# CONVERGENCE OF THE LARGEST SINGULAR VALUE OF A POLYNOMIAL IN INDEPENDENT WIGNER MATRICES 

GREG W. ANDERSON


#### Abstract

For polynomials in independent Wigner matrices we prove convergence of the largest singular value to the operator norm of the corresponding polynomial in free semicircular variables. When combined with truncation techniques, our setup yields convergence results under fourth moment hypotheses. We actually prove a more general result of the form "no eigenvalues outside the support of the limiting eigenvalue distribution" under hypotheses weaker than strict independence of the Wigner matrices. We build on ideas of Haagerup-Schultz-Thorbjørnsen on the one hand and Bai-Silverstein on the other. We refine the linearization trick so as to preserve self-adjointness and we develop a secondary trick bearing on the calculation of correction terms. Instead of Poincaré-type inequalities, we use a variety of matrix identities and $L^{p}$ estimates. The Schwinger-Dyson equation controls much of the analysis.


## Contents

1. Introduction 1
2. Formulation and discussion of the main result 3
3. Approximation of solutions of the Schwinger-Dyson equation 13
4. Tools from operator theory 18
5. Tensor products, transpositions and other algebraic tools 24
6. Construction of solutions of the Schwinger-Dyson equation 29
7. SALT block designs and random matrix estimates 32
8. The self-adjoint linearization trick 36

9 . Tools for concentration 43
10. Matrix identities 47
11. $L^{p}$ estimates for the block Wigner model 53
12. Endgame 60

References 65

## 1. Introduction

As part of a larger operator-theoretic investigation, it was shown in [8] (refining earlier work of [9]) that there are for large $N$ almost surely no eigenvalues outside an $\epsilon$-neighborhood of the support of the limiting spectral distribution of a self-adjoint polynomial in independent GUE matrices. (See [1, Chap. 5, Sec. 5] for another account of that result.) It is natural to ask if the same is true for Wigner matrices. We answer that question here in the affirmative. To a large extent this is a matter of learning to get by without the Poincaré inequality. Now the template for results of the form "no eigenvalues outside the support..." was established a number of
years earlier in the pioneering work of [2], and moreover the authors of that paper got along without the Poincaré inequality quite well-erasure of rows and columns, classical $L^{p}$ estimates and truncation arguments sufficed. Moreover they got their results under stringent fourth moment hypotheses. In this paper we channel the separately flowing streams of ideas in [2] and [8] into one river, encountering some perhaps unexpected bends.

Any discussion of largest eigenvalues of Wigner matrices must mention the classical work [3]. In that paper convergence of the largest eigenvalue of a Wigner matrix to the spectrum edge was established under stringent fourth moment hypotheses. Our general setup when suitably specialized and truncated yields results for self-adjoint polynomials in independent Wigner matrices under the same fourth moment hypotheses. Roughly speaking, we let the results of [3] do the hard work of attracting the eigenvalues to a compact neighborhood of the spectrum and then we draw them the rest of the way in by using tools including the Schwinger-Dyson equation, rather elaborate matrix identities and standard $L^{p}$ estimates.

There has been a lot of progress recently on universality in the bulk and at the edge for single Wigner matrices and sample covariance matrices. Edge-universality results in the single matrix case greatly refine and indeed render obsolete results of "no eigenvalues outside the support.." type, albeit typically under more generous moment assumptions. We mention for example [6] which proves convergence of the law of the suitably rescaled smallest eigenvalue of a sample covariance matrix with non-unity aspect ratio to the Tracy-Widom distribution. Of course many other papers could be mentioned-the area is profoundly active at the moment. It seems likely similar edge-universality results are true in the polynomial case (also in the band matrix case). From this aspirational point of view our results are crude. But we hope they could serve as a point of departure.

Generalizations of the "no eigenvalues outside the support..." result of [8] were quick to appear and continue to do so. In [19], following up on the earlier results of [9], results in the GOE and GSE cases were obtained, and they revealed a key role for "correction terms" of the sort we spend a great deal of effort in this paper to control. In [4], a generalization to non-Gaussian distributions satisfying Poincarétype inequalities was obtained. A recent preprint [14] provides a generalization involving polynomials in Gaussian Wigner matrices and deterministic matrices with convergent joint law which in particular establishes various rectangular analogues.

All the works following upon [8] including this one build on two extraordinarily powerful ideas from that paper: a counterintuitively "backwards" way of estimating the error of approximate solutions of the Schwinger-Dyson equation; and the famous linearization trick. We refine both ideas in this paper. The refinements are closely intertwined and involve a gadget we call a SALT block design.

We have been significantly influenced by the paper [12], which explores geometry and numerical analysis of the Schwinger-Dyson equation, and which could serve uninitiated readers as an introduction to the use of matricial semicircular elements. We were influenced also by [10] and [11] which develop and apply Girko's notion of deterministic equivalent. The notion of deterministic equivalent is in effect exploited here as well, but, more or less following [8], we simply harvest the needed solutions of the Schwinger-Dyson equation from Boltzmann-Fock space fully formed, thus avoiding iterative schemes for producing solutions.

Here is a rough outline of the contents of the paper. In $\S 2$ we formulate main results, indicate how to obtain results under stringent fourth moment hypotheses and finally reformulate the main result in a more convenient if technical way way involving Stieltjes transforms (see Theorem 2.6.4 below). In $\S 3$ we bring the Schwinger-Dyson equation into the picture and refine the above-mentioned idea of [8] for controlling errors of approximate solutions to the Schwinger-Dyson equation. In $\S 4$ we review elementary topics in $C^{*}$-algebra theory and supply a proof of the crucially important Proposition 2.3 .3 which equates some not a priori equivalent notions of support. Along the way we introduce many tools needed later. In $\S 5$ we add some less-than-common notions to our algebraic toolkit, and in particular, the notion of a $C^{*, T}$-algebra, which is essentially equivalent to that of a real $C^{*}$-algebra. These tools are needed throughout the remainder of the paper but are especially important for calculating corrections. In $\S 6$ we construct solutions of the Schwinger-Dyson equation by using the apparatus introduced in $\S 4$ and $\S 5$. We also introduce a secondary version of the Schwinger-Dyson equation and show how a solution of it may be extracted from the upper right corner of a solution of a suitably chosen (larger and more complicated) instance of the Schwinger-Dyson equation itself. In $\S 7$ we introduce SALT block designs and explain their precise relationship with approximation of solutions of the Schwinger-Dyson equation. In $\S 8$ we refine the linearization trick of [9] and [8] so as to preserve self-adjointness and (more to the point) to produce SALT block designs. We also introduce the secondary trick which is a kind of bootstrapping method for coercing calculations of correction terms into a format similar to that used for calculating limiting spectral distributions. In $\S 9$ we review tools we use in place of the Poincaré inequality. In $\S 10$ we present a catalog of carefully chosen matrix identities. These identities should be seen as further concentration tools. In $\S 11$ we introduce the block Wigner model and work through a rather long series of $L^{p}$ estimates and approximations. Finally, in $\S 12$ we empty out the toolbox to finish the proof of Theorem 2.6.4.

## 2. Formulation and discussion of the main result

In $\S 2.1$ we introduce general notation needed throughout the paper. In $\S 2.2$ we specify data and formulate assumptions. In $\S 2.3$ we formulate our main result. (See Theorem 2.3 .6 below.) In $\S 2.4$ we recall classical results on maximal eigenvalues holding under fourth moment hypotheses and in $\S 2.5$ we derive by truncation some consequences of our main result holding under hypotheses of the same type. Finally, in $\S 2.6$ we reformulate our main result in terms of Stieltjes transforms and an auxiliary random variable z. (See Theorem 2.6.4 below.)
2.1. Notation and terminology. Let $\mathbb{E}$ denote expectation. Let $\operatorname{Pr}$ denote probability. (We save the letters $E$ and $P$ for other purposes.) We use $\vee$ and $\wedge$ for maximum and minimum, respectively. We write $\mathbf{1}_{A}$ for the indicator of an event (or predicate) $A$. For any $\mathbb{C}$-valued random variable $Z$ and exponent $p \in[1, \infty]$, let $\|Z\|_{p}$ denote the $L^{p}$-norm of $Z$, i.e., let $\|Z\|_{p}=\left(\mathbb{E}|Z|^{p}\right)^{1 / p}$ for $p \in[1, \infty)$ and otherwise let $\|Z\|_{\infty}$ denote the essential supremum of $|Z|$. For a matrix $A$ with complex entries, let $A^{*}$ denote the transpose conjugate, $A^{\mathrm{T}}$ the transpose and $\llbracket A \rrbracket$ the largest singular value of $A$. More generally, we use $\llbracket \rrbracket$ for $C^{*}$-algebra norms. We introduce more specialized algebraic notation and terminology in the next two paragraphs.
2.1.1. Algebras and matrices. An algebra always has $\mathbb{C}$ as scalar field, is associative, and possesses a unit denoted by $1_{\mathcal{A}}$. (Other notation for the unit may also be used, e.g., simply 1.) Let $\operatorname{Mat}_{n}(\mathcal{A})$ denote the algebra of $n$-by- $n$ matrices with entries in $\mathcal{A}$. More generally, let $\operatorname{Mat}_{k \times \ell}(\mathcal{A})$ denote the space of $k$-by- $\ell$ matrices with entries in $\mathcal{A}$. The $(i, j)$-entry of a matrix $A$ is invariably denoted $A(i, j)$ (never $A_{i j}$ ). Let $\mathcal{A}^{\times}$ denote the group of invertible elements of an algebra $\mathcal{A}$, put $\mathrm{GL}_{n}(\mathcal{A})=\operatorname{Mat}_{n}(\mathcal{A})^{\times}$ (GL for general linear group) and for $A \in \operatorname{Mat}_{n}(\mathcal{A})$, let $\operatorname{tr}_{\mathcal{A}} A=\sum_{i=1}^{n} A(i, i)$. In the special case $\mathcal{A}=\mathbb{C}$ we write $\operatorname{tr}=\operatorname{tr}_{\mathbb{C}}$. Let $\mathbf{I}_{n} \in \operatorname{Mat}_{n}(\mathbb{C})$ denote the $n$-by- $n$ identity matrix and more generally, given an element $a$ of an algebra $\mathcal{A}$, let $\mathbf{I}_{n} \otimes a \in \operatorname{Mat}_{n}(\mathcal{A})$ denote the diagonal matrix with entries $a$ on the diagonal. Given a $*$-algebra $\mathcal{A}$, i.e., an algebra endowed with an involution denoted $*$, and an element $x \in \mathcal{A}$, we say that $x$ is self-adjoint if $x^{*}=x$ and we denote the set of such elements by $\mathcal{A}_{\text {sa }}$. Given a matrix $A \in \operatorname{Mat}_{k \times \ell}(\mathcal{A})$ with entries in a $*$-algebra $\mathcal{A}$, we define $A^{*} \in \operatorname{Mat}_{\ell \times k}(\mathcal{A})$ by $A^{*}(i, j)=A(j, i)^{*}$. In particular, by this $\operatorname{rule}^{\operatorname{Mat}}{ }_{n}(\mathcal{A})$ becomes a $*$-algebra whenever $\mathcal{A}$ is.
2.1.2. The noncommutative polynomial ring $\mathbb{C}\langle\mathbf{X}\rangle$. Let $\mathbb{C}\langle\mathbf{X}\rangle$ be the noncommutative polynomial ring generated over $\mathbb{C}$ by a sequence $\mathbf{X}=\left\{\mathbf{X}_{\ell}\right\}_{\ell=1}^{\infty}$ of independent noncommuting variables. By definition the family of all monomials

$$
\bigcup_{m=0}^{\infty}\left\{\mathbf{X}_{i_{1}} \cdots \mathbf{X}_{i_{m}} \mid i_{1}, \ldots, i_{m}=1,2,3, \ldots\right\}
$$

(including the empty monomial, which is identified to $1_{\mathbb{C}\langle\mathbf{X}\rangle}$ ) forms a Hamel basis for the vector space underlying $\mathbb{C}\langle\mathbf{X}\rangle$. In particular, $\mathbb{C}\langle\mathbf{X}\rangle=\bigcup_{m=1}^{\infty} \mathbb{C}\left\langle\mathbf{X}_{1}, \ldots, \mathbf{X}_{m}\right\rangle$. We equip $\mathbb{C}\langle\mathbf{X}\rangle$ with $*$-algebra structure by the rule $\mathbf{X}_{\ell}^{*}=\mathbf{X}_{\ell}$ for all $\ell$. Let $S^{\infty}$ denote the space of sequences in a set $S$. Given an algebra $\mathcal{A}$, a sequence $a \in \mathcal{A}^{\infty}$ and matrix $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)$, let $f(a) \in \operatorname{Mat}_{n}(\mathcal{A})$ denote the matrix obtained by evaluating each entry at $\mathbf{X}=a$ (and evaluating $1_{\mathbb{C}\langle\mathbf{X}\rangle}$ to $1_{\mathcal{A}}$ ). Note that if $\mathcal{A}$ is a *-algebra and $a \in \mathcal{A}_{\mathrm{sa}}^{\infty}$, then $f(a)^{*}=f^{*}(a)$, i.e., the evaluation map $f \mapsto f(a)$ is a *-algebra homomorphism. If $\mathcal{A}=\operatorname{Mat}_{N}(\mathbb{C})$, then we view $f(a)$ as an $n$-by- $n$ array of $N$-by- $N$ blocks, thus identifying it with an element of $\operatorname{Mat}_{n N}(\mathbb{C})$.
2.2. The model. We present the probabilistic data and assumptions needed to state our main result. We also indicate briefly how to construct many examples of data fulfilling our assumptions.
2.2.1. Data. For integers $\ell, N \geq 1$, fix a random element $\Xi_{\ell}^{N}$ of $\operatorname{Mat}_{N}(\mathbb{C})_{\text {sa }}$. Fix also an independent family $\{\mathcal{F}(i, j)\}_{1 \leq i \leq j<\infty}$ of $\sigma$-fields. Let $\mathcal{F}$ denote the $\sigma$-field generated by all the $\mathcal{F}(i, j)$. (See $\S 2.6 .3$ and $\S 7.2 .1$ below for augmentations of this setup needed to prove our main result but invisible in its statement.)
2.2.2. Assumptions. We assume for each $\ell$ the following:

$$
\begin{align*}
& \sup _{N} \bigvee_{i, j=1}^{N}\left\|\Xi_{\ell}^{N}(i, j)\right\|_{p}<\infty \text { for } p \in[1, \infty)  \tag{1}\\
& \sup _{N}\left\|\left[\left[\frac{\Xi_{\ell}^{N}}{\sqrt{N}}\right]\right]\right\| \|_{p}<\infty \text { for } p \in[1, \infty)  \tag{2}\\
& \left\|\limsup _{N \rightarrow \infty}\left[\left[\frac{\Xi_{\ell}^{N}}{\sqrt{N}}\right]\right]\right\|_{\infty}<\infty \tag{3}
\end{align*}
$$

Furthermore, we assume for each $\ell$ and $N$ the following:
(4) $\Xi_{\ell}^{N}$ is the upper left $N$-by- $N$ block of $\Xi_{\ell}^{N+1}$.
(5) $\quad\left(\Xi_{\ell}^{N}\right)^{\mathrm{T}}=(-1)^{\ell} \Xi_{\ell}^{N}$.
(6) $\Xi_{\ell}^{N}(i, j)$ is $\mathcal{F}(i \wedge j, i \vee j)$-measurable and $\mathbb{E} \Xi_{\ell}^{N}(i, j)=0$ for $i, j=1, \ldots, N$.
(7) $\left\|\Xi_{\ell}^{N}(i, j)\right\|_{2}=1$ for $1 \leq i<j \leq N$.

For simplicity, we insist that (4) and (5) (along with the self-adjointness of $\Xi_{\ell}^{N}$ ) hold for every sample point without exception. Finally, we assume that

$$
\begin{equation*}
\mathbb{E} \Xi_{\ell}^{N}(i, j) \Xi_{m}^{N}(i, j)=0 \text { for } 1 \leq i<j \leq N \text { and } 1 \leq \ell<m<\infty \tag{8}
\end{equation*}
$$

for all positive integers $i, j, N, \ell$ and $m$ subject to the indicated constraints.
Remark 2.2.3. In this paper we deal only with Wigner matrices having off-diagonal entries with uncorrelated real and imaginary parts. (Otherwise we would in effect have to move to a more complicated band matrix setting.) The requirement is enforced in a somewhat indirect way in our setup. In practice, this means that Wigner matrices as they occur "in nature" have to be broken down into antisymmetric and symmetric parts to be fed into our machinery. However, this extra trouble is compensated by some extra freedom: it turns out that more than just Wigner matrices can be broken down and fed in. See Remark 2.2.8 below for an example.
2.2.4. Random matrices. For each fixed $N$ we form a sequence

$$
\Xi^{N}=\left\{\Xi_{\ell}^{N}\right\}_{\ell=1}^{\infty} \in \operatorname{Mat}_{N}(\mathbb{C})_{\mathrm{sa}}^{\infty}
$$

of random hermitian matrices. We will be studying spectra of random hermitian matrices of the form

$$
f\left(\frac{\Xi^{N}}{\sqrt{N}}\right) \in \operatorname{Mat}_{n N}(\mathbb{C})_{\mathrm{sa}} \quad \text { for } f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\mathrm{sa}}
$$

in the limit $N \rightarrow \infty$.
Remark 2.2.5. It is easy to produce data fulfilling assumptions (4)-(8) and also for every $\ell$ the assumption

$$
\begin{equation*}
\sup _{N} \bigvee_{i, j=1}^{N}\left\|\Xi_{\ell}^{N}(i, j)\right\|_{\infty}<\infty \tag{9}
\end{equation*}
$$

Such data of course satisfy (1) and moreover automatically satisfy (2) and (3) by the result of Füredi-Komlós [7] paraphrased immediately below.

Proposition 2.2.6. For each $N \geq 1$, let $Y^{N}$ be a random $N-b y-N$ hermitian matrix whose entries on or above the diagonal are independent. Assume furthermore that the entries of the matrices $Y^{N}$ are essentially bounded uniformly in $N$ and have mean zero. Fix any sequence $\left\{k_{N}\right\}_{N=1}^{\infty}$ of positive integers such that $\frac{k_{N}}{\log N} \rightarrow \infty$ but $\frac{k_{N}}{N^{1 / 6}} \rightarrow 0$. Then $\sum_{N} \mathbb{E}\left[\left[\frac{Y^{N}}{c \sqrt{N}}\right]\right]^{2 k_{N}}<\infty$ for some (finite) constant $c>0$.
Here and below constants in estimates are denoted mostly by $c, C$ or $K$. The numerical values of these constants may of course vary from context to context and even from line to line.

Proof. We will use the result of Füredi-Komlós as cast in the form of the combinatorial estimate [1, Lemma 2.1.23]. By the cited lemma, for any constants

$$
\frac{c}{2}>K \geq \sup _{N} \bigvee_{i, j=1}^{N}\left\|Y^{N}(i, j)\right\|_{\infty}
$$

we deduce via "opening of the brackets" and counting of nonzero terms that

$$
\begin{aligned}
& \mathbb{E}\left[\left[\frac{Y^{N}}{c \sqrt{N}}\right]\right]^{2 k_{N}} \leq \mathbb{E} \operatorname{tr}\left(\frac{Y^{N}}{c \sqrt{N}}\right)^{2 k_{N}} \\
\leq & \frac{1}{c^{2 k_{N}}} \sum_{t=1}^{k_{N}+1} 2^{2 k_{N}}\left(2 k_{N}\right)^{3\left(2 k_{N}-2 t+2\right)} N^{t} \frac{K^{2 k_{N}}}{N^{k_{N}}} \\
= & N\left(\frac{2 K}{c}\right)^{2 k_{N}} \sum_{t=1}^{k_{N}+1}\left(\frac{2 k_{N}}{N^{1 / 6}}\right)^{6\left(k_{N}-t+1\right)}
\end{aligned}
$$

whence the result, since the last expression summed on $N$ is finite.
Remark 2.2.7. The more complicated argument presented in [1] immediately after [1, Lemma 2.1.23] gives the analogous result for Wigner-like random matrices whose entries have $L^{p}$ norms uniformly under a bound polynomial in $p$. We do not need the stronger result here for any of our proofs, but we mention it because it easily produces many more examples of data satisfying our assumptions. For example, any system $\left\{\Xi_{\ell}^{N}\right\}$ with Gaussian joint distribution satisfying (4)-(8) automatically also satisfies the remaining assumptions (1)-(3). Of course in the Gaussian case there exist noncombinatorial means to reach these same conclusions.

