{j-1} \quad \textit{and} \tag{55}$$

$$\sum_{i=1}^{2n} \frac{\partial \varphi_i}{\partial u_j}(e(A)) A^{N+i-1} = 0 \quad \textit{for } j = 1, \ldots, n \textit{ and } N \in \mathbb{N}. \tag{56}$$

Proof. Suppose at first that $f(t) \in \mathbb{C}[t]$. Then we have $\varphi(u) \in \mathbb{C}[u]^{2n}$ and $Q_1(u,t) \in \mathbb{C}[u,t]$ since high school division in this case gives the same result as Weierstrass division. Substituting $(u,t) = \left(e(A), \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}\right)$ into (53) and using the Cayley-Hamilton Theorem (47) with n replaced by $2n$, we obtain (55). Substituting $(u,t) = (e(A), A)$ into (54) and using the Cayley-Hamilton Theorem (47) again, we obtain (56). The general case $f(t) \in \mathbb{C}[[t]]$ follows from the (t)-adic to $(u_1, \ldots, u_n)$-adic continuity of Weierstrass division noted after Lemma 7.2.3. ☐

7.4. **Construction of widgets.** We present a rather involved calculation producing all the widgets needed to prove Proposition 7.4.2.

7.4.1. *Setup for the construction.* Let $F(x,y) \in \mathbb{C}[x,y]$ satisfy

$$F(0,0) = \frac{\partial F}{\partial x}(0,0) = \cdots = \frac{\partial^{2n-1} F}{\partial x^{2n-1}}(0,0) = 0 \text{ and } \frac{\partial F}{\partial y}(0,0) \neq 0. \tag{57}$$

Note that this hypothesis matches up with the conclusion of Proposition 6.2.3. Fix $f(t) \in \mathbb{C}[[t]]$ and assume that

$$f(0) = 0 \text{ and } F(t, f(t)) = 0. \tag{58}$$

Perform high school division of

$$F\left(t, \sum_{i=1}^{2n} v_i t^{i-1}\right) \in \mathbb{C}[u,v,t]$$

by $D(u,t)^2$ to obtain an identity

$$F\left(t, \sum_{i=1}^{2n} v_i t^{i-1}\right) = \sum_{i=1}^{2n} P_i(u,v) t^{i-1} + Q_2(u,v,t) D(u,t)^2 \tag{59}$$

where

$$P(u,v) = (P_1(u,v), \ldots, P_{2n}(u,v)) \in \mathbb{C}[u,v]^{2n} \text{ and } Q_2(u,v,t) \in \mathbb{C}[u,v,t].$$

Note that we could also characterize $P(u,v)$ and $Q_2(u,v,t)$ by Weierstrass division since the latter and high school division agree in this case. In the proof below we will play the two division processes off one another.

Proposition 7.4.2. *Assumptions and notation are as above. Then the $2n$-tuple $P(u,v) \in \mathbb{C}[u,v]^{2n}$ associated with $F(x,y)$ by the rule* (59) *above is a widget and furthermore the gadget $\varphi(u)$ associated with $f(t)$ by the rule* (53) *satisfies*

$$\varphi(0) = 0 \textit{ and} \tag{60}$$

$$P(u, \varphi(u)) = 0. \tag{61}$$

We break the somewhat involved proof down into many short stages, finishing in §7.4.10 below.

7.4.3. *Plan of proof.* Consider the following statements:

$$\begin{aligned} &\text{There exists unique } U(x,y) \in \mathbb{C}[[x,y]] \text{ s.t.} \\ &F(x,y) = (y - f(x))U(x,y) \text{ and } U(0,0) \neq 0. \end{aligned} \tag{62}$$

$$f(t) \in t^{2n}\mathbb{C}[[t]]. \tag{63}$$

To prove Proposition 7.4.2 it will suffice to prove in turn (62), (63), (60), (48), (49), (61), and (50).

7.4.4. *Proof of* (62). The argument here is a standard one we could omit, but we include it as a model for the remaining calculations. By (57), $F(x,y)$ is a Weierstrass divisor of multiplicity 1 with respect to y, i.e., $F(0,y) = yU_3(y)$ for some power series $U_3(y) \in \mathbb{C}[[y]]$ such that $U_3(0) \neq 0$. By Weierstrass division of y by $F(x,y)$ we write

$$y = r(x) + Q_4(x,y)F(x,y) \tag{64}$$

where

$$r(x) \in \mathbb{C}[[x]] \text{ and } Q_4(x,y) \in \mathbb{C}[[x,y]].$$

Substituting $x = 0$ above we obtain the relation

$$y = r(0) + yQ_4(0,y)U_3(y)$$

and thus deduce $Q_4(0,0) \neq 0$. Let

$$U(x,y) = Q_4(x,y)^{-1} \in \mathbb{C}[[x,y]].$$

Then we have

$$F(x,y) = (y - r(x))U(x,y).$$

Substituting $y = f(x)$ on both sides of the latter proves that

$$f(x) = r(x). \tag{65}$$

Thus the factorization (62) exists. The uniqueness of $U(x,y)$ follows by the uniqueness of the quotient produced by Weierstrass division. Thus (62) holds.

7.4.5. *Proof of* (63). By substituting $y = 0$ into (64) and using (65) we deduce

$$0 = f(x) + Q_4(x,0)F(x,0).$$

Via (57) we deduce (63).

7.4.6. *Proof of* (60). Substituting $u = 0$ on both sides of (53) one obtains the identity

$$f(t) = \sum_{i=1}^{2n} \varphi_i(0)t^{i-1} + Q_1(0,t)t^{2n},$$

thus by (63) forcing (60) to hold.

7.4.7. *Proof of* (48). Substituting $u = v = 0$ on both sides of (59) one obtains the identity

$$F(t,0) = \sum_{i=1}^{2n} P_i(0,0)t^{i-1} + Q_2(0,0,t)t^{2n},$$

thus by (57) forcing (48) to hold.

7.4.8. *Proof of* (49). Differentiation on both sides of (59) with respect to v_j followed by evaluation at $u = v = 0$ yields the relation

$$\frac{\partial F}{\partial y}(t,0)\,t^{j-1} = \sum_{i=1}^{2n} \frac{\partial P_i}{\partial v_j}(0,0)t^{i-1} + \frac{\partial Q_2}{\partial v_j}(0,0,t)t^{2n}. \tag{66}$$

Now write

$$\frac{\partial F}{\partial y}(x,0) = \sum_{i=0}^{\infty} b_i x^i$$

with coefficients $b_i \in \mathbb{C}$. By (66) we have

$$\frac{\partial P_i}{\partial v_j}(0,0) = \begin{cases} b_{i-j} & \text{if } j \le i, \\ 0 & \text{if } j > i \end{cases}$$

for $i, j = 1, \dots, 2n$. Thus we have

$$\det_{i,j=1}^{2n} \frac{\partial P_i}{\partial v_j}(0,0) = \left(\frac{\partial F}{\partial y}(0,0)\right)^{2n}.$$

The right side does not vanish by assumption (57). Thus (49) holds.

7.4.9. *Proof of* (61). We have the following chain of equalities:

$$\begin{aligned} \sum_{i=1}^{2n} P_i(u,\varphi(u))t^{i-1} + Q_2(u,\varphi(u),t)D(u,t)^2 \;&=\; F\left(t, \sum_{i=1}^{2n}\varphi(u)t^{i-1}\right) \\ =\;& \left(\sum_{i=1}^{2n}\varphi(u)t^{i-1} - f(t)\right) U\left(t, \sum_{i=1}^{2n}\varphi_i(u)t^{i-1}\right) \\ =\;& -Q_1(u,t)D(u,t)^2 U\left(t, \sum_{i=1}^{n}\varphi(u)t^{i-1}\right). \end{aligned}$$

Justifications for the steps are as follows. The first equality we obtain by substituting $v = \varphi(u)$ into (59). The second equality we obtain by specializing the factorization given in (62). The third equality we obtain directly from (53). The equality between the extreme terms of the chain of equalities above forces (61) to hold by the uniqueness of the remainder produced by Weierstrass division.