Remark 2.2.8. Let $\left\{Z_{\ell}(i, j)\right\}_{i, j, \ell=1}^{\infty}$ be an i.i.d. collection of $\mathbb{C}$-valued random variables which for some positive constant $c$ has the following properties:

$$
\begin{aligned}
& \sup _{p \in[1, \infty)} p^{-c} \mathbb{E}\left|Z_{1}(1,1)\right|^{p}<\infty, \mathbb{E}\left|Z_{1}(1,1)\right|^{2}=1, \mathbb{E} Z_{1}(1,1)^{2} \in(-1,1) \\
& \mathbb{E} Z_{1}(1,1)=0 \text { and for all } \ell, i \text { and } j, Z_{\ell}(i, j) \text { is } \mathcal{F}(i \wedge j, i \vee j) \text {-measurable. }
\end{aligned}
$$

Note that $Z_{\ell}(i, j)$ is forced to have uncorrelated real and imaginary parts. Define $Z_{\ell}^{N} \in \operatorname{Mat}_{N}(\mathbb{C})$ by $Z_{\ell}^{N}(i, j)=Z_{\ell}(i, j)$ for $i, j=1, \ldots, N$. Let $\sigma_{\ell}^{2}$ denote 8 times the variance of the imaginary (resp., real) part of $Z_{1}(1,1)$ if $\ell$ is odd (resp., even). Note that $\sigma_{\ell}>0$ for all $\ell$. Then the formula

$$
\Xi_{\ell}^{N}=\sigma_{\ell}^{-1}\left(\mathrm{i}^{\lceil\ell / 2\rceil}\left(Z_{\lceil\ell / 4\rceil}^{N}+(-1)^{\ell}\left(Z_{\lceil\ell / 4\rceil}^{N}\right)^{\mathrm{T}}\right)+\mathrm{i}^{-\lceil\ell / 2\rceil}\left(Z_{\lceil\ell / 4\rceil}^{N}+(-1)^{\ell}\left(Z_{\lceil\ell / 4\rceil}^{N}\right)^{\mathrm{T}}\right)^{*}\right)
$$

defines a family of matrices satisfying assumptions (1)-(8). Here $\lceil x\rceil$ denotes the least integer not less than $x$. Note that the matrix $Z_{\ell}^{N}$ is recoverable as a linear combination of $\left\{\Xi_{4 \ell-\nu}^{N}\right\}_{\nu=0}^{3}$.
2.3. Formulation of the main result. We state the main result after briefly sketching the operator-theoretic background.
2.3.1. Boltzmann-Fock space $\mathcal{H}$. Let $\mathcal{H}$ be a Hilbert space equipped with an orthonormal basis $\left\{v\left(i_{1} \cdots i_{k}\right)\right\}$ indexed by finite sequences of positive integers of all lengths, including the empty sequence. Let $B(\mathcal{H})$ be the $C^{*}$-algebra of bounded linear operators on $\mathcal{H}$. Let $1_{\mathcal{H}}=v(\emptyset) \in \mathcal{H}$. For integers $i>0$, let $\Sigma_{i} \in B(\mathcal{H})$ be the bounded linear operator which acts on the distinguished orthonormal basis by
the rule $\Sigma_{i} v\left(i_{1} \cdots i_{k}\right)=v\left(i i_{1} \cdots i_{k}\right)$. Then $\mathcal{H}$ is the Boltzmann-Fock space corresponding to a countable collection of particles, $1_{\mathcal{H}}$ is the vacuum state, $\Sigma_{i}$ is the $i^{\text {th }}$ raising (creation) operator and its adjoint $\Sigma_{i}^{*}$ is the $i^{\text {th }}$ lowering (annihilation) operator. (See $\S 4.4$ for review of the Boltzmann-Fock apparatus. See [1, Chap. 5] or [20] for background. See [16] for basic $C^{*}$-algebra theory.)
2.3.2. The law $\mu_{f}$. We equip $B(\mathcal{H})$ with the state $A \mapsto\left(1_{\mathcal{H}}, A 1_{\mathcal{H}}\right)$, thus making it into a $C^{*}$-probability space. More generally, we equip $\operatorname{Mat}_{n}(B(\mathcal{H}))\left(=B\left(\mathcal{H}^{n}\right)\right)$ with the state $T \mapsto \frac{1}{n}\left(1_{\mathcal{H}}, \operatorname{tr}_{B(\mathcal{H})}(T) 1_{\mathcal{H}}\right)$. Let

$$
\Xi=\left\{\Xi_{\ell}\right\}_{\ell=1}^{\infty}=\left\{\mathrm{i}^{\ell} \Sigma_{\ell}+\mathrm{i}^{-\ell} \Sigma_{\ell}^{*}\right\}_{\ell=1}^{\infty} \in B(\mathcal{H})_{\mathrm{sa}}^{\infty} .
$$

It is easy to check that $\Xi$ has a free semicircular joint law. (Indeed, it is well-known that the sequence $\left\{\Sigma_{\ell}+\Sigma_{\ell}^{*}\right\}$ has free semicircular joint law, and thus so does $\Xi$ since the latter is obtained by conjugating the former by a unitary operator diagonalized by the canonical orthonormal basis of $\mathcal{H}$.) For $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$, let $\mu_{f}$ denote the law of $f(\Xi) \in \operatorname{Mat}_{n}(B(\mathcal{H}))_{\text {sa }}$, which we invariably view as a compactly supported Borel probability measure on the real line. (See $\S 4.3$ for a quick review of $C^{*}$-probability spaces and laws of single operators. See [1, Chap. 5] for an extensive discussion of laws, including joint laws.)

Let supp $\mu$ denote the support of a Borel probability measure $\mu$ on $\mathbb{R}$, i.e., the complement of the largest open set of measure zero with respect to $\mu$. Let $\operatorname{Spec}(x)$ denote the spectrum of an element $x$ of a $C^{*}$-algebra $\mathcal{A}$. (Proposition 4.2.3 below justifies omission of reference to $\mathcal{A}$ from the notation.) These notions are bound together in our setup by the following fact.

Proposition 2.3.3. For all $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$, $\operatorname{supp} \mu_{f}=\operatorname{Spec}(f(\Xi))$.
The general result [20, Thm. 2.6.2] on Boltzmann-Fock space and elementary $C^{*}$-algebra theory together in principle imply the proposition. Nonetheless, for the reader's convenience and also as an occasion to introduce tools anyhow needed in $\S 6$ to construct and estimate solutions of the Schwinger-Dyson equation, we supply a proof of the proposition in $\S 4$ below.
2.3.4. The empirical distribution $\mu_{f}^{N}$. In general, given a hermitian matrix $A \in \operatorname{Mat}_{n}(\mathbb{C})$, the empirical distribution of its eigenvalues $\lambda_{1} \leq \cdots \leq \lambda_{n}$ is defined to be the probability measure $\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_{i}}$ on the real line. In particular, given $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$, let $\mu_{f}^{N}$ denote the (random) empirical distribution of the eigenvalues of the $N n$-by- $N n$ random hermitian matrix $f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)$.

The next result is well-known and provides the context for our main result.
Theorem 2.3.5. For all $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}, \mu_{f}^{N}$ converges weakly to $\mu_{f}$ as $N \rightarrow \infty$, almost surely.

See [1, Chap. 5] for background, references and the proof of a similar result. See also the recent preprint [15]. Yet another proof of Theorem 2.3.5 emerges here as a byproduct. (See $\S 2.6 .6$ below.)

Now we can state our main result.
Theorem 2.3.6. For all $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ and $\epsilon>0$, supp $\mu_{f}^{N}$ is contained in the $\epsilon$-neighborhood of $\operatorname{supp} \mu_{f}$ for $N \gg 0$, almost surely.

This result generalizes a random matrix (RM) result obtained in [8], refining earlier work in [9], which was a byproduct of a deeper operator-theoretic investigation. For another account of the RM result of [8] in question, in the context of an introduction to free probability, see [1, Chap. 5, Sec. 5$]$. The RM result of [8] has already inspired a variety of generalizations. In [19], following up on the earlier work [9], a generalization from the hermitian to the symmetric and symplectic Gaussian cases was obtained. In [4], a generalization to non-Gaussian distributions satisfying Poincaré-type inequalities was obtained. A recent preprint [14] provides a generalization involving polynomials in Gaussian Wigner matrices and deterministic matrices with convergent joint law. Our contribution here is to find a way to get by without any Poincaré-type inequalities. The inspiration for our workaround has been the pioneering earlier research of [2]. In the latter work essentially classical methods of erasing rows and columns as well as adroit use of classical $L^{p}$ inequalities carried the day.

The next corollary justifies the title of this paper.
Corollary 2.3.7. For every $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)$,

$$
\lim _{N \rightarrow \infty}\left[\left[f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)\right]\right]=\llbracket f(\Xi) \rrbracket \text { a.s. }
$$

Proof. After replacing $f$ by $f f^{*}$, we may assume that $f$ is self-adjoint, and furthermore that $f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)$ and $f(\Xi)$ are positive. We then need only show that the largest eigenvalue of $f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)$ converges as $N \rightarrow \infty$ to the largest element of the spectrum of $f(\Xi)$, almost surely. In any case, we have $\sup \operatorname{Spec}(f(\Xi))=\sup \operatorname{supp} \mu_{f}$ by Proposition 2.3.3. Finally, we have

$$
\sup \operatorname{supp} \mu_{f} \leq \liminf _{N \rightarrow \infty} f\left(\frac{\Xi^{N}}{\sqrt{N}}\right) \leq \limsup _{N \rightarrow \infty} f\left(\frac{\Xi^{N}}{\sqrt{N}}\right) \leq \sup \operatorname{supp} \mu_{f} \text { a.s. }
$$

by Theorem 2.3.5 on the left and Theorem 2.3.6 on the right.
2.4. Recollection of classical results. We quickly review the main results of [3].
2.4.1. The Bai-Yin model. Let $\{X(i, j)\}_{1 \leq i \leq j<\infty}$ be an independent family of real random variables such that the law of $X(i, j)$ depends only on $\mathbf{1}_{i=j}$. Assume furthermore that $X(1,1)$ and $X(1,2)$ have finite fourth moments and zero means. Let $\sigma$ be the standard deviation of $X(1,2)$. Given a positive integer $N$, let $W^{N}$ be the $N$-by- $N$ random real symmetric matrix with entries

$$
W^{N}(i, j)= \begin{cases}X(i, j) & \text { if } i \leq j \\ X(j, i) & \text { if } i>j\end{cases}
$$

To have a convenient catchphrase, let us call $\left\{W^{N}\right\}_{N=1}^{\infty}$ the Bai-Yin model for Wigner matrices. We have the following fundamental result.
Theorem 2.4.2 ([3, Thm. C ]). In the Bai-Yin model $\left\{W^{N}\right\}_{N=1}^{\infty}$, the largest eigenvalue of $\frac{W^{N}}{\sqrt{N}}$ converges to $2 \sigma$ as $N \rightarrow \infty$, almost surely.

Remark 2.4.3. By [3, Thm. A], the fourth moment hypothesis in Theorem 2.4.2 cannot be improved (while maintaining strong overall assumptions concerning the form of the joint law of the family $\left\{W^{N}\right\}$ and in particular enforcing the identification of $W^{N}$ with the upper $N$-by- $N$ block of $W^{N+1}$ ).

Remark 2.4.4. Only real symmetric matrices were treated in [3] but all the arguments carry over to the hermitian case. In particular, Theorem 2.4.2 continues to hold if we replace $W^{N}$ by $\breve{W}^{N}$, where $\breve{W}^{N}(i, j)=\left(\mathrm{i} \mathbf{1}_{i<j}-\mathrm{i} \mathbf{1}_{i>j}\right) W^{N}(i, j)$.

### 2.5. Results under fourth moment hypotheses.

2.5.1. The polynomialized Bai-Yin model. We present a straightforward polynomial generalization of the Bai-Yin model. Let $\left\{X_{k}(i, j)\right\}_{i, j, k=1}^{\infty}$ be an independent family of real random variables such that the law of $X_{k}(i, j)$ depends only on $\mathbf{1}_{i=j}$. Assume that $X_{1}(1,1)$ and $X_{1}(1,2)$ have finite fourth moments and zero means, and also for simplicity that $X_{1}(1,2)$ has unit variance. Given positive integers $\ell$ and $N$, let $W_{\ell}^{N}$ be the random $N$-by- $N$ hermitian matrix with entries

$$
W_{\ell}^{N}(i, j)=\left\{\begin{aligned}
\mathbf{1}_{i \leq j} X_{k}(i, j)+\mathbf{1}_{i>j} X_{k}(j, i) & \text { if } \ell=2 k \text { is even } \\
\mathrm{i} \mathbf{1}_{i>j} X_{k}(i, j)-\mathrm{i} \mathbf{1}_{i<j} X_{k}(j, i) & \text { if } \ell=2 k-1 \text { is odd. }
\end{aligned}\right.
$$

To have a convenient catchphrase, let us call the family $\left\{W_{\ell}^{N}\right\}$ of random matrices the polynomialized Bai-Yin model. Put $W^{N}=\left\{W_{\ell}^{N}\right\}_{\ell=1}^{\infty} \in \operatorname{Mat}_{N}(\mathbb{C})_{\mathrm{sa}}^{\infty}$ and for $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$, let $\nu_{f}^{N}$ denote the empirical distribution of eigenvalues of $f\left(\frac{W^{N}}{\sqrt{N}}\right) \in \operatorname{Mat}_{n N}(\mathbb{C})_{\text {sa }}$. We have the following result.

Corollary 2.5.2. Theorems 2.3.5 and 2.3.6 remain valid with $\mu_{f}^{N}$ replaced by $\nu_{f}^{N}$ (but with $\mu_{f}$ kept the same). Corollary 2.3.7 remains valid with $\Xi^{N}$ replaced by $W^{N}$ (but with $\Xi$ kept the same).

The proof will be completed in $\S 2.5 .6$ below after some preparation. We use the same truncation tactic as used in [2].
2.5.3. The $C$-truncated polynomialized Bai-Yin model. We continue working in the setup of $\S 2.5 .1$. Let $C>0$ be a large constant. Given a $\mathbb{C}$-valued square-integrable random variable $Z$ of mean zero, put

$$
\rho(Z)=\left\|Z \mathbf{1}_{|Z| \leq C}-\mathbb{E} Z \mathbf{1}_{|Z| \leq C}\right\|_{2}, \quad \theta(Z)=\left\|Z \mathbf{1}_{|Z|>C}-\mathbb{E} Z \mathbf{1}_{|Z|>C}\right\|_{2}
$$

and if $\rho(Z)>0$ put

$$
\operatorname{trunc}(Z)=\|Z\|_{2}\left(Z \mathbf{1}_{|Z| \leq C}-\mathbb{E} Z \mathbf{1}_{|Z| \leq C}\right) / \rho(Z)
$$

Assume now that $C>0$ is large enough so that $\rho\left(X_{1}(1,1)\right) \wedge \rho\left(X_{1}(1,2)\right)>0$. Let $\Xi_{\ell}^{N}$ be the result of applying the truncation procedure trunc to the entries of $W_{\ell}^{N}$. Let $\mathcal{F}(i, j)=\sigma\left(\left\{X_{\ell}(i, j), X_{\ell}(j, i)\right\}_{\ell=1}^{\infty}\right)$ for $1 \leq i<j<\infty$. Let us call $\left\{\Xi_{\ell}^{N}\right\} \cup\{\mathcal{F}(i, j)\}$ the $C$-truncated polynomialized Bai-Yin model.

Lemma 2.5.4. Assumptions (1)-(9) are satisfied by the $C$-truncated polynomialized Bai-Yin model $\{\mathcal{F}(i, j)\} \cup\left\{\Xi_{\ell}^{N}\right\}$. Furthermore, for each $\ell$,

$$
\begin{equation*}
\limsup _{N \rightarrow \infty}\left[\left[\frac{W_{\ell}^{N}-\Xi_{\ell}^{N}}{\sqrt{N}}\right]\right] \leq 2(\theta+1-\rho) \text { a.s. } \tag{10}
\end{equation*}
$$

where $\theta=\theta\left(X_{1}(1,2)\right)$ and $\rho=\rho\left(X_{1}(1,2)\right)$.
Proof. Remark 2.2.5 granted, assumptions (1)-(9) are trivial to verify. It remains only to prove (10). We have in any case a bound

$$
\left[\left[\frac{W_{\ell}^{N}-\Xi_{\ell}^{N}}{\sqrt{N}}\right]\right] \leq\left[\left[\frac{W_{\ell}^{N}-\rho \Xi_{\ell}^{N}}{\sqrt{N}}\right]\right]+\left[\left[\frac{(1-\rho) \Xi_{\ell}^{N}}{\sqrt{N}}\right]\right] \text { a.s. }
$$

The terms on the right almost surely tend as $N \rightarrow \infty$ to limits $2 \theta$ and $2(1-\rho)$, respectively, by Theorem 2.4.2 and Remark 2.4.4. The claim (10) is proved.

Lemma 2.5.5. For $A, B \in \operatorname{Mat}_{N}(\mathbb{C})_{\text {sa }}$ let $\lambda_{i}(A)$ and $\lambda_{i}(B)$ denote the $i^{\text {th }}$ largest eigenvalue, respectively. Then we have (i) $\bigvee_{i=1}^{N}\left|\lambda_{i}(A)-\lambda_{i}(B)\right| \leq \llbracket A-B \rrbracket$ and (ii) the corresponding empirical distributions are within distance $\llbracket A-B \rrbracket$ as measured in the Lipschitz bounded metric.

Recall that the distance of probability measures $\mu$ and $\nu$ on the real line in the Lipschitz bounded metric is the supremum of $\left|\int \varphi d \mu-\int \varphi d \nu\right|$ where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ ranges over functions with supremum norm and Lipschitz constant both $\leq 1$. Recall also that the Lipschitz-bounded metric is compatible with weak convergence.

Proof. (i) This is well-known. See [13] or [18]. (ii) For any test function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ with sup norm and Lipschitz constant both $\leq 1$, since $|\varphi(x)-\varphi(y)| \leq|x-y|$, we have $\left|\int \varphi d \mu_{A}-\int \varphi d \mu_{B}\right| \leq \llbracket A-B \rrbracket$ by part (i) of the lemma.
2.5.6. Proof of Corollary 2.5.2. Fix $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\mathrm{sa}}$ and $\epsilon>0$ arbitrarily. With a large constant $C>0$ to be aptly chosen presently, let $\left\{\Xi_{\ell}^{N}\right\}$ be the $C$-truncated polynomialized Bai-Yin model. By property (3) for $\left\{\Xi_{\ell}^{N}\right\}$ and its analogue for $\left\{W_{\ell}^{N}\right\}$, which holds by Theorem 2.4.2, along with estimate (10) and dominated convergence, we can choose $C$ large enough to guarantee that

$$
\limsup _{N \rightarrow \infty}\left[\left[f\left(\frac{W^{N}}{\sqrt{N}}\right)-f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)\right]\right]<\frac{\epsilon}{2} \text { a.s. }
$$

By Lemma 2.5.5(i), almost surely for $N \gg 0$, we have that $\operatorname{supp} \nu_{f}^{N}$ is contained in the $\frac{\epsilon}{2}$-neighborhood of $\operatorname{supp} \mu_{f}^{N}$, and in turn, by Theorem 2.3.6, almost surely for $N \gg 0$, we have that $\mu_{f}^{N}$ is contained in the $\frac{\epsilon}{2}$-neighborhood of $\operatorname{supp} \mu_{f}$. Thus the analogue of Theorem 2.3.6 is proved. A similar argument using Lemma 2.5.5(ii) proves the analogue of Theorem 2.3.5. Finally, the analogue of Corollary 2.3.7 is proved by almost verbatim repetition of the proof of that corollary.
2.6. Reformulation of main results. We rewrite our main result in a form involving Stieltjes transforms and an auxiliary random variable $\mathbf{z}$.
2.6.1. Further notation. Given a complex number $z \in \mathbb{C}$, let $\Re z=\frac{z+z^{*}}{2}$ and $\Im z=\frac{z-z^{*}}{2 \mathrm{i}}$. Let $\mathfrak{h}=\{z \in \mathbb{C} \mid \Im z>0\}$, which is the classical upper half-plane.
2.6.2. Stieltjes transforms. In general, given a probability measure $\mu$ on the real line, recall that the Stieltjes transform is defined by the formula

$$
S_{\mu}(z)=\int \frac{\mu(d t)}{t-z} \text { for } z \in \mathbb{C} \backslash \operatorname{supp} \mu
$$

Recall also that

$$
\begin{equation*}
S_{\mu}\left(z^{*}\right) \equiv S_{\mu}(z)^{*} \text { and }\left|S_{\mu}(z) \Im z\right| \leq 1 \tag{11}
\end{equation*}
$$

In particular, $S_{\mu}$ is real-valued on $\mathbb{R} \backslash \operatorname{supp} \mu$.
2.6.3. The auxiliary random variable $\mathbf{z}$. Let $m$ be an even positive integer. Let $\mathbf{z}$ be an $\mathfrak{h}$-valued random variable independent of $\mathcal{F}$ the law of which is specified by the integration formula

$$
\mathbb{E} \varphi(\mathbf{z})=\int_{0}^{\infty} \int_{-\infty}^{\infty} \varphi(x+\mathrm{i} y) \frac{e^{-\left(x^{2}+y^{2}\right) / 2} y^{m}}{(m-1)!!\pi} d x d y
$$

We call $m$ the strength of the repulsion of $\mathbf{z}$ from the real axis. For simplicity we assume that $\Im \mathbf{z}>0$ holds without exception. In general we allow $m$ to vary from one appearance of $\mathbf{z}$ to the next. Results below involving $\mathbf{z}$ are often stated with hypotheses to the effect that $m$ be sufficiently large. As we will see the exact distribution of $\mathbf{z}$ is not too important. But it is quite important that $\|1 / \Im \mathbf{z}\|_{p}<\infty$ for $p \in[1, m+1)$. Thus, by choosing the strength of the repulsion of $\mathbf{z}$ from the real axis large enough, the random variable $1 / \Im \mathbf{z}$ can be made to possess as many finite moments as we like.

We will prove the following technical result. This is the "actual" main result of the paper. We keep all notation and assumptions for Theorem 2.3.6 along with the notation introduced immediately above.

Theorem 2.6.4. Fix $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ arbitrarily. Then there exists a sequence

$$
\left\{\operatorname{bias}^{N}: \mathbb{C} \backslash \operatorname{supp} \mu_{f} \rightarrow \mathbb{C}\right\}_{N=1}^{\infty}
$$

of (deterministic) analytic functions satisfying

$$
\operatorname{bias}^{N}\left(z^{*}\right) \equiv \operatorname{bias}^{N}(z)^{*}
$$

such that for every $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N} N^{1 / 2}\left\|S_{\mu_{f}^{N}}(\mathbf{z})-S_{\mu_{f}}(\mathbf{z})\right\|_{p}<\infty,  \tag{12}\\
& \sup _{N} N^{3 / 2}\left\|S_{\mu_{f}^{N+1}}(\mathbf{z})-S_{\mu_{f}^{N}}(\mathbf{z})\right\|_{p}<\infty,  \tag{13}\\
& \sup _{N}\left\|\operatorname{bias}^{N}(\mathbf{z})\right\|_{p}<\infty \text { and }  \tag{14}\\
& \sup _{N} N^{2}\left\|\mathbb{E}\left(S_{\mu_{f}^{N}}(\mathbf{z}) \mid \mathbf{z}\right)-S_{\mu_{f}}(\mathbf{z})-\frac{\operatorname{bias}^{N}(\mathbf{z})}{N}\right\|_{p}<\infty, \tag{15}
\end{align*}
$$

provided the strength of the repulsion of $\mathbf{z}$ from the real axis is sufficiently great, depending on $p$.

The proof of Theorem 2.6.4 commences in $\S 4$ and takes up the remainder of the paper. An operator-theoretic description of $\left\{\operatorname{bias}^{N}\right\}$ will be developed below similar to if rather more complicated than that given for $S_{\mu_{f}}(z)$. (See Remark 8.1.5 below.) The remainder of $\S 2.6$ is devoted to recovering Theorems 2.3.5 and 2.3.6 from Theorem 2.6.4.

Remark 2.6.5. Fix a point $z_{0} \in \mathfrak{h}$ arbitrarily. For any analytic function $g: \mathfrak{h} \rightarrow \mathbb{C}$, we can recover the value $g\left(z_{0}\right)$ as the average of $g(z)$ over the disc $\left|z-z_{0}\right| \leq \frac{1}{2} \Im z_{0}$. Thus statement (12) for, say, $p=4$ implies that $S_{\mu_{f}^{N}}\left(z_{0}\right) \rightarrow_{N \rightarrow \infty} S_{\mu_{f}}\left(z_{0}\right)$, almost surely, by Jensen's inequality in conditional form and the Borel-Cantelli lemma.
2.6.6. Proof of Theorem 2.3.5 with Theorem 2.6.4 granted. By assumption (3), there exists a constant $A>0$ such that $\operatorname{supp} \mu_{f}^{N} \subset[-A, A]$ for $N \gg 0$, almost surely. By Remark 2.6.5, we have $S_{\mu_{f}^{N}}(\mathrm{i}+1 / k) \rightarrow_{N \rightarrow \infty} S_{\mu_{f}}(\mathrm{i}+1 / k)$, almost surely, for every integer $k>0$. The latter statement by standard subsequencing arguments (which we omit) implies that $\mu_{f}^{N}$ converges weakly to $\mu_{f}$, almost surely.

To derive Theorem 2.3.6 from Theorem 2.6.4 we need two lemmas.
Lemma 2.6.7. Let $\left\{Y_{N}\right\}_{N=1}^{\infty}$ be a sequence of nonnegative random variables on a common probability space. Assume that $\sup _{N} N \mathbb{E} Y_{N}<\infty$. Assume furthermore that $\sup _{N} N^{1 / 2}\left\|Y_{N+1}-Y_{N}\right\|_{4}<\infty$. Then $Y_{N} \rightarrow_{N \rightarrow \infty} 0$, almost surely.
Proof. We have $Y_{\left[k^{5 / 4}\right\rfloor} \rightarrow_{k \rightarrow \infty} 0$, almost surely, by the Chebychev inequality and the Borel-Cantelli lemma. (Here and below $\lfloor x\rfloor$ denotes the greatest integer not exceeding $x$.) Put $[N]=\vee_{k=1}^{\infty}\left\lfloor k^{5 / 4}\right\rfloor \mathbf{1}_{k^{5 / 4}<N}$. Clearly, we have $Y_{[N]} \rightarrow_{N \rightarrow \infty} 0$, almost surely. Since $N-[N]=O\left(N^{1 / 5}\right)$, we have $\left\|Y_{N}-Y_{[N]}\right\|_{4}=O\left(N^{3 / 10}\right)$ by the Minkowski inequality. Thus $Y_{N}-Y_{[N]} \rightarrow_{N \rightarrow \infty} 0$, almost surely, by the Chebychev inequality and the Borel-Cantelli lemma. The result follows.

Recall that the support of a function $\varphi$, denoted $\operatorname{supp} \varphi$, is the complement of the largest open set on which the function vanishes identically.
Lemma 2.6.8. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be infinitely differentiable and compactly supported. Then there exists a function $\Upsilon: \mathbb{C} \rightarrow \mathbb{C}$ with the following properties:
(I) $\Upsilon$ is infinitely differentiable and compactly supported. Furthermore, $\Upsilon$ satisfies $\operatorname{supp} \Upsilon \cap \mathbb{R}=\operatorname{supp} \varphi$ and has the symmetry $\Upsilon\left(z^{*}\right) \equiv \Upsilon(z)^{*}$.
(II) For any open set $D \subset \mathbb{C}$ such that $D^{*}=D \supset \operatorname{supp} \Upsilon$ and analytic function $b: D \rightarrow \mathbb{C}$ such that $b\left(z^{*}\right) \equiv b(z)^{*}$, we have $\Re \mathbb{E} \Upsilon(\mathbf{z}) b(\mathbf{z})=0$.
(III) For probability measures $\mu$ on $\mathbb{R}$, we have $\Re \mathbb{E} \Upsilon(\mathbf{z}) S_{\mu}(\mathbf{z})=\int \varphi d \mu$.

The lemma mildly refines a procedure buried in the proof of [1, Lemma 5.5.5].
Proof. We identify $\mathbb{C}$ with $\mathbb{R}^{2}$ in the customary way. We switch back and forth between writing $x+\mathrm{i} y$ and $(x, y)$ as it suits us. To begin the construction, let $\theta: \mathbb{R} \rightarrow[0,1]$ be an even infinitely differentiable function supported in the interval $[-1,1]$ and identically equal to 1 on the subinterval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Let $m$ denote the strength of the repulsion of $\mathbf{z}$ from the real axis. Put

$$
\Gamma(x, y)=\frac{1}{2 \pi} \theta(y) \sum_{j=0}^{m} \frac{(\mathrm{i} y)^{j}}{j!} \varphi^{(j)}(x)
$$

noting that $\Gamma$ is supported in $\operatorname{supp} \varphi \times[-1,1]$. Put $\Gamma^{\prime}(x, y)=\left(\frac{\partial}{\partial x}+\mathrm{i} \frac{\partial}{\partial y}\right) \Gamma(x, y)$, noting that $\Gamma^{\prime}\left(z^{*}\right) \equiv \Gamma^{\prime}(z)^{*}$. The significance of the differential operator $\frac{\partial}{\partial x}+\mathrm{i} \frac{\partial}{\partial y}$ is that it kills all analytic functions, i.e., it codes the Cauchy-Riemann equations. The sum defining $\Gamma(x, y)$ is contrived so that

$$
\Gamma^{\prime}(x, y)=\frac{1}{2 \pi} \frac{(\mathrm{i} y)^{m}}{m!} \varphi^{(m+1)}(x) \text { for }(x, y) \in \mathbb{R} \times\left(-\frac{1}{2}, \frac{1}{2}\right)
$$

Let $\rho(x, y)=\frac{y^{m} e^{-\left(x^{2}+y^{2}\right) / 2}}{(m-1)!!\pi}$. Then we have $2 \Gamma^{\prime}(x, y)=\Upsilon(x, y) \rho(x, y)$ for some function $\Upsilon$ satisfying (I). For any Borel measurable function $h: \mathbb{C} \rightarrow \mathbb{C}$ satisfying $h(z)^{*} \equiv h\left(z^{*}\right)$ almost everywhere with respect to Lebesgue measure we have

$$
\begin{equation*}
\Re \mathbb{E} \Upsilon(\mathbf{z}) h(\mathbf{z})=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma^{\prime}(x, y) h(x, y) d x d y \tag{16}
\end{equation*}
$$

provided that the integral on the right is absolutely convergent, as follows directly from the definition of $\Upsilon$. Furthermore, for any compact set $T \subset \mathbb{R}^{2}$ with a polygonal boundary and analytic function $h$ defined in a neighborhood of $T$ we have

$$
\begin{equation*}
\int_{T} \Gamma^{\prime} h d x d y=-\mathrm{i} \int_{\partial T} \Gamma h(d x+\mathrm{i} d y) \tag{17}
\end{equation*}
$$

by Green's theorem and the fact that $h$ is killed by $\frac{\partial}{\partial x}+\mathrm{i} \frac{\partial}{\partial y}$. To prove (II), take $T$ such that supp $\Gamma \subset T \backslash \partial T \subset T \subset D$ and take $h=b$. Then formulas (16) and (17) yield the result. To prove (III), assume at first that $\mu=\delta_{t}$ for some real $t$ and hence $S_{\mu}(z)=\frac{1}{t-z}$. Take $T$ to be an annulus centered at $t$ and take $h=\frac{1}{t-z}$. In the limit as the inner radius tends to 0 and the outer radius tends to $\infty$, formulas (16) and (17) yield the result. Finally, to get (III) in general, use Fubini's theorem - the hypotheses of the latter hold by (11) and the fact that $m \geq 1$.
2.6.9. Proof of Theorem 2.3.6 with Theorem 2.6.4 granted. Take $\mathbf{z}$ to have a strength of repulsion from the real axis large enough so that all statements of Theorem 2.6.4 hold for the given matrix $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ in the case $p=4$. As in the proof of Theorem 2.3.5, fix $A>0$ such that $\operatorname{supp} \mu_{f}^{N} \subset[-A, A]$ for $N \gg 0$, almost surely. Fix $\epsilon>0$ arbitrarily. Fix an infinitely differentiable function $\varphi: \mathbb{R} \rightarrow[0,1]$ with the following support properties:

- $\varphi$ is identically equal to 1 on $[-A, A]$ minus the $\epsilon$-neighborhood of $\operatorname{supp} \mu_{f}$.
- $\varphi$ is supported in some compact set disjoint from supp $\mu_{f}$.

For $N>0$ consider the nonnegative random variable $Y_{N}=n N \int \varphi d \mu_{f}^{N}$ the value of which for $N \gg 0$ bounds the number of eigenvalues of the random hermitian matrix $f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)$ straying outside the $\epsilon$-neighborhood of $\operatorname{supp} \mu_{f}$, almost surely. It will be enough to show that $Y_{N} \rightarrow_{N \rightarrow \infty} 0$, almost surely. Now by Lemma 2.6.8 and Fubini's theorem, for some compactly supported infinitely differentiable function $\Upsilon: \mathbb{C}^{2} \rightarrow \mathbb{C}$ with support disjoint from supp $\mu_{f}$, we have for each $N>0$ the representation $Y_{N}=n N \Re \mathbb{E}\left(\Upsilon(\mathbf{z}) S_{\mu_{f}^{N}}(\mathbf{z}) \mid \mathcal{F}\right)$, almost surely. Furthermore, by similar reasoning, for any analytic function $b: \mathbb{C} \backslash \operatorname{supp} \mu_{f} \rightarrow \mathbb{C}$ satisfying $b\left(z^{*}\right) \equiv b(z)^{*}$, we have $\Re \mathbb{E}(\Upsilon(\mathbf{z}) b(\mathbf{z}) \mid \mathcal{F})=0$, almost surely. From statements (12) and (13) with $p=4$ we deduce that $\sup _{N} N^{\frac{1}{2}}\left\|Y_{N+1}-Y_{N}\right\|_{4}<\infty$ via Jensen's inequality in conditional form. From statements (14) and (15) we deduce that $\sup _{N} N \mathbb{E} Y_{N}<\infty$. Thus $Y_{N} \rightarrow_{N \rightarrow \infty} 0$, almost surely, by Lemma 2.6.7, which finishes the proof.

## 3. Approximation of solutions of the Schwinger-Dyson equation

In this section we refine a powerful idea from [8] concerning approximation of solutions of the Schwinger-Dyson (SD) equation, working in a setup directed toward exploiting a refinement of the linearization trick presented later in the paper which preserves self-adjointness. See Lemma 3.4.3 below for a short paraphrase of the idea from [8] in a simplified geometry. See Proposition 3.5.2 below for an adaptation of the idea tailored for use in the proof of Theorem 2.6.4. We will make a few forward-looking references to material recorded in $\S 4$ and $\S 5$ but there is no danger of circular reasoning.

### 3.1. Block algebras.

Definition 3.1.1. Provisionally, we define a block algebra to be a finite-dimensional $C^{*}$-algebra isomorphic to $\operatorname{Mat}_{s}(\mathbb{C})$ for some positive integer $s$. Given a block algebra $\mathcal{S}$, let $B(\mathcal{S})$ denote the Banach algebra of linear maps of $\mathcal{S}$ to itself, equipped with the usual operator norm, again denoted by $\llbracket \cdot \rrbracket$.

Presently we will refine this definition by adding more structure, none of which for the moment is needed. See $\S 5.3$ below for the upgrade. See $\S 5.1$ below for a detailed discussion of norming rules and in particular for the rule by which $B(\mathcal{S})$ is normed.

### 3.2. The Schwinger-Dyson equation and its differentiated form.

Definition 3.2.1. Let $\mathcal{S}$ be a block algebra. Let $\mathcal{D} \subset \mathcal{S}$ be a (nonempty) open subset. Let $\Phi \in B(\mathcal{S})$ be a linear map. We say that an analytic function $G: \mathcal{D} \rightarrow \mathcal{S}$ satisfies the Schwinger-Dyson (SD) equation with covariance map $\Phi$ if

$$
1_{\mathcal{S}}+(\Lambda+\Phi(G(\Lambda))) G(\Lambda)=0
$$

for all $\Lambda \in \mathcal{D}$. Necessarily one has $G(\Lambda) \in \mathcal{S}^{\times}$for all $\Lambda \in \mathcal{D}$.
See [1, Chap. 5, Secs.4-5] for applications in random matrix theory. Also see [1] for references to the matrix model literature. Of course the Schwinger-Dyson equation plays a huge role in [8] and all recent similar works. See [12] for a viewpoint on the SD equation which influenced us a lot. For more background see [20] and [17].
3.2.2. Notation for derivatives. Given an analytic function $G: \mathcal{D} \rightarrow \mathcal{S}$ defined on an open subset $\mathcal{D}$ of a block algebra $\mathcal{S}$ and $\Lambda \in \mathcal{D}$, we define

$$
\mathbf{D}[G](\Lambda)=\left(\left.\zeta \mapsto \frac{d}{d t} G(\Lambda+t \zeta)\right|_{t=0}\right) \in B(\mathcal{S})
$$

For $\zeta \in \mathcal{S}$ we write $\mathbf{D}[G](\Lambda ; \zeta)=\mathbf{D}[G](\Lambda)(\zeta)$ to compress notation a bit.
Proposition 3.2.3. Let $\mathcal{S}$ be a block algebra and let $\mathcal{D} \subset \mathcal{S}$ be an open set. Let $G: \mathcal{D} \rightarrow \mathcal{S}$ be a solution of the $S D$ equation with covariance map $\Phi \in B(\mathcal{S})$. Then for every $\Lambda \in \mathcal{D}$ and $\zeta \in \mathcal{S}$ we have

$$
\begin{align*}
\zeta & =G(\Lambda)^{-1} \mathbf{D}[G](\Lambda ; \zeta) G(\Lambda)^{-1}-\Phi(\mathbf{D}[G](\Lambda ; \zeta))  \tag{18}\\
& =\mathbf{D}[G]\left(\Lambda ; G(\Lambda)^{-1} \zeta G(\Lambda)^{-1}-\Phi(\zeta)\right) \\
0 & =G(\Lambda)+\mathbf{D}[G](\Lambda ; \Lambda)+2 \mathbf{D}[G](\Lambda ; \Phi(G(\Lambda))) \tag{19}
\end{align*}
$$

We have immediate use for (18) in $\S 3$. The specialization (19) will be crucial in the endgame for proving statement (13) of Theorem 2.6.4.

Proof. To compress notation further we write $G=G(\Lambda)$ and $G^{\prime}=\mathbf{D}[G](\Lambda)$. By differentiation of the SD equation we obtain $\left(\zeta+\Phi\left(G^{\prime}(\zeta)\right)\right) G+(\Lambda+\Phi(G)) G^{\prime}(\zeta)=0$ and hence $\zeta=G^{-1} G^{\prime}(\zeta) G^{-1}-\Phi\left(G^{\prime}(\zeta)\right)$. Thus the first equality in (18) holds. Now for any linear operators $A$ and $B$ on a finite-dimensional vector space we have $A B=1 \Rightarrow B A=1$. Thus $\zeta=G^{\prime}\left(G^{-1} \zeta G^{-1}-\Phi(\zeta)\right)$, and hence the second equality in (18) holds. Finally, (19) follows by taking $\zeta=G(\Lambda)$ in (18).

### 3.3. SD tunnels.

Definition 3.3.1. Suppose we are given

- a solution $G: \mathcal{D} \rightarrow \mathcal{S}$ of the SD equation with covariance map $\Phi \in B(\mathcal{S})$,
- a point $\Lambda_{0} \in \mathcal{D}$ and
- (finite) constants $\mathfrak{T}>0$ and $\mathfrak{G} \geq 1$.

Put

$$
\mathcal{T}=\left\{\Lambda_{0}+\mathrm{i} t 1_{\mathcal{S}}+\zeta \mid t \in[0, \infty) \text { and } \zeta \in \mathcal{S} \text { s.t. } \llbracket \zeta \rrbracket \leq 1 / \mathfrak{G}\right\}
$$

Suppose that the following conditions hold:

$$
\begin{align*}
& \mathcal{T} \subset \mathcal{D} .  \tag{20}\\
& \sup _{\Lambda \in \mathcal{T}} \llbracket G(\Lambda) \rrbracket \leq \mathfrak{G} .  \tag{21}\\
& \sup _{\substack{\Lambda, \Lambda^{\prime} \in \mathcal{T} \\
\text { s.t. } \Lambda \neq \Lambda^{\prime}}} \frac{\llbracket G(\Lambda)-G\left(\Lambda^{\prime}\right) \rrbracket}{\llbracket \Lambda-\Lambda^{\prime} \rrbracket} \leq \mathfrak{G}^{2} .  \tag{22}\\
& \sup _{\substack{\Lambda, \Lambda^{\prime} \in \mathcal{T} \\
\text { s.t. } \Lambda \neq \Lambda^{\prime}}}^{\llbracket G(\Lambda)-G\left(\Lambda^{\prime}\right)-\mathbf{D}[G]\left(\Lambda^{\prime} ; \Lambda-\Lambda^{\prime}\right) \rrbracket}  \tag{23}\\
& \llbracket \Lambda-\Lambda^{\prime} \rrbracket^{2} \tag{24}
\end{align*} \mathfrak{G}^{3} .
$$

In this situation we say that the collection $\left(G: \mathcal{D} \rightarrow \mathcal{S}, \Phi, \Lambda_{0}, \mathfrak{T}, \mathfrak{G}\right)$ is a SchwingerDyson (SD) tunnel.

A major goal of the self-adjoint linearization trick developed below is to produce many examples of SD tunnels. See Remark 7.1.3 below for all examples of SD tunnels needed for the proof of Theorem 2.6.4.

Remark 3.3.2. If $\left(G: \mathcal{D} \rightarrow \mathcal{S}, \Phi, \Lambda_{0}, \mathfrak{T}, \mathfrak{G}\right)$ is an SD tunnel, then for every $t \in[0, \infty)$, so is $\left(G: \mathcal{D} \rightarrow \mathcal{S}, \Phi, \Lambda_{0}+\mathrm{i} t 1_{\mathcal{S}}, \mathfrak{T}, \mathfrak{G}\right)$.
3.4. The tunnel estimates. We explain how SD tunnels control errors.
3.4.1. Setup for the tunnel estimates.

- Let $\left(G: \mathcal{D} \rightarrow \mathcal{S}, \Phi, \Lambda_{0}, \mathfrak{T}, \mathfrak{G}\right)$ be an SD tunnel.
- Let $F=\left(t \mapsto F_{t}\right):[0, \mathfrak{T}] \rightarrow \mathcal{S}$ be a continuous function.