7.4.10. *Proof of* (50). Clearly $\gamma = \varphi(e(A))$ by (61) and the uniqueness asserted in Lemma 7.1.3. Recall also that $\frac{\partial P}{\partial v}(e(A),\gamma)$ by the cited lemma is invertible in $\mathrm{Mat}_{2n}(\mathbb{C}[[1/z]])$. By implicit differentiation we find that

$$\frac{\partial \varphi}{\partial u}(e(A)) = \frac{\partial P}{\partial v}(e(A),\gamma)^{-1}\frac{\partial P}{\partial u}(e(A),\gamma).$$

Thus (50) follows from (56). The proof of Proposition 7.4.2 is complete. □

7.5. **Proof of Proposition 7.1.5.** With $f(t)$ expanded as on line (51), we write

$$f(t) = \sum_{i=0}^{2N-1} c_i t^i + t^N f_N(t) \;\text{ and }\; f_N(t) = \sum_{i=2N}^{\infty} c_i t^{i-N}.$$

By Proposition 6.2.3, for every $N \gg 0$ the irreducible equation

$$F_N(x,y) = F_{f_N(1/z)}(x,y) \in \mathbb{C}[x,y]$$

is such that both (57) and (58) hold with (f, F) replaced by (f_N, F_N). For the rest of the proof fix N large enough in the preceding sense. Let $\varphi(u) \in \mathbb{C}[[u]]^{2n}$ be the gadget associated with $f_N(t)$ by the procedure given before Lemma 7.3.2. Let $P(u, v) \in \mathbb{C}[u, v]^n$ be the widget associated with $f_N(t)$ by the procedure given before Proposition 7.4.2. Arbitrarily fix $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ of negative spectral valuation and any $B \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$. Let $\gamma \in (1/z)\mathbb{C}[[1/z]]^{2n}$ be the unique vector such that $P(e(A), \gamma) = 0$. We then have

$$\sum_{j=2N}^{\infty} c_j \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^j = \sum_{j=1}^{2n} \varphi_j(e(A)) \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{N+j-1} = \sum_{j=1}^{2n} \gamma_j \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{N+j-1}$$

at the first step by (55) and at the second step by (60), (61) and the uniqueness guaranteed by Lemma 7.1.3. Thus the pair $(N, P(u, v))$ has property (52), as desired. The proof of Proposition 7.1.5 is complete. □

8. Proof of the main result

We prove Proposition 2.5.5 by checking hypotheses in Proposition 2.4.2, thereby completing the proof of Theorem 1.

8.1. **Review of the setup for Proposition 2.5.5.** Let us start simply by repeating statements (6) and (9) here for the reader's convenience:

$$I_n + a^{(0)}g + \sum_{\theta=1}^{q} \sum_{j=2}^{\infty} \kappa_j^{(\theta)} (a^{(\theta)}g)^j = 0. \tag{67}$$

(68) The linear map

$$\left(h \mapsto a^{(0)}h + \sum_{\theta=1}^{q} \sum_{j=2}^{\infty} \sum_{\nu=0}^{j-1} \kappa_j^{(\theta)} (a^{(\theta)}g)^{\nu} (a^{(\theta)}h)(a^{(\theta)}g)^{j-1-\nu} \right)$$

$: \mathrm{Mat}_n(\mathbb{C}((1/z))) \to \mathrm{Mat}_n(\mathbb{C}((1/z)))$ is invertible.

Concerning the data appearing above, we have by (7) and (12) that

$$a^{(0)} \in \mathrm{Mat}_n(\mathbb{C}(z)), \;\; a^{(1)}, \ldots, a^{(q)} \in \mathrm{Mat}_n(\mathbb{C}) \;\text{ and }\; g \in \mathrm{Mat}_n(\mathbb{C}((1/z))). \tag{69}$$

We furthermore have

$$e(a^{(1)}g), \ldots, e(a^{(q)}g) \in (1/z)\mathbb{C}[[1/z]]^n \tag{70}$$

by assumption (8) and Lemma 6.2.1. Here we have used the notation $e(\cdot)$ defined in §7. We continue below to make free use of notation introduced in §7. By (7) and (13) the following statement holds:

(71) The power series $\sum_{j=2}^{\infty} \kappa_j^{(\theta)} t^j \in \mathbb{C}[[t]]$ is algebraic for $\theta = 1, \ldots, q$.