For $t \in[0, \mathfrak{T}]$ we put

$$
\begin{aligned}
& \Lambda_{t}=\Lambda_{0}+\mathrm{i} t 1_{\mathcal{S}}, \quad G_{t}=G\left(\Lambda_{t}\right), \quad G_{t}^{\prime}=\mathbf{D}[G]\left(\Lambda_{t}\right) \\
& E_{t}=1_{\mathcal{S}}+\left(\Lambda_{t}+\Phi\left(F_{t}\right)\right) F_{t}, \quad V_{t}=G_{t}^{\prime}\left(E_{t} G_{t}^{-1}\right)=G_{t}^{\prime}\left(\Phi\left(G_{t} E_{t}\right)\right)+G_{t} E_{t}
\end{aligned}
$$

The last equality, which will be useful below, is an instance of (18). We also define constants

$$
\mathfrak{C}=4(1+\llbracket \Phi \rrbracket), \quad \mathfrak{F}=1 \vee \sup _{t \in[0, \mathfrak{T}]} \llbracket F_{t} \rrbracket, \quad \mathfrak{A}=\sup _{t \in[0, \mathfrak{T}]} \llbracket E_{t} \rrbracket .
$$

The quantity $\mathfrak{A}$ is a natural measure of the failure of $F$ to satisfy the SD equation. We emphasize that we assume nothing of the function $F$ beyond continuity.

Proposition 3.4.2. Data and notation are as above. For $t \in[0, \mathfrak{T}]$ we have

$$
\begin{align*}
\llbracket F_{t}-G_{t} \rrbracket & \leq \mathfrak{C} \mathfrak{G}^{2} \mathfrak{F}\left(\llbracket E_{t} \rrbracket+\mathbf{1}_{\mathfrak{C G F A} \geq 1}+\mathbf{1}_{\llbracket F_{\mathfrak{I}} \rrbracket \geq 1}\right),  \tag{25}\\
\llbracket F_{t}+V_{t}-G_{t} \rrbracket & \leq \mathfrak{C}^{2} \mathfrak{G}^{5} \mathfrak{F}^{2}\left(\llbracket E_{t} \rrbracket^{2}+\mathbf{1}_{\mathfrak{C G F A} \geq 1}+\mathbf{1}_{\llbracket F_{\mathfrak{I}} \rrbracket \geq 1}\right) . \tag{26}
\end{align*}
$$

For the proof we need one absolutely stunning lemma.

Lemma 3.4.3. If

$$
\begin{align*}
& \mathfrak{C} \mathfrak{G} \mathfrak{F} \mathfrak{A}<1 \text { and }  \tag{27}\\
& \llbracket F_{\mathfrak{T} \rrbracket} \rrbracket<1, \tag{28}
\end{align*}
$$

then for every $t \in[0, \mathfrak{T}]$, the inverse $H_{t}=-\left(\Lambda_{t}+\Phi\left(F_{t}\right)\right)^{-1}$ exists,

$$
\begin{align*}
& \llbracket \Phi\left(H_{t} E_{t}\right) \rrbracket<1 / \mathfrak{G}, \quad(\text { hence }) \Lambda_{t}-\Phi\left(H_{t} E_{t}\right) \in \mathcal{D} \text { and }  \tag{29}\\
& H_{t}-H_{t} E_{t}=F_{t}=G\left(\Lambda_{t}-\Phi\left(H_{t} E_{t}\right)\right)-H_{t} E_{t} . \tag{30}
\end{align*}
$$

Proof. Fix $t \in[0, \mathfrak{T}]$ arbitrarily. Hypothesis (27) implies that $\llbracket E_{t} \rrbracket<1 / 2$. By Lemma 4.1.1 below it follows that $H_{t}$ is well-defined and satisfies

$$
\begin{equation*}
\llbracket H_{t} \rrbracket \leq 2 \llbracket F_{t} \rrbracket . \tag{31}
\end{equation*}
$$

Thus claim (29) holds by (20), (27) and (31). It remains only to prove claim (30), and since the first equality in (30) holds by definition of $H_{t}$, we have only to prove the second equality. By the Weierstrass Approximation Theorem, we may assume that $F$ depends polynomially and a fortiori analytically on $t$, i.e., $F$ is the restriction to $[0, \mathfrak{T}]$ of an analytic function defined in a neighborhood of $[0, \mathfrak{T}]$ in the complex plane. Put

$$
\widehat{H}_{t}=G\left(\Lambda_{t}-\Phi\left(H_{t} E_{t}\right)\right) \text { and } \widehat{F}_{t}=\widehat{H}_{t}-H_{t} E_{t}
$$

Note that $\widehat{F}_{t}$ depends analytically on $t$. It is enough to prove $F_{t} \equiv \widehat{F}_{t}$. In any case, since $G$ satisfies the SD equation with covariance map $\Phi$, we have

$$
1_{\mathcal{S}}+\left(\Lambda_{t}-\Phi\left(H_{t} E_{t}\right)+\Phi\left(\widehat{H}_{t}\right)\right) \widehat{H}_{t}=1_{\mathcal{S}}+\left(\Lambda_{t}+\Phi\left(\widehat{F}_{t}\right)\right) \widehat{H}_{t}=0
$$

and hence $\widehat{H}_{t}=-\left(\Lambda_{t}+\Phi\left(\widehat{F}_{t}\right)\right)^{-1}$. We thus have

$$
F_{t}-\widehat{F}_{t}=H_{t}-\widehat{H}_{t}=H_{t} \Phi\left(F_{t}-\widehat{F}_{t}\right) \widehat{H}_{t}=H_{t} \Phi\left(F_{t}-\widehat{F}_{t}\right) G\left(\Lambda_{t}-\Phi\left(H_{t} E_{t}\right)\right)
$$

where at the second step we use the resolvent identity (32). Finally, by (24), (28), (29) and (31) we have

$$
\llbracket H_{\mathfrak{T}} \rrbracket \llbracket \Phi \rrbracket \llbracket G\left(\Lambda_{\mathfrak{T}}-\Phi\left(H_{\mathfrak{T}} E_{\mathfrak{T}}\right)\right) \rrbracket<1,
$$

hence the difference $F_{t}-\widehat{F}_{t}$ vanishes identically for $t$ near $\mathfrak{T}$ and hence $F_{t} \equiv \widehat{F}_{t}$ by analytic continuation.
3.4.4. Proof of Proposition 3.4.2. We may assume that $\mathbf{1}_{\mathfrak{C G F} \mathfrak{A} \geq 1}+\mathbf{1}_{\llbracket F_{\mathfrak{\nwarrow}} \rrbracket \geq 1}=0$, for otherwise crude estimates suffice. But then the hypotheses of Lemma 3.4.3 are fulfilled. Thus it follows via (22), (29), (30), and (31) that

$$
\llbracket F_{t}-G_{t} \rrbracket \leq \llbracket H_{t}-G_{t} \rrbracket+\llbracket H_{t} E_{t} \rrbracket \leq \mathfrak{G}^{2}\left(2 \llbracket \Phi \rrbracket \mathfrak{F} \llbracket E_{t} \rrbracket\right)+2 \mathfrak{F} \llbracket E_{t} \rrbracket
$$

whence (25). To prove (26), we begin by noting the identity

$$
\begin{aligned}
F_{t}+V_{t}-G_{t}= & G\left(\Lambda_{t}-\Phi\left(H_{t} E_{t}\right)\right)+G_{t}^{\prime}\left(\Phi\left(H_{t} E_{t}\right)\right)-G_{t} \\
& +\left(G_{t}-H_{t}\right) E_{t}+G_{t}^{\prime}\left(\Phi\left(\left(G_{t}-H_{t}\right) E_{t}\right)\right)
\end{aligned}
$$

Then, reasoning as above, but now, instead of (22), using (23) and (24) along with the bound $\llbracket H_{t}-G_{t} \rrbracket \leq \mathfrak{G}^{2}\left(2 \llbracket \Phi \rrbracket \mathfrak{F} \llbracket E_{t} \rrbracket\right)$ obtained en passant above, we find that

$$
\begin{aligned}
& \llbracket F_{t}+V_{t}-G_{t} \rrbracket \\
\leq & \mathfrak{G}^{3}\left(2 \llbracket \Phi \rrbracket \mathfrak{F} \llbracket E_{t} \rrbracket\right)^{2}+\mathfrak{G}^{2}\left(2 \llbracket \Phi \rrbracket \mathfrak{F} \llbracket E_{t} \rrbracket\right) \llbracket E_{t} \rrbracket+\mathfrak{G}^{3} \llbracket \Phi \rrbracket\left(\mathfrak{G}^{2}\left(2 \llbracket \Phi \rrbracket \mathfrak{F} \llbracket E_{t} \rrbracket\right)\right) \llbracket E_{t} \rrbracket,
\end{aligned}
$$

whence (26).

Remark 3.4.5. Formula (30) is not an obvious target to shoot at! This surprising approach-to deform the given solution of the SD equation back toward the approximation-is exactly what we learned from [8]. The importance and utility of this idea cannot be overestimated.
3.5. The modified tunnel estimates. We put the tunnel estimates in a form tailored to the needs of the proof of Theorem 2.6.4.
3.5.1. Setup for the modified tunnel estimates.

- Let $\left(G: \mathcal{D} \rightarrow \mathcal{S}, \Phi, \Lambda_{0}, \mathfrak{T}, \mathfrak{G}\right)$ be an SD tunnel.
- Let $\mathfrak{L} \in[1, \infty)$ be a constant.
- Let $F=\left(t \mapsto F_{t}\right):[0, \infty) \rightarrow \mathcal{S}$ be a Lipschitz continuous function with Lipschitz constant bounded by $\mathfrak{L}$ and satisfying $\sup _{t \in[\mathfrak{T}, \infty)} \llbracket F_{t} \rrbracket \leq 1 / 2$.
For $t \in[0, \infty)$ we put

$$
\Lambda_{t}=\Lambda_{0}+\mathrm{i} t 1_{\mathcal{S}}, \quad E_{t}=1_{\mathcal{S}}+\left(\Lambda_{t}+\Phi\left(F_{t}\right)\right) F_{t}
$$

and we define constants

$$
\mathfrak{C}=99\left(1+\llbracket \Lambda_{0} \rrbracket+\llbracket \Phi \rrbracket\right), \quad \mathfrak{E}=\frac{1}{2} \llbracket E_{0} \rrbracket+\frac{1}{2} \int_{0}^{\infty} \llbracket E_{t} \rrbracket e^{-t} d t .
$$

The quantity $\mathfrak{E}$ is a perhaps less natural measure of the failure of $F$ to satisfy the SD equation but it has the advantage of being a sort of moment and thus more accessible to control by classical $L^{p}$ estimates. In practice the Lipschitz constant of $F$ will also be a quantity over which we have control.

Proposition 3.5.2. Data, notation and assumptions are as above. We have

$$
\begin{aligned}
\llbracket F_{0}-G\left(\Lambda_{0}\right) \rrbracket & \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \mathfrak{L}\right)^{6}\left(\mathfrak{E}+\mathfrak{E}^{2}\right), \\
{\left[\left[F_{0}+\mathbf{D}[G]\left(\Lambda_{0} ; E_{0} G\left(\Lambda_{0}\right)^{-1}\right)-G\left(\Lambda_{0}\right)\right]\right] } & \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \mathfrak{L}\right)^{12}\left(\mathfrak{E}^{2}+\mathfrak{E}^{4}\right),
\end{aligned}
$$

almost surely.
Proof. We begin by noting the crude bound

$$
\sup _{t \in[0, \infty)} \llbracket F_{t} \rrbracket \leq 2 e^{\mathfrak{T}} \mathfrak{L}
$$

We next claim that

$$
\sup _{t \in[0, \infty)} e^{-t} \llbracket E_{t} \rrbracket \leq \sqrt{e^{\mathfrak{T}} \mathfrak{C} \mathfrak{L}^{2}\left(\mathfrak{E}+\mathfrak{E}^{2}\right)} .
$$

Call the left side above $\mathfrak{B}$. Note that since $e^{-t} \llbracket E_{t} \rrbracket$ depends continuously on $t$ and tends to 0 at $\infty$, we have $\mathfrak{B}=e^{-t_{0}} \llbracket E_{t_{0}} \rrbracket<\infty$ for some $t_{0} \in[0, \infty)$. Now fix $0 \leq s<t<\infty$ arbitrarily. We have

$$
\begin{aligned}
& \left|e^{-s} \llbracket E_{s} \rrbracket-e^{-t} \llbracket E_{t} \rrbracket\right| \\
\leq & e^{-s} \llbracket E_{s} \rrbracket\left(1-e^{s-t}\right)+e^{-t} \llbracket E_{s}-E_{t} \rrbracket \\
\leq & \left(\mathfrak{B}+(1+\llbracket \Phi \rrbracket \mathfrak{L})\left(2 e^{\mathfrak{T}} \mathfrak{L}\right)+\left(\llbracket \Lambda_{0} \rrbracket+e^{-t} t+\llbracket \Phi \rrbracket\left(2 e^{\mathfrak{T}} \mathfrak{L}\right)\right) \mathfrak{L}\right)|s-t| \\
\leq & \left(\mathfrak{B}+e^{\mathfrak{T}} \mathfrak{C} \mathfrak{L}^{2} / 16\right)|s-t| .
\end{aligned}
$$

Thus $\mathfrak{B}+e^{\mathfrak{T}} \mathfrak{C} \mathfrak{L}^{2} / 16$ bounds the Lipschitz constant of $t \mapsto e^{-t} \llbracket E_{t} \rrbracket$. It follows that there exists a right triangle of altitude $\mathfrak{B}$ and base $\frac{\mathfrak{B}}{\mathfrak{B}+e^{\mathfrak{Z}} \mathfrak{C}^{2} / 16}$ under the graph of $e^{-t} \llbracket E_{t} \rrbracket$, and hence $2 \mathfrak{E} \geq \frac{1}{2} \frac{\mathfrak{B}}{\mathfrak{B}+e^{\mathfrak{F}} \mathfrak{C}^{2} / 16}$. The claim follows after some algebraic manipulation which we omit.

Now for the continuous path $\left(t \mapsto F_{t}\right):[0, \mathfrak{T}] \rightarrow \mathcal{S}$ let the constants $\mathfrak{F}$ and $\mathfrak{A}$ be as defined in Proposition 3.4.2. Since

$$
\mathfrak{F} \leq 2 e^{\mathfrak{T}} \mathfrak{L}, \quad \mathfrak{A} \leq e^{3 \mathfrak{T} / 2} \sqrt{\mathfrak{C} \mathfrak{L}^{2}\left(\mathfrak{E}+\mathfrak{E}^{2}\right)}, \quad \llbracket F_{\mathfrak{T}} \rrbracket \leq 1 / 2
$$

we have by Proposition 3.4.2 that

$$
\begin{aligned}
& \llbracket F_{0}-G\left(\Lambda_{0}\right) \rrbracket \leq \mathfrak{C}^{2}\left(2 e^{\mathfrak{T}} \mathfrak{L}\right)\left(2 \mathfrak{E}+\mathbf{1}_{\mathfrak{C} \mathfrak{G}\left(2 e^{\mathfrak{T}} \mathfrak{L}\right) e^{3 \mathfrak{T} / 2} \sqrt{\mathfrak{C N}^{2}\left(\mathfrak{E}+\mathfrak{E}^{2}\right)} \geq 1}\right), \\
& {\left[\left[F_{0}+\mathbf{D}[G]\left(\Lambda_{0} ; E_{0} G\left(\Lambda_{0}\right)^{-1}\right)-G\left(\Lambda_{0}\right)\right]\right] } \\
\leq & \mathfrak{C}^{2} \mathfrak{G}^{5}\left(2 e^{\mathfrak{T}} \mathfrak{L}\right)^{2}\left(4 \mathfrak{E}^{2}+\mathbf{1}_{\mathfrak{C} G\left(2 e^{\mathfrak{T}} \mathfrak{L}\right) e^{3 \mathfrak{T} / 2} \sqrt{\mathfrak{C}^{2}\left(\mathfrak{E}+\mathfrak{E}^{2}\right)} \geq 1}\right),
\end{aligned}
$$

whence the result after using Chebychev bounds and simplifying brutally.

## 4. Tools from operator theory

We review some elementary topics from $C^{*}$-algebra theory and in particular cobble together a proof of Proposition 2.3.3 from standard ingredients. These same ingredients will be used in $\S 6$ to construct and estimate solutions of the Schwinger-Dyson equation. With the latter goal in mind, we also derive an abstract algebraic version of the Schwinger-Dyson equation by (in effect) manipulating blockdecomposed matrices. (See Proposition 4.6 .4 below.) A tool used in that proof (see Proposition 4.5.2 below) has multiple uses in the sequel.
4.1. Warmup exercises. We record without proof several elementary facts used below. Recall that we only use algebras $\mathcal{A}$ possessing a unit $1_{\mathcal{A}}$.

Lemma 4.1.1. Let $x$ and $y$ be elements of a Banach algebra with $x$ invertible and $2\left[\left[x^{-1}\right]\right] \llbracket y \rrbracket \leq 1$. Then $x-y$ is invertible and $\left[\left[(x-y)^{-1}\right]\right] \leq 2\left[\left[x^{-1}\right]\right]$.
Here and below we invariably use $\llbracket \cdot \rrbracket$ to denote the norm on a Banach algebra.
4.1.2. We note the resolvent identity

$$
\begin{equation*}
x^{-1}-y^{-1}=y^{-1}(y-x) x^{-1}=x^{-1}(y-x) y^{-1} \quad\left(x, y \in \mathcal{A}^{\times}\right) \tag{32}
\end{equation*}
$$

holding in any algebra $\mathcal{A}$ and its infinitesimal variant $\frac{d}{d t} x^{-1}=-x^{-1} \frac{d x}{d t} x^{-1}$. We also need the iterated version

$$
\begin{equation*}
x^{-1}-y^{-1}=y^{-1}(y-x) y^{-1}+y^{-1}(y-x) y^{-1}(y-x) x^{-1} \quad\left(x, y \in \mathcal{A}^{\times}\right) . \tag{33}
\end{equation*}
$$

Lemma 4.1.3. Let $\mathcal{A}$ be a Banach algebra. Let $K \subset \mathcal{A}$ be a compact (resp., $\sigma$-compact) subset. Then the set $\left\{x \in \mathcal{A} \mid x-z \in \mathcal{A}^{\times}\right.$for $\left.z \in K\right\}$ is open (resp., Borel measurable) in $\mathcal{A}$.
4.2. Positivity. We recall basic facts about positive elements of $C^{*}$-algebras.
4.2.1. Positive elements and their square roots. If an element $x$ of a $C^{*}$-algebra $\mathcal{A}$ is self-adjoint with nonnegative spectrum, we write $x \geq 0$; and if furthermore $x$ is invertible, then we write $x>0$. Elements satisfying $x \geq 0$ are called positive. Elements of the form $x x^{*}$ are automatically positive. For $x \in \mathcal{A}$ such that $x \geq 0$, there exists unique $y \in \mathcal{A}$ such that $y \geq 0$ and $y^{2}=x$ (see [16, Thm. 2.2.1]), in which case we write $x^{1 / 2}=y$.
4.2.2. $C^{*}$-subalgebras and $G N S$. Let $\mathcal{A}$ be a $C^{*}$-algebra. We say that a closed subspace $\mathcal{A}_{0} \subset \mathcal{A}$ is a $C^{*}$-subalgebra if $\mathcal{A}_{0}$ is stable under $*$, closed under multiplication and furthermore $1_{\mathcal{A}} \in \mathcal{A}_{0}$, in which case $\mathcal{A}_{0}$ is a $C^{*}$-algebra in its own right for which $1_{\mathcal{A}_{0}}=1_{\mathcal{A}}$. Each $C^{*}$-algebra is isomorphic to a $C^{*}$-subalgebra of $B(H)$ for some Hilbert space $H$ via the GNS construction (see [16, §3.4]).

Proposition 4.2.3. For any $C^{*}$-algebra $\mathcal{A}$ and $C^{*}$-subalgebra $\mathcal{A}_{0} \subset \mathcal{A}$ we have $\mathcal{A}_{0} \cap \mathcal{A}^{\times}=\mathcal{A}_{0}^{\times}$.
(See [16, Thm. 2.1.11].) Thus the spectrum of $x \in \mathcal{A}_{0}$ is the same whether viewed in $\mathcal{A}_{0}$ or $\mathcal{A}$. In particular, $x$ is positive in $\mathcal{A}_{0}$ if and only if positive in $\mathcal{A}$.

Proposition 4.2.4. For every element $x$ of a $C^{*}$-algebra $\mathcal{A}$, if $x$ is normal, and in particular, if $x$ is self-adjoint, then $\llbracket x \rrbracket$ equals the spectral radius of $x$. Consequently, $\llbracket x \rrbracket^{2}$ equals the spectral radius of $x x^{*}$ and $x^{*} x$.
(See [16, Thm. 2.1.1 and Cor. 2.1.2].) It follows that a $*$-algebra can be normed as $C^{*}$-algebra in at most one way. We always use that norm when it exists.
4.2.5. Real and imaginary parts. Given any $*$-algebra and $Z \in \mathcal{A}$ we write $\Re Z=$ $\frac{Z+Z^{*}}{2}$ and $\Im Z=\frac{Z-Z^{*}}{2 \mathrm{i}}$. (This generalizes the notation we already have for real and imaginary parts of a complex number.)

The next elementary result plays an vitally important role in the paper.
Lemma 4.2.6. Let $\mathcal{A}$ be a $C^{*}$-algebra. Let $A \in \mathcal{A}$ satisfy $\Im A \geq 0$ and let $z \in \mathfrak{h}$. Then $A+z 1_{\mathcal{A}} \in \mathcal{A}^{\times}$and $\left[\left[\left(A+z 1_{\mathcal{A}}\right)^{-1}\right]\right] \leq 1 / \Im z$.

Proof. To abbreviate we write $1=1_{\mathcal{A}}, z=z 1_{\mathcal{A}}$, and so on. After replacing $A$ by $(A+\Re z) / \Im z$ we may assume without loss of generality that $z=\mathrm{i}$. Write $A=X+\mathrm{i} Y$ with $X=\Re A$ and $Y=\Im A$. Since $Y \geq 0$, we have $1+Y>0$, and hence we can write $A+\mathrm{i}=(1+Y)^{1 / 2}(W+\mathrm{i})(1+Y)^{1 / 2}$ where $W=(1+Y)^{-1 / 2} X(1+Y)^{-1 / 2} \in \mathcal{A}_{\text {sa }}$. Since both $(1+Y)^{1 / 2}$ and $W+\mathrm{i}$ are normal and have spectra disjoint from the open unit disc centered at the origin, both are invertible with inverse of norm $\leq 1$ by Proposition 4.2.4. Thus $A+\mathrm{i}$ is invertible with inverse of norm $\leq 1$.
4.3. States and spectral theory. We recall some basic definitions and results pertaining to $C^{*}$-probability spaces. Much of this background is covered in [16]. The rest of it is more or less implicit in [16] and [20] but hard to extract. Some of this material is also covered in [1] but unfortunately Lemma 4.3 .6 below is not. For the reader's convenience we supply short proofs of some key statements which are part of standard " $C^{*}$-know-how" but hard to pin down in the literature.
4.3.1. States. Let $\mathcal{A}$ be a $C^{*}$-algebra. Let $\phi: \mathcal{A} \rightarrow \mathbb{C}$ be any linear functional (perhaps not bounded). One calls $\phi$ positive if for every $A \in \mathcal{A}$, if $A \geq 0$, then $\phi(A) \geq 0$, in which case $\phi$ is automatically bounded and satisfies $\phi\left(x^{*}\right)=\phi(x)^{*}$. One calls $\phi$ a state if $\phi$ is positive and $\phi\left(1_{\mathcal{A}}\right)=1$, in which case $\llbracket \phi \rrbracket=1$. One calls a state $\phi$ faithful if for every $A \in \mathcal{A}$, if $A \geq 0$ and $A \neq 0$, then $\phi(A)>0$. Note that by Proposition 4.2.3, for any $C^{*}$-subalgebra $\mathcal{A}_{0} \subset \mathcal{A}$ and state $\phi$ on $\mathcal{A}$ the restriction of $\phi$ to $\mathcal{A}_{0}$ is again a state. (All of this is covered in [16, Chap. 3].)
Definition 4.3.2. A pair $(\mathcal{A}, \phi)$ consisting of a $C^{*}$-algebra $\mathcal{A}$ and a state $\phi$ is called a $C^{*}$-probability space. We call $(\mathcal{A}, \phi)$ faithful if $\phi$ is faithful.
4.3.3. Laws of noncommutative random variables. Given a $C^{*}$-probability space $(\mathcal{A}, \phi)$ and self-adjoint $A \in \mathcal{A}$, there exists a unique Borel probability measure $\mu_{A}$ on the spectrum of $A$, called the law of $A$, such that $\phi(f(A))=\int f d \mu_{A}$ for every continuous $\mathbb{C}$-valued function $f$ on the spectrum of $A$, where $f(A)$ is defined by means of the functional calculus at $A$, i.e., the inverse Gelfand transform, and $\mu_{A}$ is provided by the Riesz representation theorem. For convenience we always extend the law $\mu_{A}$ to a Borel probability measure on the real line supported on $\operatorname{Spec}(A)$. (See [1, Chap. 5] for background on laws.) We note the important formula

$$
S_{\mu_{A}}(z)=\phi\left(\left(A-z 1_{\mathcal{A}}\right)^{-1}\right)
$$

for the Stieltjes transform of the law $\mu_{A}$ which holds for every $z \in \mathbb{C}$ belonging neither to the support of $\mu_{A}$ nor to the spectrum of $A$. A simple and useful criterion for equality of the latter two sets is provided by the next result.

Lemma 4.3.4. Let $(\mathcal{A}, \phi)$ be a faithful $C^{*}$-probability space. Then, for every $A \in \mathcal{A}_{\text {sa }}, \operatorname{supp} \mu_{A}=\operatorname{Spec}(A)$.
Proof. Let $K=\operatorname{Spec}(A) \subset \mathbb{R}$, noting that $K$ is compact. Let $\mathcal{A}_{0} \subset \mathcal{A}$ be the $C^{*}$-subalgebra generated by $A$ and put $\phi_{0}=\left.\phi\right|_{\mathcal{A}_{0}}$, which is a faithful state on $\mathcal{A}_{0}$. By the theory of the Gelfand transform, $\mathcal{A}_{0}$ can be identified with the $C^{*}$-algebra of continuous complex-valued functions defined on $K$. Under this identification the operator $A$ becomes the identity function $\operatorname{Spec}(A) \rightarrow \mathbb{R}$ and $\phi_{0}$ becomes the linear functional represented by $\mu_{A}$. By Urysohn's Lemma, $\phi_{0}$ cannot be faithful unless $\operatorname{supp} \mu_{A}=K$.

Lemma 4.3.5. If $(\mathcal{A}, \phi)$ is a faithful $C^{*}$-probability space, then so is $\left(\operatorname{Mat}_{n}(\mathcal{A}), \phi_{n}\right)$, where $\phi_{n}(A)=\frac{1}{n} \sum_{i=1}^{n} \phi(A(i, i))$.

Proof. There is exactly one way to norm the $*$-algebra $\operatorname{Mat}_{n}(\mathcal{A})$ as a $C^{*}$-algebra. (See [16, Thm. 3.4.2] and also §5.1.6 below.) Following our convention to norm every *-algebra as a $C^{*}$-algebra when possible, we thus $\operatorname{regard} \operatorname{Mat}_{n}(\mathcal{A})$ as a $C^{*}$-algebra. For $0 \neq A \in \operatorname{Mat}_{n}(\mathcal{A})$ such that $A \geq 0$,

$$
\phi_{n}(A)=\phi_{n}\left(A^{1 / 2} A^{1 / 2}\right)=\frac{1}{n} \sum_{i, j=1}^{n} \phi\left(A^{1 / 2}(i, j) A^{1 / 2}(i, j)^{*}\right) .
$$

This formula first of all make it clear that $\phi_{n}$ is a state and hence that $\left(\operatorname{Mat}_{n}(\mathcal{A}), \phi_{n}\right)$ is a $C^{*}$-probability space. But furthermore, at least one term on the right is $>0$ since $\phi$ is faithful and $A^{1 / 2} \neq 0$. Thus $\phi_{n}$ is faithful.
Lemma 4.3.6. Let $H$ be a Hilbert space, let $v \in H$ be a unit vector, and consider the vectorial state $\phi=(A \mapsto(v, A v)): B(\mathcal{H}) \rightarrow \mathbb{C}$ associated with $v$. Let $\mathcal{A}, \hat{\mathcal{A}} \subset B(H)$ be $C^{*}$-subalgebras such that $A \hat{A}=\hat{A} A$ for all $A \in \mathcal{A}$ and $\hat{A} \in \hat{\mathcal{A}}$. Assume furthermore that the vector $v$ is cyclic for $\hat{\mathcal{A}}$, i.e., that the set $\{\hat{A} v \mid \hat{A} \in \hat{\mathcal{A}}\}$ is dense in $H$. Then $\left.\phi\right|_{\mathcal{A}}$ is faithful.

This is the most important point of the proof of Proposition 2.3.3.
Proof. Fix $A \in \mathcal{A}$ such that $A \geq 0$ and $A \neq 0$. Clearly there exists $h \in H$ such that $\left(h, A^{1 / 2} h\right)>0$. Thus by hypothesis there exists $\hat{A} \in \hat{\mathcal{A}}$ such that

$$
0<\left(\hat{A} v, A^{1 / 2} \hat{A} v\right)=\left(v, \hat{A}^{*} \hat{A} A^{1 / 2} v\right)=\phi\left(\hat{A}^{*} \hat{A} A^{1 / 2}\right) \leq \phi\left(\left(\hat{A}^{*} A\right)^{2}\right)^{1 / 2} \phi(A)^{1 / 2}
$$

The last inequality holds by Cauchy-Schwarz and forces $\phi(A)>0$.
4.4. Boltzmann-Fock space. We now have a closer look at the noncommutative probability space $\left(B(\mathcal{H}), A \mapsto\left(1_{\mathcal{H}}, A 1_{\mathcal{H}}\right)\right)$ first mentioned in $\S 2.3 .1$ and we complete the proof of Proposition 2.3.3. Essentially we are just summarizing in condensed form enough material from [20] to be able to exploit the powerful insight expressed by [20, Remark 2.6.7].
4.4.1. Definition of $\mathcal{H}$ and the $C^{*}$-probability space $(B(\mathcal{H}), \phi)$. Recall that $\mathcal{H}$ is a Hilbert space canonically equipped with an orthonormal basis $\left\{v\left(i_{1} \cdots i_{k}\right)\right\}$ indexed by all finite sequences of positive integers, including the empty sequence. Recall that $1_{\mathcal{H}}=v(\emptyset) \in \mathcal{H}$. We equip $B(\mathcal{H})$ with the vectorial state $\phi^{\mathrm{BF}}$ defined by $\phi^{\mathrm{BF}}(A)=\left(1_{\mathcal{H}}, A 1_{\mathcal{H}}\right)$, thus making it into a noncommutative probability space. Context permitting, we drop the superscript and write $\phi=\phi^{\mathrm{BF}}$.
4.4.2. Raising and lowering operators. Recall that $\Sigma_{i} \in B(\mathcal{H})$ acts by the rule $\Sigma_{i} v\left(i_{1} \cdots i_{k}\right)=v\left(i i_{1} \cdots i_{k}\right)$. Let $p_{\mathcal{H}} \in B(\mathcal{H})$ denote orthogonal projection to the linear span of $1_{\mathcal{H}}$. It is easy to verify the following relations, where $i$ and $j$ are any positive integers:
(34) $p_{\mathcal{H}} \Sigma_{i}=0=\Sigma_{i}^{*} p_{\mathcal{H}}, \quad \Sigma_{i}^{*} \Sigma_{j}=\delta_{i j} 1_{B(\mathcal{H})}, \llbracket \Sigma_{i} \rrbracket=\llbracket \Sigma_{i}^{*} \rrbracket=1$,
(35) $\phi\left(\Sigma_{i}\right)=\phi\left(\Sigma_{i}^{*}\right)=0, \phi\left(\Sigma_{i} \Sigma_{j}\right)=\phi\left(\Sigma_{i} \Sigma_{j}^{*}\right)=\phi\left(\Sigma_{i}^{*} \Sigma_{j}^{*}\right)=0, \phi\left(\Sigma_{i}^{*} \Sigma_{j}\right)=\delta_{i j}$.
4.4.3. Right raising and lowering operators. For each integer $i>0$, let $\hat{\Sigma}_{i} \in B(\mathcal{H})$ be defined by the action $\hat{\Sigma}_{i} v\left(i_{1} \cdots i_{k}\right)=v\left(i_{1} \cdots i_{k} i\right)$ on the canonical orthonormal basis of $\mathcal{H}$. In direct analogy to (34) we have

$$
\begin{equation*}
p_{\mathcal{H}} \hat{\Sigma}_{i}=0=\hat{\Sigma}_{i}^{*} p_{\mathcal{H}}, \quad \hat{\Sigma}_{i}^{*} \hat{\Sigma}_{j}=\delta_{i j} 1_{B(\mathcal{H})}, \quad\left[\left[\hat{\Sigma}_{i}\right]\right]=\left[\left[\hat{\Sigma}_{i}^{*}\right]\right]=1 . \tag{36}
\end{equation*}
$$

We also have right analogues of the relations (35) but we will not need them. It is easy to verify the following relations, where $i$ and $j$ are any positive integers:

$$
\begin{equation*}
\hat{\Sigma}_{i} \Sigma_{j}=\Sigma_{j} \hat{\Sigma}_{i}, \quad \hat{\Sigma}_{j}^{*} \Sigma_{i}=\Sigma_{i} \hat{\Sigma}_{j}^{*}+\delta_{i j} p_{\mathcal{H}}, \quad \Sigma_{i} p_{\mathcal{H}}=\hat{\Sigma}_{i} p_{\mathcal{H}} \tag{37}
\end{equation*}
$$

Note that every relation above implies another by taking adjoints on both sides.
4.4.4. Proof of Proposition 2.3.3. Let $\mathcal{A} \subset B(\mathcal{H})$ be the $C^{*}$-subalgebra generated by the sequence $\Xi=\left\{\Xi_{\ell}\right\}=\left\{\mathrm{i}^{\ell} \Sigma_{\ell}+\mathrm{i}^{-\ell} \Sigma_{\ell}^{*}\right\}$. By Lemmas 4.3.4 and 4.3.5, it is enough to show that $\left.\phi\right|_{\mathcal{A}}$ is faithful. Let $\hat{\Xi}_{\ell}=\mathrm{i}^{\ell} \hat{\Sigma}_{\ell}+\mathrm{i}^{-\ell} \hat{\Sigma}_{\ell}^{*} \in B(\mathcal{H})_{\mathrm{sa}}$ for positive integers $\ell$ and let $\hat{\mathcal{A}} \subset B(\mathcal{H})$ be the $C^{*}$-subalgebra of $B(\mathcal{H})$ generated by the sequence $\left\{\hat{\Xi}_{\ell}\right\}$. Using (37), one verifies that $\Xi_{\ell} \hat{\Xi}_{m}=\hat{\Xi}_{m} \Xi_{\ell}$ for all $\ell$ and $m$. (Here we are using the powerful insight of [20, Remark 2.6.7].) Thus every element of $\mathcal{A}$ commutes with every element of $\hat{\mathcal{A}}$. It is also easy to see that $1_{\mathcal{H}}$ is cyclic for $\hat{\mathcal{A}}$. Therefore $\left.\phi\right|_{\mathcal{A}}$ is faithful by Lemma 4.3.6.

We conclude our discussion of Boltzmann-Fock space by recording the following technical result for use in $\S 6$.

Lemma 4.4.5. Fix a positive integer $m$. Let

$$
x \in\left\{1_{B(\mathcal{H})}\right\} \cup\left\{\Sigma_{j}, \Sigma_{j}^{*} \mid j=1, \ldots, m\right\} .
$$

## The following hold:

- $\hat{\Sigma}_{i}^{*} x \hat{\Sigma}_{j}=\delta_{i j} x$ for all $i$ and $j$.
- $p_{\mathcal{H}} x \hat{\Sigma}_{i}=p_{\mathcal{H}} x \hat{\Sigma}_{i} p_{\mathcal{H}}$ and $p_{\mathcal{H}} \hat{\Sigma}_{i}^{*} x p_{\mathcal{H}}=\hat{\Sigma}_{i}^{*} x p_{\mathcal{H}}$ for all $i$.
- $x$ commutes with $\hat{\Sigma}_{i}$ and $\hat{\Sigma}_{i}^{*}$ for all $i>m$.
- $x$ commutes with $p_{\mathcal{H}}+\sum_{i=1}^{m} \hat{\Sigma}_{i} \hat{\Sigma}_{i}^{*}$.

Proof. The first three statements follow straightforwardly from (34), (36) and (37), so we just supply a proof for the last statement. We write $[A, B]=A B-B A$. Note that $[A, B C]=[A, B] C+B[A, C]$. Fix $j \in\{1, \ldots, m\}$. We then have

$$
\begin{aligned}
& {\left[\Sigma_{j}, p_{\mathcal{H}}+\sum_{i=1}^{m} \hat{\Sigma}_{i} \hat{\Sigma}_{i}^{*}\right]=\left[\Sigma_{j}, p_{\mathcal{H}}\right]+\sum_{i=1}^{m}\left(\left[\Sigma_{j}, \hat{\Sigma}_{i}\right] \hat{\Sigma}_{i}^{*}+\hat{\Sigma}_{i}\left[\Sigma_{j}, \hat{\Sigma}_{i}^{*}\right]\right) } \\
= & \Sigma_{j} p_{\mathcal{H}}+\hat{\Sigma}_{j}\left[\Sigma_{j}, \hat{\Sigma}_{j}^{*}\right]=\Sigma_{j} p_{\mathcal{H}}-\hat{\Sigma}_{j} p_{\mathcal{H}}=0
\end{aligned}
$$

The analogous relation with $\Sigma_{j}^{*}$ in place of $\Sigma_{j}$ follows by taking adjoints.
4.5. Projections and inverses. We discuss a broadened interpretation of the familiar formula

$$
\left[\begin{array}{ll}
a & b  \tag{38}\\
c & d
\end{array}\right]^{-1}=\left[\begin{array}{cc}
0 & 0 \\
0 & d^{-1}
\end{array}\right]+\left[\begin{array}{c}
1 \\
-d^{-1} c
\end{array}\right]\left(a-b d^{-1} c\right)^{-1}\left[\begin{array}{ll}
1 & -b d^{-1}
\end{array}\right]
$$

for inverting a block-decomposed two-by-two matrix.
4.5.1. Projections and $\pi$-inverses. Let $\mathcal{A}$ be a $*$-algebra. A projection $\pi \in \mathcal{A}$ by definition satisfies $\pi=\pi^{*}=\pi^{2}$. A family $\left\{\pi_{i}\right\}$ of projections is called orthonormal if $\pi_{i} \neq 0$ and $\pi_{i} \pi_{j}=\delta_{i j} \pi_{i}$ for all $i$ and $j$. Given $x \in \mathcal{A}$ and a projection $0 \neq \pi \in \mathcal{A}$, we denote by $x_{\pi}^{-1}$ the inverse of $\pi x \pi$ in the $*$-algebra $\pi \mathcal{A} \pi$, if it exists, in which case it is uniquely defined. We call $x_{\pi}^{-1}$ the $\pi$-inverse of $x$. Note that $x_{\pi}^{-1}=(\pi x \pi)_{\pi}^{-1}$.
Proposition 4.5.2. Let $\mathcal{A}$ be $a *$-algebra. Let $\left\{\pi, \pi^{\perp}\right\}$ be an orthonormal system of projections in $\mathcal{A}$ and put $\sigma=\pi+\pi^{\perp}$. Let $x \in \mathcal{A}$ be such that $\pi^{\perp} x \pi^{\perp} \in\left(\pi^{\perp} \mathcal{A} \pi^{\perp}\right)^{\times}$. Then we have

$$
\begin{equation*}
\sigma x \sigma \in(\sigma \mathcal{A} \sigma)^{\times} \Leftrightarrow \pi\left(x-x x_{\pi^{\perp}}^{-1} x\right) \pi \in(\pi \mathcal{A} \pi)^{\times} \tag{39}
\end{equation*}
$$

and under these equivalent conditions we have

$$
\begin{align*}
\pi x_{\sigma}^{-1} \pi & =\left(x-x x_{\pi^{\perp}}^{-1} x\right)_{\pi}^{-1},  \tag{40}\\
x_{\sigma}^{-1}-x_{\pi^{\perp}}^{-1} & =\left(\pi-x_{\pi^{\perp}}^{-1} x \pi\right) x_{\sigma}^{-1}\left(\pi-\pi x x_{\pi^{\perp}}^{-1}\right) . \tag{41}
\end{align*}
$$

This bit of folklore has at least three distinct important uses in the paper.
Proof. It is easy to see that the linear map

$$
\left(A \mapsto\left[\begin{array}{c}
\pi  \tag{42}\\
\pi^{\perp}
\end{array}\right] A\left[\begin{array}{ll}
\pi & \pi^{\perp}
\end{array}\right]\right): \sigma \mathcal{A} \sigma \rightarrow \operatorname{Mat}_{2}(\sigma \mathcal{A} \sigma)
$$

commutes with matrix multiplication, preserves the involution $*$ and is one-to-one. The image $\mathcal{B}$ of the map (42) is a $*$-algebra isomorphic to $\sigma \mathcal{A} \sigma$. We remark that

$$
1_{\mathcal{B}}=\left[\begin{array}{cc}
\pi & 0 \\
0 & \pi^{\perp}
\end{array}\right]=\left[\begin{array}{cc}
1_{\pi \mathcal{A} \pi} & 0 \\
0 & 1_{\pi^{\perp} \mathcal{A} \pi^{\perp}}
\end{array}\right] .
$$

Let $\left[\begin{array}{ll}a & b \\ c & d\end{array}\right] \in \mathcal{B}$ be the image of $\sigma x \sigma \in \sigma \mathcal{A} \sigma$ under (42). Since $d \in\left(\pi^{\perp} \mathcal{A} \pi^{\perp}\right)^{\times}$, we have a factorization

$$
\left[\begin{array}{ll}
a & b  \tag{43}\\
c & d
\end{array}\right]=\left[\begin{array}{cc}
1 & b d^{-1} \\
0 & 1
\end{array}\right]\left[\begin{array}{cc}
a-b d^{-1} c & 0 \\
0 & d
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
d^{-1} c & 1
\end{array}\right]
$$

holding in the algebra $\mathcal{B}$. Here we have abused notation by writing

$$
d^{-1}=d_{\pi^{\perp}}^{-1}, \quad 1=1_{\pi \mathcal{A} \pi}, \quad 1=1_{\pi^{\perp} \mathcal{A} \pi^{\perp}} .
$$

This cannot cause confusion because the dropped subscripts can be inferred from position in the matrix. We abuse notation similarly below. By (43) we have

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in \mathcal{B}^{\times} \Leftrightarrow a-b d^{-1} c \in(\pi \mathcal{A} \pi)^{\times}
$$

which is equivalent to (39). Let us now take (39) for granted. After some trivial algebraic manipulation we obtain the formula (38) which under the present interpretation is equivalent to the conjunction of (40) and (41).
Remark 4.5.3. The point of Proposition 4.5.2 is to make formula (38) available for use without having to resort to block-decomposed matrices.
4.6. Cuntz frames and quasi-circularity. We elaborate upon a suggestion made in the last exercise of [1]. We fix a $*$-algebra $\mathcal{A}$.
Definition 4.6.1. Suppose we are given a collection $\{\pi\} \cup\left\{\rho_{i}\right\}_{i=1}^{\infty}$ of elements of $\mathcal{A}$ satisfying the following conditions:
(44) $\quad \pi$ is a nonzero projection, $\pi \rho_{i}=0$ and $\rho_{i}^{*} \rho_{j}=\delta_{i j} 1_{\mathcal{A}}$ for all $i$ and $j$.