8.2. **Application of Proposition 7.1.5.** Proposition 7.1.5 provides us with an arbitrarily large positive integer N, widgets

$$P^{(1)}(u, v), \ldots, P^{(q)}(u, v) \in \mathbb{C}[u, v]^{2n}$$

and vectors

$$\gamma^{(1)}, \ldots, \gamma^{(q)} \in (1/z)\mathbb{C}[[1/z]]^{2n}$$

such that

$$P^{(\theta)}(e(a^{(\theta)}g),\gamma^{(\theta)}) = 0, \tag{72}$$

$$\sum_{j=2N}^{\infty} \kappa_j^{(\theta)}(a^{(\theta)}g)^j = \sum_{j=1}^{2n} \gamma_j^{(\theta)}(a^{(\theta)}g)^{N+j-1}, \quad \text{and} \tag{73}$$

$$\begin{aligned} &\sum_{j=2N}^{\infty}\sum_{\nu=0}^{j-1} \kappa_j^{(\theta)}(a^{(\theta)}g)^{\nu}(a^{(\theta)}h)(a^{(\theta)}g)^{j-1-\nu} \\ &= \sum_{j=1}^{2n}\sum_{\nu=0}^{N+j-2} \gamma_j^{(\theta)}(a^{(\theta)}g)^{\nu}(a^{(\theta)}h)(a^{(\theta)}g)^{N+j-2-\nu} \end{aligned} \tag{74}$$

for $\theta = 1,\dots,q$ and any $h \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$.

8.3. **Polynomial version of** (67)**.** We now rewrite (67) with the help of widgets as a solution in $\mathbb{C}((1/z))$ of a system of $3nq+n^2$ polynomial equations in $3nq+n^2$ variables with coefficients in $\mathbb{C}(z)$.

8.3.1. *Variables.* We employ the family of variables

$$\{U_i\}_{i=1}^{qn} \cup \{V_i\}_{i=1}^{2qn} \cup \{\xi_i\}_{i=1}^{n^2}.$$

Let

$$U = (U_1,\dots,U_{qn}), \quad V = (V_1,\dots,V_{2qn}), \quad \text{and} \quad \xi = (\xi_1,\dots,\xi_{n^2}).$$

Let

$$\Xi = \begin{bmatrix} \xi_1 & \dots & \xi_n \\ \vdots & & \vdots \\ \xi_{n^2-n+1} & \dots & \xi_{n^2} \end{bmatrix} \in \mathrm{Mat}_n(\mathbb{C}[\xi]).$$

We break the U's and V's down into groups by introducing the following notation:

$$\begin{aligned} U_i^{(\theta)} &= U_{i+(\theta-1)q} \ \text{ for } i=1,\dots,n \text{ and } \theta = 1,\dots,q. \\ V_i^{(\theta)} &= V_{i+(\theta-1)q} \ \text{ for } i=1,\dots,2n \text{ and } \theta = 1,\dots,q. \\ U^{(\theta)} &= (U_1^{(\theta)},\dots,U_n^{(\theta)}) \ \text{ and } \ V^{(\theta)} = (V_1^{(\theta)},\dots,V_{2n}^{(\theta)}) \ \text{ for } \theta = 1,\dots,q. \end{aligned}$$

8.3.2. *A special matrix.* We define

$$\begin{aligned} &\mathcal{H}(V,\xi) \\ = \ & I_n + a^{(0)}\Xi + \sum_{\theta=1}^{q}\left(\sum_{k=2}^{2N-1}\kappa_k^{(\theta)}(a^{(\theta)}\Xi)^k + \sum_{k=1}^{2n} V_k^{(\theta)}(a^{(\theta)}\Xi)^{N+k-1}\right) \\ \in \ & \mathrm{Mat}_n(\mathbb{C}[V,\xi]). \end{aligned}$$

8.3.3. *Polynomials.* We define $3qn + n^2$ polynomials belonging to $\mathbb{C}[U, V, \xi]$ as follows.

$$\begin{aligned}
& F_{i+n(\theta-1)}(U,\xi) \\
= \ & F_i^{(\theta)}(U^{(\theta)},\xi) = U_i^{(\theta)} - e_i(a^{(\theta)}\Xi) \ \text{ for } i = 1,\dots,n \text{ and } \theta = 1,\dots,q, \\
\\
& G_{i+2n(\theta-1)}(U,V) \\
= \ & P^{(\theta)}(U^{(\theta)},V^{(\theta)}) \ \text{ for } i = 1,\dots,2n \text{ and } \theta = 1,\dots,q, \text{ and} \\
\\
& H_{i+n(j-1)}(V,\Xi) \\
= \ & \mathcal{H}(V,\Xi)(i,j) \ \text{ for } i,j = 1,\dots,n.
\end{aligned}$$

8.3.4. *Presentation of the system of equations.* Let

$$\begin{aligned}
F(U,\xi) &= \begin{bmatrix} F_1(U,\xi) & \dots & F_{qn}(U,\xi) \end{bmatrix}^{\mathrm{T}}, \\
G(U,V) &= \begin{bmatrix} G_1(U,V) & \dots & G_{2qn}(U,V) \end{bmatrix}^{\mathrm{T}}, \\
H(V,\xi) &= \begin{bmatrix} H_1(V,\xi) & \dots & H_{n^2}(V,\xi) \end{bmatrix}^{\mathrm{T}}.
\end{aligned}$$

Then our system of polynomial equations takes the form

$$F(U,\xi) = 0, \quad G(U,V) = 0, \quad H(V,\xi) = 0. \tag{75}$$

Note that this system has all coefficients in $\mathbb{C}(z)$ by (69) and the definitions.

8.3.5. *The solution Υ_0.* We claim that the following formulas specify a solution in $\mathbb{C}((1/z))$ of the system of equations (75):

$$U_i^{(\theta)} = e_i(a^{(\theta)}g) \ \text{ for } i = 1,\dots,n \text{ and } \theta = 1,\dots,q. \tag{76}$$

$$V_i^{(\theta)} = \gamma_i^{(\theta)} \ \text{ for } i = 1,\dots,2n \text{ and } \theta = 1,\dots,q. \tag{77}$$

$$\xi_{i+n(j-1)} = \Xi(i,j) = g(i,j) \ \text{ for } i,j = 1,\dots,n. \tag{78}$$

The equation $F(U,\xi) = 0$ is obviously satisfied. The equation $G(U,V) = 0$ is satisfied because it merely restates the system of equations (72). Finally, one verifies that $H(U,\xi) = 0$ is satisfied by using (67), (73), (76), and (77). The claim is proved. The solution of (75) specified by (76), (77), and (78) will be denoted by Υ_0.

8.4. **Analysis of the Jacobian determinant.** Now we study the Jacobian matrix

$$\begin{bmatrix}
\frac{\partial F}{\partial U}(U,\xi) & 0 & \frac{\partial F}{\partial \xi}(U,\xi) \\
\frac{\partial G}{\partial U}(U,V) & \frac{\partial G}{\partial V}(U,V) & 0 \\
0 & \frac{\partial H}{\partial V}(V,\xi) & \frac{\partial H}{\partial \xi}(V,\xi)
\end{bmatrix} \in \mathrm{Mat}_{n^2+3qn}(\mathbb{C}[U,V,\xi]). \tag{79}$$

for the system of equations (75). Let

$$\begin{bmatrix}
I_n & 0 & b_{13} \\
b_{21} & b_{22} & 0 \\
0 & b_{32} & b_{33}
\end{bmatrix} \in \mathrm{Mat}_{n^2+3qn}(\mathbb{C}((1/z))) \tag{80}$$