We call $\{\pi\} \cup\left\{\rho_{i}\right\}_{i=1}^{\infty}$ a Cuntz frame in $\mathcal{A}$. Note that $\{\pi\} \cup\left\{\rho_{i} \rho_{i}^{*}\right\}_{i=1}^{\infty}$ is automatically an orthonormal system of projections.
Remark 4.6.2. The relations $\rho_{i}^{*} \rho_{j}=\delta_{i j} 1_{\mathcal{A}}$ are those defining the Cuntz algebra [5], hence our choice of terminology.
4.6.3. Quasi-circular operators. Suppose we are given a Cuntz frame $\{\pi\} \cup\left\{\rho_{i}\right\}_{i=1}^{\infty}$ in $\mathcal{A}$. We say that an operator $A \in \mathcal{A}$ is quasi-circular (with respect to the given Cuntz frame) if the following statements hold:

$$
\begin{align*}
& \rho_{i}^{*} A \rho_{j}=\delta_{i j} A \text { for all } i \text { and } j .  \tag{45}\\
& \pi A \rho_{i} \pi=\pi A \rho_{i} \text { and } \pi \rho_{i}^{*} A \pi=\rho_{i}^{*} A \pi \text { for all } i .  \tag{46}\\
& \text { There exists an integer } k_{A} \geq 0 \text { such that } A \text { commutes }  \tag{47}\\
& \text { with } \pi+\sum_{i=1}^{k_{A}} \rho_{i} \rho_{i}^{*} \text { and also with } \rho_{i} \text { and } \rho_{i}^{*} \text { for all } i>k_{A} .
\end{align*}
$$

Proposition 4.6.4. Let $\{\pi\} \cup\left\{\rho_{i}\right\}_{i=1}^{\infty}$ be a Cuntz frame in $\mathcal{A}$. Let $A \in \mathcal{A}^{\times}$be quasi-circular with respect to the given frame. Choose any integer $k \geq k_{A}$. Then

$$
\begin{equation*}
\pi A^{-1} \pi=\left(\pi A \pi-\sum_{i=1}^{k} \pi A \rho_{i} \pi A^{-1} \pi \rho_{i}^{*} A \pi\right)_{\pi}^{-1} \tag{48}
\end{equation*}
$$

In particular, one automatically has $\pi A^{-1} \pi \in(\pi \mathcal{A} \pi)^{\times}$.
Identity (48) is an abstract algebraic version of the Schwinger-Dyson equation. See the proof of Proposition 6.1.4 below for the application.

Proof. Consider the projections $\sigma=\pi+\sum_{i=1}^{k} \rho_{i} \rho_{i}^{*}$ and $\pi^{\perp}=\sigma-\pi$. We claim that

$$
\begin{equation*}
A_{\pi^{\perp}}^{-1}=\sum_{i=1}^{k} \rho_{i} A^{-1} \rho_{i}^{*} . \tag{49}
\end{equation*}
$$

In any case, we have $\pi^{\perp} A \pi^{\perp}=\sum_{i=1}^{k} \rho_{i} A \rho_{i}^{*}$ by (45). Furthermore, we have

$$
\left(\sum_{i=1}^{k} \rho_{i} A \rho_{i}^{*}\right)\left(\sum_{j=1}^{k} \rho_{j} A^{-1} \rho_{j}^{*}\right)=\pi^{\perp}=\left(\sum_{i=1}^{k} \rho_{i} A^{-1} \rho_{i}^{*}\right)\left(\sum_{j=1}^{k} \rho_{j} A \rho_{j}^{*}\right)
$$

by (44). Thus claim (49) holds. To prove (48), we calculate as follows:

$$
\begin{aligned}
\pi A^{-1} \pi & =\pi \sigma A^{-1} \sigma \pi=\pi A_{\sigma}^{-1} \pi=\left(A-A A_{\pi^{\perp}}^{-1} A\right)_{\pi}^{-1} \\
& =\left(\pi A \pi-\pi A A_{\pi^{\perp}}^{-1} A \pi\right)_{\pi}^{-1}=\left(\pi A \pi-\sum_{i=1}^{k} \pi A \rho_{i} A^{-1} \rho_{i}^{*} A \pi\right)_{\pi}^{-1} \\
& =\left(\pi A \pi-\sum_{i=1}^{k} \pi A \rho_{i} \pi A^{-1} \pi \rho_{i}^{*} A \pi\right)_{\pi}^{-1} .
\end{aligned}
$$

The first step is simply an exploitation of orthonormality of $\left\{\pi, \pi^{\perp}\right\}$. Since $A$ commutes with $\sigma$ by (47), we have $\sigma A^{-1} \sigma A=\sigma=1_{\sigma A \sigma}=A \sigma A^{-1} \sigma$, which justifies the second step. The third step is an application of (40) and the fourth step is a trivial consequence of the definition of $\pi$-inverse. The fifth step is an application of (49) and the last step is an application of (46). The proof of (48) is complete.

Remark 4.6.5. The preceding calculation will obviate consideration of combinatorics of free semicircular variables in the sequel. We present this approach as counterpoint to the nowadays standard combinatorial approach discussed briefly in [1, Chap. 5] and developed in great detail in [17].

## 5. Tensor products, transpositions and other algebraic tools

We now add to our algebraic toolkit a variety of notions needed to deal with the Schwinger-Dyson equation. The less common notions, e.g., that of a transposition, are needed to deal with the secondary version of the Schwinger-Dyson equation and ultimately with correction terms.
5.1. Tensor products and norming rules. We rehearse the most basic rules of calculation and estimation used in the paper.
5.1.1. Tensor products of vector spaces and algebras. Given vector spaces $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$, let $\mathcal{A} \otimes \mathcal{B}$ denote the tensor product of $\mathcal{A}$ and $\mathcal{B}$ formed over $\mathbb{C}$. If $\mathcal{A}$ and $\mathcal{B}$ are both algebras, we invariably endow $\mathcal{A} \otimes \mathcal{B}$ with algebra structure by the rule $\left(a_{1} \otimes b_{1}\right)\left(a_{2} \otimes b_{2}\right)=a_{1} a_{2} \otimes b_{1} b_{2}$. If $\mathcal{A}$ and $\mathcal{B}$ are both $*$-algebras, we invariably endow $\mathcal{A} \otimes \mathcal{B}$ with $*$-algebra structure by the rule $(a \otimes b)^{*}=a^{*} \otimes b^{*}$.
5.1.2. Tensor notation for building matrices. Let $\mathcal{A}$ be an algebra. We identify the algebra $\operatorname{Mat}_{n}(\mathbb{C}) \otimes \mathcal{A}$ with $\operatorname{Mat}_{n}(\mathcal{A})$ by the rule $(X \otimes a)(i, j)=x(i, j) a$ and more generally use the same rule to identify the space $\operatorname{Mat}_{k \times \ell}(\mathbb{C}) \otimes \mathcal{A}$ with the space of rectangular matrices $\operatorname{Mat}_{k \times \ell}(\mathcal{A})$. Furthermore, in the case $\mathcal{A}=\operatorname{Mat}_{s}(\mathbb{C})$, we identify $X \otimes a$ with an element of $\operatorname{Mat}_{k s \times \ell s}(\mathbb{C})$ by viewing $X \otimes a$ as a $k$-by- $\ell$ arrangement of $s$-by- $s$ blocks $X(i, j) a$. In other words, we identify $X \otimes a$ with the usual Kronecker product of $X$ and $a$.
5.1.3. Banach spaces. Banach spaces always have $\mathbb{C}$ as scalar field, and bounded (multi)linear maps between Banach spaces are always $\mathbb{C}$-(multi)linear, unless explicitly noted otherwise. To avoid collision with the notation $\|\cdot\|_{p}$, we let $\llbracket \cdot \rrbracket_{\mathcal{V}}$ denote the norm of a Banach space $\mathcal{V}$ and context permitting (nearly always), we drop the subscript.
5.1.4. (Multi)linear maps between Banach spaces. Given Banach spaces $\mathcal{V}$ and $\mathcal{W}$, let $B(\mathcal{V}, \mathcal{W})$ denote the space of bounded linear maps $\mathcal{V} \rightarrow \mathcal{W}$. Let $B(\mathcal{V})=B(\mathcal{V}, \mathcal{V})$ and let $\mathcal{V}^{\star}$ denote the linear dual of $\mathcal{V}$. Given $T \in B(\mathcal{V}, \mathcal{W})$, let $\llbracket T \rrbracket=\llbracket T \rrbracket_{B(\mathcal{V}, \mathcal{W})}$ be the best constant such that $\llbracket T v \rrbracket \leq \llbracket T \rrbracket \llbracket v \rrbracket$. We always use the norm on $B(\mathcal{V}, \mathcal{W})$ so defined. More generally, let $B\left(\mathcal{V}_{1}, \ldots, \mathcal{V}_{r} ; \mathcal{W}\right)$ denote the space of bounded $r$-linear maps $\mathcal{V}_{1} \times \cdots \times \mathcal{V}_{r} \rightarrow \mathcal{W}$ and given $T \in B\left(\mathcal{V}_{1}, \ldots, \mathcal{V}_{r} ; \mathcal{W}\right)$, let $\llbracket T \rrbracket=\llbracket T \rrbracket_{B\left(\mathcal{V}_{1}, \ldots, \mathcal{V}_{r} ; \mathcal{W}\right)}$ be the best constant such that $\llbracket T\left(v_{1}, \ldots, v_{r}\right) \rrbracket \leq \llbracket T \rrbracket \llbracket v_{1} \rrbracket \cdots \llbracket v_{r} \rrbracket$. We always use the norm on $B\left(\mathcal{V}_{1}, \ldots, \mathcal{V}_{r} ; \mathcal{W}\right)$ so defined.
5.1.5. Matrix spaces over $C^{*}$-algebras. Let $\mathcal{A}$ be any $C^{*}$-algebra. We have already noted in the proof of Lemma 4.3 .5 that there is a unique way to norm the $*$-algebra $\operatorname{Mat}_{n}(\mathcal{A})$ as a $C^{*}$-algebra. In turn, we always norm the space of rectangular matrices $\operatorname{Mat}_{k \times \ell}(\mathcal{A})$ by the formula $\llbracket A \rrbracket=\llbracket A A^{*} \rrbracket^{1 / 2}$. Note that

$$
\begin{equation*}
\bigvee_{i=1}^{k} \bigvee_{j=1}^{\ell} \llbracket A(i, j) \rrbracket \leq \llbracket A \rrbracket \leq \sum_{m=-\infty}^{\infty} \bigvee_{i=1}^{k} \bigvee_{j=1}^{\ell} \llbracket A(i, j) \rrbracket \mathbf{1}_{i-j=m} \tag{50}
\end{equation*}
$$

Moreover, given $B \in \operatorname{Mat}_{\ell \times m}(\mathcal{A})$, we have $\llbracket A B \rrbracket \leq \llbracket A \rrbracket \llbracket B \rrbracket$. In particular, for every square or rectangular matrix $A$ with complex number entries, $\llbracket A \rrbracket$ is the largest singular value of $A$.
5.1.6. Tensor products of $C^{*}$-algebras. Given $C^{*}$-algebras $\mathcal{A}$ and $\mathcal{B}$ with at least one of them finite-dimensional, the $*$-algebra $\mathcal{A} \otimes \mathcal{B}$ has exactly one $C^{*}$-algebra norm. To see this, only existence requires comment since uniqueness we have already noted after Proposition 4.2.4. We proceed as follows. Firstly, we observe that since $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{B} \otimes \mathcal{A}$ are isomorphic $*$-algebras, we may assume that $\mathcal{A}$ is finite-dimensional. Then, after reducing to the case $\mathcal{A}=\operatorname{Mat}_{n}(\mathbb{C})$ and $\mathcal{B}=B(H)$ by using the GNS construction, we can make identifications $\mathcal{A} \otimes \mathcal{B}=\operatorname{Mat}_{n}(B(H))=B\left(H^{n}\right)$ yielding the desired norm. Thus existence is settled. The preceding argument shows that for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have $\llbracket a \otimes b \rrbracket=\llbracket a \rrbracket \llbracket b \rrbracket$. In a similar vein we have the following useful general observation.

Lemma 5.1.7. Let $\mathcal{S}$ be a finite-dimensional $C^{*}$-algebra. Let $\left\{e_{i}\right\}_{i=1}^{n}$ be any linearly independent family of elements of $\mathcal{S}$. Then for all $C^{*}$-algebras $\mathcal{A}$ and families $\left\{a_{i}\right\}_{i=1}^{n}$ of elements of $\mathcal{A}$ we have

$$
\frac{1}{C} \bigvee_{i=1}^{n} \llbracket a_{i} \rrbracket \leq\left[\left[\sum_{i=1}^{n} e_{i} \otimes a_{i}\right]\right]=\left[\left[\sum_{i=1}^{n} a_{i} \otimes e_{i}\right]\right] \leq C \sum_{i=1}^{n} \llbracket a_{i} \rrbracket
$$

for a constant $C \geq 1$ depending only on $\mathcal{S}$ and $\left\{e_{i}\right\}$.
Proof. We may assume that $\mathcal{S}=\operatorname{Mat}_{s}(\mathbb{C})$. Furthermore, there is no loss of generality to assume that $n=s^{2}$ and thus that $\left\{e_{i}\right\}_{i=1}^{s^{2}}$ is a basis for $\operatorname{Mat}_{s}(\mathbb{C})$. Finally, there is no loss of generality to assume that $\left\{e_{i}\right\}_{i=1}^{s^{2}}$ consists of elementary matrices, in which case the lemma at hand reduces to (50).

### 5.2. Transpositions.

Definition 5.2.1. Let $\mathcal{A}$ be a $*$-algebra. A transposition $a \mapsto a^{\mathrm{T}}$ of $\mathcal{A}$ is a $\mathbb{C}$-linear map such that $\left(a^{\mathrm{T}}\right)^{\mathrm{T}}=a,\left(a^{*}\right)^{\mathrm{T}}=\left(a^{\mathrm{T}}\right)^{*}$ and $(a b)^{\mathrm{T}}=b^{\mathrm{T}} a^{\mathrm{T}}$ for all $a, b \in \mathcal{A}$. Necessarily $1_{\mathcal{A}}^{\mathrm{T}}=1_{\mathcal{A}}$. $\mathrm{A} *$-algebra (resp., $C^{*}$-algebra) equipped with a transposition T will be called a (*, T)-algebra (resp., $C^{*, \mathrm{~T}}$-algebra).

Remark 5.2.2. Of course $\operatorname{Mat}_{n}(\mathbb{C})$ is a $C^{*, T}$-algebra. More generally, for any Hilbert space $H$ equipped with an orthonormal basis $\left\{h_{i}\right\}$, there exists a unique structure of $C^{*, \mathrm{~T}}$-algebra for $B(H)$ such that $\left(h_{i}, A h_{j}\right)=\left(h_{j}, A^{\mathrm{T}} h_{i}\right)$ for all operators $A \in B(H)$ and indices $i$ and $j$. The concept of $C^{*, T}$-algebra is essentially equivalent to that of a real $C^{*}$-algebra.
5.2.3. Transpositions, tensor products and matrices. Given $(*, \mathrm{~T})$-algebras $\mathcal{A}$ and $\mathcal{B}$, we invariably equip $\mathcal{A} \otimes \mathcal{B}$ with a transposition by the rule $(a \otimes b)^{\mathrm{T}}=a^{\mathrm{T}} \otimes b^{\mathrm{T}}$, thus equipping $\mathcal{A} \otimes \mathcal{B}$ with the structure of $(*, \mathrm{~T})$-algebra. Note that if $\mathcal{A}$ and $\mathcal{B}$ are $C^{*, T}$-algebras at least one of which is finite-dimensional, then $\mathcal{A} \otimes \mathcal{B}$ is again a $C^{*, T}$-algebra. For any $(*, \mathrm{~T})$-algebra $\mathcal{A}$ and matrix $A \in \operatorname{Mat}_{k \times \ell}(\mathcal{A})$, we define $A^{\mathrm{T}} \in \operatorname{Mat}_{\ell \times k}(\mathcal{A})$ by $A^{\mathrm{T}}(i, j)=A(j, i)^{\mathrm{T}}$. Thus, in particular, $\operatorname{Mat}_{n}(\mathcal{A})$ is automatically a $(*, \mathrm{~T})$-algebra (resp., $C^{*, \mathrm{~T}}$-algebra) whenever $\mathcal{A}$ is.
5.2.4. Transpositions on $\mathbb{C}\langle\mathbf{X}\rangle$ and $B(\mathcal{H})$. We equip the noncommutative polynomial ring $\mathbb{C}\langle\mathbf{X}\rangle$ with a transposition by the rule $\mathbf{X}_{\ell}^{\mathrm{T}}=(-1)^{\ell} \mathbf{X}_{\ell}$ for every $\ell$. The $C^{*}$-algebra $B(\mathcal{H})$ is canonically equipped with a transposition because BoltzmannFock space $\mathcal{H}$ is canonically equipped with an orthonormal basis.
Remark 5.2.5. The evaluation maps

$$
\begin{aligned}
\left(f \mapsto f\left(\frac{\Xi^{N}}{\sqrt{N}}\right)\right): \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle) & \rightarrow \operatorname{Mat}_{n N}(\mathbb{C}) \\
(f \mapsto f(\Xi)): \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle) & \rightarrow \operatorname{Mat}_{n}(B(\mathcal{H}))
\end{aligned}
$$

figuring in Theorems 2.3.5, 2.3.6 and 2.6.4 are ( $*, \mathrm{~T}$ )-algebra homomorphisms. One verifies this in the former case by using assumption (5) which (recall) says that $\left(\Xi_{\ell}^{N}\right)^{\mathrm{T}}=(-1)^{\ell} \Xi_{\ell}^{N}$. One verifies this in the latter case by noting that $\Sigma_{\ell}^{\mathrm{T}}=\Sigma_{\ell}^{*}$ which, since (recall) $\Xi_{\ell}=\mathrm{i}^{\ell} \Sigma_{\ell}+\mathrm{i}^{-\ell} \Sigma_{\ell}^{*}$, implies $\Xi_{\ell}^{\mathrm{T}}=(-1)^{\ell} \Xi_{\ell}$.
Lemma 5.2.6. If $x$ is an element of a $C^{*, T}$-algebra $\mathcal{A}$, then $\left(x^{-1}\right)^{\mathrm{T}}=\left(x^{\mathrm{T}}\right)^{-1}$, $x \in \mathcal{A}_{\mathrm{sa}} \Rightarrow x^{\mathrm{T}} \in \mathcal{A}_{\mathrm{sa}}, \operatorname{Spec}(x)=\operatorname{Spec}\left(x^{\mathrm{T}}\right), x \geq 0 \Rightarrow x^{\mathrm{T}} \geq 0$ and $\left[\left[x^{\mathrm{T}}\right]\right]=\llbracket x \rrbracket$.
Proof. The first two claims are obvious. The third claim follows from the first. The second and third claims imply the fourth. The fifth holds for self-adjoint $x$ by Proposition 4.2.4 along with second and third claims. The fifth claim holds in general because $\left[\left[x^{\mathrm{T}}\right]\right]^{2}=\left[\left[\left(x^{\mathrm{T}}\right)^{*} x^{\mathrm{T}}\right]\right]=\left[\left[\left(x x^{*}\right)^{\mathrm{T}}\right]\right]=\llbracket x x^{*} \rrbracket=\llbracket x \rrbracket^{2}$.
Definition 5.2.7. Given a $C^{*, T}$-algebra $\mathcal{A}$ and a state $\phi \in \mathcal{A}^{\star}$, we say that $\phi$ is T -stable if $\phi\left(A^{\mathrm{T}}\right)=\phi(A)$ for all $A \in \mathcal{A}$. A pair $(\mathcal{A}, \phi)$ consisting of a $C^{*, \mathrm{~T}}$-algebra and a T-stable state $\phi$ will be called a $C^{*, T}$-probability space.
Remark 5.2.8. It is easy to see that both $\left(\operatorname{Mat}_{N}(\mathbb{C}), \frac{1}{N} \operatorname{tr}\right)$ and $\left(B(\mathcal{H}), \phi^{\mathrm{BF}}\right)$ are in fact $C^{*, T}$-probability spaces.