be the result of evaluating (79) at the point Υ_0. To prove Proposition 2.5.5 and thereby to complete the proof of Theorem 1, we have by Proposition 2.4.2 only to prove that the determinant of the matrix (80) does not vanish. Now provided that $\det b_{22} \neq 0$, we have a matrix identity

$$\left[\begin{array}{ccc} I_n & 0 & b_{13} \\ b_{21} & b_{22} & 0 \\ 0 & b_{32} & b_{33} \end{array}\right] \left[\begin{array}{ccc} I_n & 0 & 0 \\ -b_{22}^{-1}b_{21} & I_{2n} & 0 \\ 0 & 0 & I_{n^2} \end{array}\right] \left[\begin{array}{ccc} I_n & 0 & -b_{13} \\ 0 & I_{2n} & 0 \\ 0 & 0 & I_{n^2} \end{array}\right]$$
$$= \left[\begin{array}{ccc} I_n & 0 & 0 \\ 0 & b_{22} & 0 \\ -b_{32}b_{22}^{-1}b_{21} & b_{32} & b_{32}b_{22}^{-1}b_{21}b_{13} + b_{33} \end{array}\right].$$

Thus it will be enough to prove that

$$\det b_{22} \neq 0, \tag{81}$$
$$b_{32}b_{22}^{-1}b_{21} = 0, \quad \text{and} \tag{82}$$
$$\det b_{33} \neq 0. \tag{83}$$

8.5. **Proof of** (81)**.** We have by the definitions

$$b_{22} = \sum_{\theta=1}^{q} \mathbf{e}_{\theta\theta} \otimes \frac{\partial P^{(\theta)}}{\partial v}(e(a^{(\theta)}g), \gamma^{(\theta)}). \tag{84}$$

Thus (81) holds by Lemma 7.1.3.

8.6. **Proof of** (82)**.** For b_{21} and b_{32} we have formulas similar to (84), namely

$$\begin{aligned} b_{21} &= \sum_{\theta=1}^{q} \mathbf{e}_{\theta\theta} \otimes \frac{\partial P^{(\theta)}}{\partial u}(e(a^{(\theta)}g), \gamma^{(\theta)}) \text{ and} \\ b_{32} &= \sum_{\theta=1}^{q} \mathbf{e}_{\theta\theta} \otimes \left[\begin{array}{ccc} ((a^{(\theta)}g)^N)^\flat & \ldots & ((a^{(\theta)}g)^{N+2n-1})^\flat \end{array}\right]. \end{aligned}$$

Thus (82) holds by property (50) of a widget.

8.7. **Proof of** (83)**.** We have for $i, j = 1, \ldots, n$ that

$$\begin{aligned} &\frac{\partial \mathcal{H}(V,\xi)}{\partial \xi_{i+(j-1)n}} \\ &= a^{(0)}\mathbf{e}_{ij} + \sum_{\theta=1}^{q}\sum_{k=2}^{2N-1}\sum_{\nu=0}^{k-1} \kappa_k^{(\theta)}(a^{(\theta)}\Xi)^\nu (a^{(\theta)}\mathbf{e}_{ij})(a^{(\theta)}\Xi)^{k-1-\nu} \\ &\quad + \sum_{\theta=1}^{q}\sum_{k=1}^{2n}\sum_{\nu=0}^{N+k-2} V_k^{(\theta)}(a^{(\theta)}\Xi)^\nu (a^{(\theta)}\mathbf{e}_{ij})(a^{(\theta)}\Xi)^{N+k-2-\nu} \end{aligned}$$

and hence after evaluating both sides at Υ_0 and using (74), we find that b_{33} is a matrix describing with respect to the basis

$$\mathbf{e}_{11}, \ldots, \mathbf{e}_{1n}, \ldots, \mathbf{e}_{n1}, \ldots, \mathbf{e}_{nn} \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$$

the invertible linear map considered in (68). Thus (83) holds. Thus the proof of Proposition 2.5.5 is complete and with it the proof of Theorem 1.

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