### 5.3. Block algebras ("version 2.0").

Definition 5.3.1. A block algebra is a $C^{*}, \mathrm{~T}$-algebra isomorphic to the matrix algebra $\operatorname{Mat}_{s}(\mathbb{C})$ for some integer $s>0$. A basis $\left\{e_{i j}\right\}_{i, j=1}^{s}$ for $\mathcal{S}$ such that $e_{i j} e_{i^{\prime} j^{\prime}}=\delta_{j i^{\prime}} e_{i j^{\prime}}$ and $e_{i j}^{*}=e_{j i}=e_{i j}^{\mathrm{T}}$ will be called standard.
Remark 5.3.2. A choice of standard basis of a block algebra is the same thing as a choice of a $C^{*, T}$-algebra isomorphism with $\operatorname{Mat}_{s}(\mathbb{C})$.

Remark 5.3.3. The tensor product of block algebras is again a block algebra. Furthermore, for every block algebra $\mathcal{S}$, the tensor product algebra $\mathbb{C}\langle\mathbf{X}\rangle \otimes \mathcal{S}$ (resp., $B(\mathcal{H}) \otimes \mathcal{S})$ is a $(*, \mathrm{~T})$-algebra (resp., $C^{*, \mathrm{~T}}$-algebra).
Remark 5.3.4. Each block algebra $\mathcal{S}$ is equipped with a unique state $\tau_{\mathcal{S}}$ satisfying $\tau_{\mathcal{S}}\left(e_{i j}\right)=(\operatorname{dim} \mathcal{S})^{-1 / 2} \delta_{i j}$ for any standard basis $\left\{e_{i j}\right\}$. Necessarily $\tau_{\mathcal{S}}$ is T-stable. More generally, for each projection $e \in \mathcal{S}$, there exists a unique state $\tau_{\mathcal{S}, e} \in \mathcal{S}^{\star}$ such that $\left.\tau_{\mathcal{S}, e}\right|_{e \mathcal{S} e}=\tau_{e S}$.
5.3.5. The bullet map. Given a block algebra $\mathcal{S}$, we define a linear isomorphism $\left(A \mapsto A^{\bullet}\right): \mathcal{S}^{\otimes 2} \rightarrow B(\mathcal{S})$ by the formula $(x \otimes y)^{\bullet}=(z \mapsto x z y)$. That the bullet map is indeed a linear isomorphism one can check by calculating with a standard basis. This map in general neither preserves norms nor algebra structure.
5.3.6. The half-transpose map. Given a block algebra $\mathcal{S}$, we define a linear isomorphism $\left(A \mapsto A^{1 \otimes \mathrm{~T}}\right) \in B\left(\mathcal{S}^{\otimes 2}\right)$ by the formula $(x \otimes y)^{1 \otimes \mathrm{~T}}=x \otimes y^{\mathrm{T}}$. This map in general neither preserves norms nor algebra structure.
Remark 5.3.7. Strangely enough, the composite map

$$
\left((x \otimes y) \mapsto\left((x \otimes y)^{1 \otimes \mathrm{~T}}\right) \bullet\right): \mathcal{S}^{\otimes 2} \rightarrow B(\mathcal{S})
$$

is an isomorphism of algebras, as one verifies by calculating with a standard basis. This observation is the key to calculating correction terms. (But this map still does not in general preserve norms.)

## 5.4. $\mathcal{S}$-linear forms.

Definition 5.4.1. Let $\mathcal{S}$ be any block algebra. An $\mathcal{S}$-linear form $L$ is an element of the tensor product algebra $\mathbb{C}\langle\mathbf{X}\rangle \otimes \mathcal{S}$ of the form $L=\sum_{\ell=1}^{\infty} \mathbf{X}_{\ell} \otimes a_{\ell}$ for some elements $a_{\ell} \in \mathcal{S}$ vanishing for $\ell \gg 0$. We refer to the sum $\sum_{\ell} \mathbf{X}_{\ell} \otimes a_{\ell}$ as the Hamel expansion of $L$ and to the elements $a_{\ell} \in \mathcal{S}$ as the Hamel coefficients of $L$. Given a sequence $\xi=\left\{\xi_{\ell}\right\}_{\ell=1}^{\infty} \in \mathcal{A}^{\infty}$ in an algebra $\mathcal{A}$, we define $L(\xi)=\sum_{\ell} \xi_{\ell} \otimes a_{\ell} \in \mathcal{A} \otimes \mathcal{S}$, calling this the evaluation of $L$ at $\xi$. It is especially important to notice that if $\mathcal{A}=\operatorname{Mat}_{N}(\mathbb{C})$, then $L(\xi) \in \operatorname{Mat}_{N}(\mathbb{C}) \otimes \mathcal{S}=\operatorname{Mat}_{N}(\mathcal{S})$. (This is the reason for putting the tensor factors in $\mathbb{C}\langle\mathbf{X}\rangle \otimes \mathcal{S}$ in the "wrong" order.)
Definition 5.4.2. Let $\mathcal{S}$ be a block algebra and let $L$ be an $\mathcal{S}$-linear form with Hamel expansion $L=\sum \mathbf{X}_{\ell} \otimes a_{\ell}$. We define $\Phi_{L} \in B(\mathcal{S})$ by the formula $\Phi_{L}(\zeta)=\sum a_{\ell} \zeta a_{\ell}$ for $\zeta \in \mathcal{S}$ and we define $\Psi_{L}=\sum(-1)^{\ell} a_{\ell}^{\otimes 2} \in \mathcal{S}^{\otimes 2}$. We call $\Phi_{L}$ the covariance map attached to $L$. We call $\Psi_{L}$ the covariance tensor attached to $L$.
Definition 5.4.3. For any ( $*, \mathrm{~T}$ )-algebra $\mathcal{A}$, let $\mathcal{A}_{\text {alt }}^{\infty}$ denote the space of sequences $\xi=\left\{\xi_{\ell}\right\}_{\ell=1}^{\infty}$ in $\mathcal{A}$ such that $\xi_{\ell}^{\mathrm{T}}=(-1)^{\ell} \xi_{\ell}$ for all $\ell$. Also put $\mathcal{A}_{\mathrm{salt}}^{\infty}=\mathcal{A}_{\mathrm{sa}}^{\infty} \cap \mathcal{A}_{\mathrm{alt}}^{\infty}$.

Remark 5.4.4. Let $\mathcal{A}$ be a $(*, \mathrm{~T})$-algebra, $\xi=\left\{\xi_{\ell}\right\}_{\ell=1}^{\infty} \in \mathcal{A}_{\text {salt }}^{\infty}$ a sequence and $L$ an $\mathcal{S}$-linear form. Then we have $L^{\mathrm{T}}(\xi)=L(\xi)^{\mathrm{T}}$ and $L^{*}(\xi)=L(\xi)^{*}$. In particular, this observation applies to the sequences $\Xi^{N} \in \operatorname{Mat}_{N}(\mathbb{C})_{\text {salt }}^{\infty}$ and $\Xi \in B(\mathcal{H})_{\text {salt }}^{\infty}$ figuring prominently in Theorem 2.6.4.

## 5.5. $\mathcal{S}$-(bi)linear constructions.

5.5.1. $\mathcal{S}$-linear extension of states. Given any $C^{*}$-probability space $(\mathcal{A}, \phi)$ and block algebra $\mathcal{S}$, we define the $\mathcal{S}$-linear extension $\phi_{\mathcal{S}}: \mathcal{A} \otimes \mathcal{S} \rightarrow \mathcal{S}$ of $\phi$ by the formula

$$
\phi_{\mathcal{S}}(x \otimes y)=\phi(x) y .
$$

Note that since $\phi$ commutes with the involution, the same is true for $\phi_{\mathcal{S}}$, i.e.,

$$
\begin{equation*}
\phi_{\mathcal{S}}\left(A^{*}\right)=\phi_{\mathcal{S}}(A)^{*} \tag{51}
\end{equation*}
$$

for $A \in \mathcal{A} \otimes \mathcal{S}$. Suppose now that $(\mathcal{A}, \phi)$ is a $C^{*, T}$-probability space. Note that since $\phi$ is T-stable, $\phi_{\mathcal{S}}$ commutes with T, i.e.,

$$
\begin{equation*}
\phi_{\mathcal{S}}\left(A^{\mathrm{T}}\right)=\phi_{\mathcal{S}}(A)^{\mathrm{T}} \tag{52}
\end{equation*}
$$

for $A \in \mathcal{A} \otimes \mathcal{S}$.

Remark 5.5.2. Consider the case $(\mathcal{A}, \phi)=\left(\operatorname{Mat}_{N}(\mathbb{C}), \frac{1}{N} \operatorname{tr}\right)$. We have

$$
\phi_{\mathcal{S}}=\frac{1}{N} \operatorname{tr}_{\mathcal{S}}: \operatorname{Mat}_{N}(\mathcal{S}) \rightarrow \mathcal{S}
$$

Thus the ad hoc construction $\operatorname{tr}_{\mathcal{S}}$ fits into a more general conceptual framework.
Remark 5.5.3. Objects like $\phi_{\mathcal{S}}$ are the stock-in-trade of operator-valued free probability theory. See [12] for an interesting introduction to this point of view in the context of some practical problems of computation, and further references. See [17] for in depth treatment of such topics.
Remark 5.5.4. The $\mathcal{S}$-linear extension $\phi_{\mathcal{S}}^{\mathrm{BF}}$ of the state $\phi^{\mathrm{BF}}$ with which $B(\mathcal{H})$ is canonically equipped satisfies
(53) $\quad\left(p_{\mathcal{H}} \otimes 1_{\mathcal{S}}\right) A\left(p_{\mathcal{H}} \otimes 1_{\mathcal{S}}\right)=p_{\mathcal{H}} \otimes \phi_{\mathcal{S}}^{\mathrm{BF}}(A)$, hence $\left[\left[\phi_{\mathcal{S}}^{\mathrm{BF}}(A)\right]\right] \leq \llbracket A \rrbracket$
for all $A \in B(\mathcal{H}) \otimes \mathcal{S}$ and hence $\left[\left[\phi_{\mathcal{S}}^{\mathrm{BF}}\right]\right]=1$. In fact, in full generality, we have $\llbracket \phi_{\mathcal{S}} \rrbracket=1$ by a similar argument using the GNS construction, which we omit.
5.5.5. $\mathcal{S}$-bilinear extension of states. Let $\mathcal{S}$ be a block algebra and let $(\mathcal{A}, \phi)$ be a $C^{*}$-probability space. We define the $\mathcal{S}$-bilinear extension

$$
\phi_{\mathcal{S}, \mathcal{S}}: \mathcal{A} \otimes \mathcal{S} \times \mathcal{A} \otimes \mathcal{S} \rightarrow \mathcal{S}^{\otimes 2}
$$

of $\phi$ by the formula

$$
\phi_{\mathcal{S}, \mathcal{S}}\left(x_{1} \otimes y_{1}, x_{2} \otimes y_{2}\right)=\phi\left(x_{1} x_{2}\right) y_{1} \otimes y_{2} .
$$

Remark 5.5.6. Let $\mathcal{S}$ be a block algebra and consider the $C^{*}$-probability space $\left(\operatorname{Mat}_{N}(\mathbb{C}), \frac{1}{N} \operatorname{tr}\right)$. For $R_{1}, R_{2} \in \operatorname{Mat}_{N}(\mathcal{S})$ we have

$$
\phi_{\mathcal{S}, \mathcal{S}}\left(R_{1}, R_{2}\right)=\frac{1}{N} \sum_{i, j=1}^{N} R_{1}(i, j) \otimes R_{2}(j, i) \in \mathcal{S}^{\otimes 2}
$$

Remark 5.5.7. Consider the $C^{*}$-algebra embeddings

$$
\left.\begin{array}{l}
\iota^{(1)}=\left(x \otimes y \mapsto x \otimes y \otimes 1_{\mathcal{S}}\right) \\
\iota^{(2)}=\left(x \otimes y \mapsto x \otimes 1_{\mathcal{S}} \otimes y\right)
\end{array}\right\}: \mathcal{A} \otimes \mathcal{S} \rightarrow \mathcal{A} \otimes \mathcal{S}^{\otimes 2}
$$

One has

$$
\begin{equation*}
\phi_{\mathcal{S}, \mathcal{S}}(A, B)=\phi_{\mathcal{S}^{\otimes 2}}\left(\iota^{(1)}(A) \iota^{(2)}(B)\right) \tag{54}
\end{equation*}
$$

and thus $\llbracket \phi_{\mathcal{S}, \mathcal{S}} \rrbracket=1$ since $\llbracket \phi_{\mathcal{S} \otimes^{2} \rrbracket} \rrbracket=1$. In a similar vein, we have the formula

$$
\begin{equation*}
\phi_{\mathcal{S}, \mathcal{S}}(A, B)^{\bullet}(\zeta)=\phi_{\mathcal{S}}\left(A\left(1_{\mathcal{A}} \otimes \zeta\right) B\right) \tag{55}
\end{equation*}
$$

we will use below to study the secondary Schwinger-Dyson equation.
Remark 5.5.8. Consider the $C^{*, T}$-probability space $(\mathcal{A}, \phi)=\left(\operatorname{Mat}_{N}(\mathbb{C}), \frac{1}{N} \operatorname{tr}\right)$. Let $\mathcal{S}$ be any block algebra. Let $R \in \operatorname{Mat}_{N}(\mathcal{S})$ be any matrix. We have

$$
\begin{align*}
\phi_{\mathcal{S}, \mathcal{S}}(R, R)^{\bullet} & =\left(\zeta \mapsto \frac{1}{N} \sum_{i, j=1}^{N} R(i, j) \zeta R(j, i)\right)  \tag{56}\\
\phi_{\mathcal{S}, \mathcal{S}}\left(R, R^{\mathrm{T}}\right)^{1 \otimes \mathrm{~T}} & =\frac{1}{N} \sum_{i, j=1}^{N} R(i, j)^{\otimes 2} \tag{57}
\end{align*}
$$

Puzzling expressions of the form on the right are ubiquitous below; those on the left may also appear puzzling but are tractable in the operator-theoretic context.

## 6. Construction of solutions of the Schwinger-Dyson equation

We construct solutions of the Schwinger-Dyson equation by using the BoltzmannFock apparatus reviewed in $\S 4$ above along with the $\mathcal{S}$-linear machinery introduced in $\S 5$ above. (See Proposition 6.1.4 below.) We also construct solutions of a secondary version of the Schwinger-Dyson equation. (See Proposition 6.2 .2 below.) We then apply all these constructions to define an object which ultimately determines the sequence $\left\{\right.$ bias $\left.^{N}\right\}$ figuring in Theorem 2.6.4.
6.1. The solution of the SD equation attached to an $\mathcal{S}$-linear form. Fix a block algebra $\mathcal{S}$ and an $\mathcal{S}$-linear form $L$ with Hamel expansion $L=\sum \mathbf{X}_{\ell} \otimes a_{\ell}$. Recall that by definition $\Phi_{L}=\left(\zeta \mapsto \sum a_{\ell} \zeta a_{\ell}\right) \in B(\mathcal{S})$.
6.1.1. The nonempty open set $\mathcal{D}_{L}$. We define the set

$$
\mathcal{D}_{L}=\left\{\Lambda \in \mathcal{S} \mid L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda \in(B(\mathcal{H}) \otimes \mathcal{S})^{\times}\right\} \subset \mathcal{S}
$$

It is clear that $\mathcal{D}_{L}$ is nonempty and Lemma 4.1.1 implies that $\mathcal{D}_{L}$ is open.

### 6.1.2. The special function $G_{L}$. For $\Lambda \in \mathcal{D}_{L}$ we put

$$
G_{L}(\Lambda)=\phi_{\mathcal{S}}^{\mathrm{BF}}\left(\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right) \in \mathcal{S},
$$

where $\phi_{\mathcal{S}}^{\mathrm{BF}}$ is the $\mathcal{S}$-linear extension of $\phi^{\mathrm{BF}} \in B(\mathcal{H})^{\star}$. By direct manipulation of series expansions one can verify that $G_{L}: \mathcal{D}_{L} \rightarrow \mathcal{S}$ is an analytic function. By means of the resolvent identity (32) in infinitesimal form, one can verify that

$$
\mathbf{D}\left[G_{L}\right](\Lambda ; \zeta)=\phi_{\mathcal{S}}^{\mathrm{BF}}\left(\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}(1 \otimes \zeta)\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right)
$$

for all $\zeta \in \mathcal{S}$. Note also that

$$
\begin{align*}
& \Lambda \in \mathcal{D}_{L} \Leftrightarrow \Lambda^{*} \in \mathcal{D}_{L^{*}} \Rightarrow G_{L}(\Lambda)^{*}=G_{L^{*}}\left(\Lambda^{*}\right)  \tag{58}\\
& \Lambda \in \mathcal{D}_{L} \Leftrightarrow \Lambda^{\mathrm{T}} \in \mathcal{D}_{L^{\mathrm{T}}} \Rightarrow G_{L}(\Lambda)^{\mathrm{T}}=G_{L^{\mathrm{T}}}\left(\Lambda^{\mathrm{T}}\right) \tag{59}
\end{align*}
$$

Relation (58) holds by the symmetry (51) along with the observation that $*$ commutes with inversion. Relation (59) can be verified by a straightforward calculation exploiting Lemma 5.2.6 and relation (52).
6.1.3. Estimates for $G_{L}$. For any $\mathcal{S}$-linear form $L$ and points $\Lambda, \Lambda_{1}, \Lambda_{2} \in \mathcal{D}_{L}$, we have estimates

$$
\begin{align*}
& \llbracket G_{L}(\Lambda) \rrbracket \leq\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right]\right]  \tag{60}\\
& \llbracket G_{L}\left(\Lambda_{1}\right)-G_{L}\left(\Lambda_{2}\right) \rrbracket  \tag{61}\\
\leq & \llbracket \Lambda_{1}-\Lambda_{2} \rrbracket\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{1}\right)^{-1}\right]\right]\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{2}\right)^{-1}\right]\right] \\
& \llbracket \mathbf{D}\left[G_{L}\right](\Lambda) \rrbracket \leq\left[\left(\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right]\right]^{2},  \tag{62}\\
& \llbracket G_{L}\left(\Lambda_{1}\right)-G_{L}\left(\Lambda_{2}\right)-\mathbf{D}\left[G_{L}\right]\left(\Lambda_{2} ; \Lambda_{1}-\Lambda_{2}\right) \rrbracket  \tag{63}\\
\leq & \left.\llbracket \Lambda_{1}-\Lambda_{2} \rrbracket \rrbracket^{2}\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{1}\right)^{-1}\right]\right]\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{2}\right)^{-1}\right]\right]^{2}
\end{align*}
$$

which follow directly from the resolvent identity (32), the iterated resolvent identity (33), the estimate (53) and the definitions.

We arrive finally at the main result of this section.
Proposition 6.1.4. The function $G_{L}: \mathcal{D}_{L} \rightarrow \mathcal{S}$ is a solution of the $S D$ equation with covariance map $\Phi_{L}$.

Proof. We specialize Proposition 4.6 .4 by taking

$$
\begin{gathered}
\mathcal{A}=B(\mathcal{H}) \otimes \mathcal{S}, \quad \pi=p_{\mathcal{H}} \otimes 1_{\mathcal{S}}, \quad \rho_{\ell}=\hat{\Sigma}_{\ell} \otimes 1_{\mathcal{S}} \quad \text { and } \\
A=L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda=-1_{B(\mathcal{H})} \otimes \Lambda+\sum_{\ell}\left(\mathrm{i}^{\ell} \Sigma_{\ell} \otimes a_{\ell}+\mathrm{i}^{-\ell} \Sigma_{\ell}^{*} \otimes a_{\ell}\right)
\end{gathered}
$$

To verify that the family $\{\pi\} \cup\left\{\rho_{i}\right\}_{\ell=1}^{\infty}$ is a Cuntz frame in $\mathcal{A}$ we use (36). To verify that $A$ is quasi-circular we use Lemma 4.4.5. Now in view of (53), the left side of (48) specializes to $p_{\mathcal{H}} \otimes G_{L}(\Lambda)$ and moreover necessarily $G_{L}(\Lambda) \in \mathcal{S}^{\times}$. But we also have

$$
\pi A \pi=-p_{\mathcal{H}} \otimes \Lambda, \quad \pi A \rho_{\ell} \pi=\mathrm{i}^{\ell} p_{\mathcal{H}} \otimes a_{\ell}, \quad \pi \rho_{\ell}^{*} A \pi=\mathrm{i}^{-\ell} p_{\mathcal{H}} \otimes a_{\ell}
$$

as one verifies by using (34) and (37). Thus the inverse in the algebra $\pi \mathcal{A} \pi$ of the right side of (48) specializes to $-p_{\mathcal{H}} \otimes\left(\Lambda+\Phi_{L}\left(G_{L}(\Lambda)\right)\right)$.
Remark 6.1.5. Proposition 6.1 .4 is essentially well-known apart from one small detail. For comparison with a typical proof, see [1, Chap. 5, Secs. 4,5] (main text, not the exercises), and in particular [1, Chap. 5, Lemma 5.5.10]. That proof falls a bit short of proving Proposition 6.1.4 as stated because it relies on an analytic continuation argument to extend a generating function identity proved by combinatorics throughout a connected open set. But we do not know a priori that $\mathcal{D}_{L}$ is connected. (It would be a surprise if it were not but we leave the question aside.) Thus we have presented the operator-theoretic proof of Proposition 6.1.4 suggested by the last exercise in [1] (which does not otherwise seem to be present in the literature in detail) because it makes connectedness of $\mathcal{D}_{L}$ a non-issue.
6.2. The secondary SD equation. We construct solutions of a secondary form of the Schwinger-Dyson equation by using the secondary trick in germinal form. (See Proposition 6.2 .2 below and its proof.) Later the secondary trick will be developed much farther. Using some special examples of solutions of the secondary SchwingerDyson equation we then construct a object which ultimately is going to determine the heretofore mysterious sequence $\left\{\right.$ bias $\left.^{N}\right\}$ figuring in Theorem 2.6.4.
6.2.1. The special function $G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)$. Let $\mathcal{S}$ be a block algebra. For $j=1,2$, let $L_{j}$ be an $\mathcal{S}$-linear form and let $\Lambda_{j} \in \mathcal{D}_{L_{j}}$ be a point. We define

$$
G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)=\phi_{\mathcal{S}, \mathcal{S}}^{\mathrm{BF}}\left(\left(L_{1}(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{1}\right)^{-1},\left(L_{2}(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{2}\right)^{-1}\right)
$$

where $\phi_{\mathcal{S}, \mathcal{S}}^{\mathrm{BF}}$ is the $\mathcal{S}$-bilinear extension of $\phi^{\mathrm{BF}}$. It is easy to see that $G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)$ depends analytically on $\left(\Lambda_{1}, \Lambda_{2}\right)$. By Remark 5.5.7 we have
(64) $\llbracket G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right) \rrbracket \leq\left[\left[\left(L_{1}(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{1}\right)^{-1}\right]\right]\left[\left[\left(L_{2}(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{2}\right)^{-1}\right]\right]$,
which is an estimate straightforwardly analogous to (60).
Proposition 6.2.2. Let $\mathcal{S}$ be a block algebra. For $j=1,2$, let $L_{j}=\sum \mathbf{X}_{\ell} \otimes a_{\ell j}$ be an $\mathcal{S}$-linear form and let $\Lambda_{j} \in \mathcal{D}_{L_{j}}$ be a point. Then the secondary SD equation

$$
\begin{align*}
& G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)  \tag{65}\\
= & \left(\left(\left(G_{L_{1}}\left(\Lambda_{1}\right)^{-1} \otimes G_{L_{2}}\left(\Lambda_{2}\right)^{-1}-\sum a_{\ell 1} \otimes a_{\ell 2}\right)^{1 \otimes \mathrm{~T}}\right)^{-1}\right)^{1 \otimes \mathrm{~T}}
\end{align*}
$$

holds. In particular, the expression on the right side is well-defined.
It is worth noting as a consistency check that the expression on the right side remains invariant if we replace the transposition T by any other transposition of $\mathcal{S}$.

Proof. By Remark 5.3.7 it suffices to prove that
(66) $\zeta=G_{L_{1}}\left(\Lambda_{1}\right)^{-1} G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)^{\bullet}(\zeta) G_{L_{2}}\left(\Lambda_{2}\right)^{-1}-\sum a_{\ell 1} G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)^{\bullet}(\zeta) a_{\ell 2}$
holds for all $\zeta \in \mathcal{S}$. Let $\mathcal{M}_{2}$ be a block algebra equipped with a standard basis $\left\{e_{i j}\right\}_{i, j=1}^{2}$. Fix $\zeta \in \mathcal{S}$ arbitrarily and put

$$
\Lambda=\Lambda_{1} \otimes e_{11}+\Lambda_{2} \otimes e_{22}+\zeta \otimes e_{12} \in \mathcal{S} \otimes \mathcal{M}_{2}
$$

Consider also the $\mathcal{S} \otimes \mathcal{M}_{2}$-linear form

$$
L=L_{1} \otimes e_{11}+L_{2} \otimes e_{22}
$$

To compress notation put

$$
A_{j}=L_{j}(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda_{j} \in(\mathcal{A} \otimes \mathcal{S})^{\times}
$$

for $j=1,2$ and put

$$
A=L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda \in \mathcal{A} \otimes \mathcal{S} \otimes \mathcal{M}_{2}
$$

In fact $A \in\left(\mathcal{A} \otimes \mathcal{S} \otimes \mathcal{M}_{2}\right)^{\times}$, and more precisely

$$
A^{-1}=A_{1}^{-1} \otimes e_{11}+A_{2}^{-1} \otimes e_{22}+\left(A_{1}^{-1}\left(1_{\mathcal{A}} \otimes \zeta\right) A_{2}^{-1}\right) \otimes e_{12}
$$

as one immediately verifies. Thus by the trivial identity (55) we have

$$
G_{L}(\Lambda)=G_{L_{1}}\left(\Lambda_{1}\right) \otimes e_{11}+G_{L_{2}}\left(\Lambda_{2}\right) \otimes e_{22}+G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)^{\bullet}(\zeta) \otimes e_{12}
$$

By Proposition 6.1.4, the SD equation

$$
0=1_{\mathcal{S}} \otimes 1_{\mathcal{M}_{2}}+\left(\Lambda+\Phi_{L}\left(G_{L}(\Lambda)\right)\right) G_{L}(\Lambda)
$$

is satisfied. By expanding the right side in the form $\cdots+b \otimes e_{12}+\ldots$ we find that

$$
\begin{aligned}
0= & \left(\Lambda_{1}+\Phi_{L_{1}}\left(G_{L_{1}}\left(\Lambda_{1}\right)\right)\right) G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)^{\bullet}(\zeta) \\
& +\left(\zeta+\sum a_{\ell 1} G_{L_{1}, L_{2}}\left(\Lambda_{1}, \Lambda_{2}\right)^{\bullet}(\zeta) a_{\ell 2}\right) G_{L_{2}}\left(\Lambda_{2}\right),
\end{aligned}
$$

which yields (66) after some further manipulation which we omit.
Remark 6.2.3. Fix an $\mathcal{S}$-linear form $L$ and a point $\Lambda \in \mathcal{D}_{L}$. Then we have

$$
\begin{equation*}
\mathbf{D}\left[G_{L}\right](\Lambda)=G_{L, L}(\Lambda, \Lambda)^{\bullet} \tag{67}
\end{equation*}
$$

as one verifies by exploiting the infinitesimal form of the resolvent identity (32). Note that the equation (66) in the case $\left(L_{1}, L_{2}, \Lambda_{1}, \Lambda_{2}\right)=(L, L, \Lambda, \Lambda)$ specializes to the equation (18) obtained through differentiation.
Remark 6.2.4. Fix an $\mathcal{S}$-linear form $L$ and a point $\Lambda \in \mathcal{D}_{L}$. Let $\Psi_{L}$ be as in Definition 5.4.2. Recall that if $L=\sum \mathbf{X}_{\ell} \otimes a_{\ell}$ is the Hamel expansion of $L$ then $\Psi_{L}=\sum(-1)^{\ell} a_{\ell}^{\otimes 2}$. Then we have

$$
\begin{equation*}
\left(\left(G_{L}(\Lambda)^{-1}\right)^{\otimes 2}-\Psi_{L}\right)^{-1}=G_{L, L^{\mathrm{T}}}\left(\Lambda, \Lambda^{\mathrm{T}}\right)^{1 \otimes \mathrm{~T}} \tag{68}
\end{equation*}
$$

by the secondary SD equation (65) in the case $\left(L_{1}, L_{2}, \Lambda_{1}, \Lambda_{2}\right)=\left(L, L^{\mathrm{T}}, \Lambda, \Lambda^{\mathrm{T}}\right)$ along with the symmetry (59). In turn, we have

$$
\begin{equation*}
\left[\left[\left(\left(G_{L}(\Lambda)^{-1}\right)^{\otimes 2}-\Psi_{L}\right)^{-1}\right]\right] \leq \llbracket 1 \otimes \mathrm{~T} \rrbracket\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right]\right]^{2} \tag{69}
\end{equation*}
$$

by Remark 5.5.7, (64), and Lemma 5.2.6.
6.3. The universal correction. We construct an object which by means of the self-adjoint linearization trick developed below determines the functions $\left\{\operatorname{bias}^{N}\right\}$ figuring in Theorem 2.6.4. Before doing so we must introduce tensor cumulants and tensor shuffles.
6.3.1. A tensor generalization of fourth cumulants. Let $Y$ be any $\mathcal{S}$-valued random variable such that $\|\llbracket Y \rrbracket\|_{4}<\infty$ and $\mathbb{E} Y=0$. Let $Z$ be an independent copy of $Y$. We define

$$
\begin{aligned}
\mathbf{C}^{(4)}(Y)= & \mathbb{E}\left(Y^{*} \otimes Y \otimes Y^{*} \otimes Y\right)-\mathbb{E}\left(Y^{*} \otimes Y \otimes Z^{*} \otimes Z\right) \\
& -\mathbb{E}\left(Y^{*} \otimes Z \otimes Z^{*} \otimes Y\right)-\mathbb{E}\left(Y^{*} \otimes Z \otimes Y^{*} \otimes Z\right) \in \mathcal{S}^{\otimes 4}
\end{aligned}
$$

6.3.2. Shuffle notation. For positive integers $k$ we define bilinear maps
$[\cdot, \cdot]_{k}: \mathcal{S}^{\otimes k} \times \mathcal{S}^{\otimes k} \rightarrow \mathcal{S}^{\otimes 2 k}, \quad\left[x_{1} \otimes \cdots \otimes x_{k}, y_{1} \otimes \cdots \otimes y_{k}\right]_{k}=x_{1} \otimes y_{1} \otimes \cdots \otimes x_{k} \otimes y_{k}$,

$$
\langle\cdot, \cdot\rangle_{k}: \mathcal{S}^{\otimes k} \times \mathcal{S}^{\otimes k} \rightarrow \mathcal{S}, \quad\left\langle x_{1} \otimes \cdots \otimes x_{k}, y_{1} \otimes \cdots \otimes y_{k}\right\rangle_{k}=x_{1} y_{1} \cdots x_{k} y_{k}
$$

6.3.3. Definition of $\operatorname{Bias}_{L}^{N}$. Let $L=\sum \mathbf{X}_{\ell} \otimes a_{\ell}$ be a self-adjoint $\mathcal{S}$-linear form. Let $\Lambda \in \mathcal{D}_{L}$ be a point. To abbreviate notation we write

$$
\begin{gathered}
\Phi=\Phi_{L} \in B(\mathcal{S}), \quad \Psi=\Psi_{L} \in \mathcal{S}^{\otimes 2}, \quad X^{N}=L\left(\Xi^{N}\right)=\sum \Xi_{\ell}^{N} \otimes a_{\ell} \in \operatorname{Mat}_{N}(\mathcal{S})_{\mathrm{sa}}, \\
G=G_{L}(\Lambda) \in \mathcal{S}^{\times}, \quad G^{\prime}=\mathbf{D}\left[G_{L}\right](\Lambda) \in B(\mathcal{S}), \quad \check{G}=\left(\left(G^{-1}\right)^{\otimes 2}-\Psi_{L}\right)^{-1} \in\left(\mathcal{S}^{\otimes 2}\right)^{\times} .
\end{gathered}
$$

By Remark 6.2.4, the object $\check{G}$ above is well-defined. We now define

$$
\begin{aligned}
\widehat{\operatorname{Bias}}_{L}^{N}(\Lambda)= & \left\langle[\Psi, \Psi]_{2},\left[\check{G}, G^{\otimes 2}\right]_{2}\right\rangle_{4}-\Phi(G) G \\
& +\frac{1}{N} \sum_{i=1}^{N}\left\langle\mathbb{E} X^{N}(i, i)^{\otimes 2}, G^{\otimes 2}\right\rangle_{2}-\frac{1}{N^{3 / 2}} \sum_{i=1}^{N}\left\langle\mathbb{E} X^{N}(i, i)^{\otimes 3}, G^{\otimes 3}\right\rangle_{3} \\
& +\frac{1}{N^{2}} \sum_{\substack{i, j=1 \\
i \neq j}}^{N}\left\langle\mathbf{C}^{(4)}\left(X^{N}(i, j)\right), G^{\otimes 4}\right\rangle_{4}, \\
\operatorname{Bias}_{L}^{N}(\Lambda)= & G^{\prime}\left(\widehat{\operatorname{Bias}}_{L}^{N} G^{-1}\right) .
\end{aligned}
$$

The analytic functions

$$
\widehat{\operatorname{Bias}}_{L}^{N}, \operatorname{Bias}_{L}^{N}: \mathcal{D}_{L} \rightarrow \mathcal{S}
$$

thus defined we call the unwrapped universal correction and universal correction indexed by $L$ and $N$, respectively. We only define the former function to expedite certain calculations-the latter function is the theoretically important one with good symmetry properties. It is a straightforward if tedious matter to verify that $\operatorname{Bias}_{L}^{N}$ commutes with the $C^{*}$-algebra involution just as $G_{L}$ does. For a constant $c$ independent of $N, L$ and $\Lambda$ we have

$$
\begin{equation*}
\sup _{N}\left[\left[\operatorname{Bias}_{L}^{N}(\Lambda)\right]\right] \leq c\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right]\right]^{5} \tag{70}
\end{equation*}
$$

by estimates (60), (62) and (69) along with assumption (1).

## 7. SALT BLOCK DESIGNS AND RANDOM MATRIX ESTIMATES

We introduce a general algebraic/analytic notion of crucial importance in this paper. We then immediately supply its main application in the paper, which is to serve as a "hypothesis-checking machine" for Proposition 3.5.2 in a certain situation arising in the proof of Theorem 2.6.4.

### 7.1. SALT block designs.

Definition 7.1.1. A $S A L T$ block design is a quadruple $(\mathcal{S}, L, \Theta, e)$ consisting of

- a block algebra $\mathcal{S}$,
- a self-adjoint $\mathcal{S}$-linear form $L$,
- an element $\Theta \in \mathcal{S}$ (perhaps not self-adjoint), and
- a projection $e \in \mathcal{S}$
such that for every $C^{*, T}$-algebra $\mathcal{A}$, sequence $\xi \in \mathcal{A}_{\text {salt }}^{\infty}$, point $z \in \mathfrak{h}$ and parameter value $t \geq 0$ we have

$$
\begin{align*}
& L(\xi)-1_{\mathcal{A}} \otimes\left(\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}\right) \in(\mathcal{A} \otimes \mathcal{S})^{\times} \text {and }  \tag{71}\\
& {\left[\left[\left(L(\xi)-1_{\mathcal{A}} \otimes\left(\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}\right)\right)^{-1}\right]\right] \leq c_{0}(1+\llbracket L(\xi) \rrbracket)^{c_{1}}(1+1 / \Im z)^{c_{2}}} \tag{72}
\end{align*}
$$

for some constants $c_{0}, c_{1}, c_{2} \geq 1$ depending only on $(\mathcal{S}, L, \Theta, e)$ and thus independent of $\mathcal{A}, \xi, z$ and $t$. We declare any finite constant $\mathfrak{T} \geq \llbracket \Theta \rrbracket+2\left(1+\llbracket \Phi_{L} \rrbracket\right)$ to be a cutoff for the design, where $\Phi_{L} \in B(\mathcal{S})$ is as in Definition 5.4.2. We emphasize that we invariably choose the constants $c_{0}, c_{1}$ and $c_{2}$ not less than 1 (rather than merely nonnegative) because in practice this simplifies the derivation of various crude upper bounds we will need.

We will take up the problem of constructing such gadgets in $\S 8$ below.
Remark 7.1.2. In the situation of (71) and (72), simply because $L(\xi)$ is self-adjoint, we automatically have

$$
\left[\left[\left(L(\xi)-1_{\mathcal{A}} \otimes\left(\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}\right)\right)^{-1}\right]\right] \leq \frac{1}{2\left(1+\llbracket \Phi_{L} \rrbracket\right)} \wedge \frac{1}{t-\mathfrak{T}} \text { for } t>\mathfrak{T}
$$

by Lemma 4.2.6.
Remark 7.1.3. Let $(\mathcal{S}, L, \Theta, e), c_{0}, c_{1}, c_{2}$ and $\mathfrak{T}$ be as in Definition 7.1.1. Put

$$
\mathfrak{G}(z)=2 c_{0}(1+\llbracket L(\Xi) \rrbracket)^{c_{1}}(1+1 / \Im z)^{c_{2}} \text { for } z \in \mathfrak{h}
$$

Now fix $z \in \mathfrak{h}, t \in[0, \infty)$ and $\zeta \in \mathcal{S}$ such that $\llbracket \zeta \rrbracket \leq 1 / \mathfrak{G}(z)$ arbitrarily and put

$$
\Lambda=\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}+\zeta
$$

We then have $\Lambda \in \mathcal{D}_{L}$ and

$$
\llbracket G_{L}(\Lambda) \rrbracket \leq\left[\left[\left(L(\Xi)-1_{B(\mathcal{H})} \otimes \Lambda\right)^{-1}\right]\right] \leq\left\{\begin{aligned}
\mathfrak{G}(z) & \text { in general }, \\
\frac{1}{2\left(1+\llbracket \Phi_{L} \rrbracket\right)} \wedge \frac{1}{t-\mathfrak{T}} & \text { for } t \geq \mathfrak{T},
\end{aligned}\right.
$$

by Lemma 4.1.1, estimate (60), (71), (72) and Remark 7.1.2. Given also $\Lambda^{\prime} \in \mathcal{D}_{L}$ with "primed" variables, we have

$$
\begin{aligned}
& \llbracket G_{L}(\Lambda)-G_{L}\left(\Lambda^{\prime}\right) \rrbracket \leq \llbracket \Lambda-\Lambda^{\prime} \rrbracket \mathfrak{G}(z) \mathfrak{G}\left(z^{\prime}\right) \\
& \llbracket \mathbf{D}\left[G_{L}\right](\Lambda) \rrbracket \leq \mathfrak{G}(z)^{2} \\
& \llbracket G_{L}(\Lambda)-G_{L}\left(\Lambda^{\prime}\right)-\mathbf{D}\left[G_{L}\right]\left(\Lambda^{\prime} ; \Lambda-\Lambda^{\prime}\right) \rrbracket \leq \llbracket \Lambda-\Lambda^{\prime} \rrbracket^{2} \mathfrak{G}(z) \mathfrak{G}\left(z^{\prime}\right)^{2}
\end{aligned}
$$

by (61), (62) and (63), respectively. In particular, it follows that the collection

$$
\left(G_{L}: \mathcal{D}_{L} \rightarrow \mathcal{S}, \Phi_{L}, \Theta+z e, \mathfrak{T}, \mathfrak{G}(z)\right)
$$

is an SD tunnel for each fixed $z \in \mathfrak{h}$. We also have a bound

$$
\left.\llbracket\left[\left(\left(G_{L}(\Lambda)^{-1}\right)^{\otimes 2}-\Psi_{L}\right)^{-1}\right]\right] \leq \llbracket 1 \otimes \mathrm{~T} \rrbracket \mathfrak{G}(z)^{2}
$$

by (69) and a bound

$$
\begin{equation*}
\left[\left[\operatorname{Bias}_{L}^{N}(\Theta+z e)\right]\right] \leq c \mathfrak{G}(z)^{5} \tag{73}
\end{equation*}
$$

for a constant $c$ independent of $L, N$ and $z$ by (70).
Remark 7.1.4. Again let $(\mathcal{S}, L, \Theta, e), c_{0}, c_{1}, c_{2}$ and $\mathfrak{T}$ be as in Definition 7.1.1. Let $Y \in \operatorname{Mat}_{N}(\mathcal{S})_{\mathrm{sa}}$ be of the form $Y=L(\eta)$ for some $\eta \in \operatorname{Mat}_{N}(\mathbb{C})_{\text {salt }}^{\infty}$. Then for every $z \in \mathfrak{h}$ and $t \in[0, \infty)$ we have

$$
\begin{aligned}
& Y-\mathbf{I}_{N} \otimes\left(\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}\right) \in \mathrm{GL}_{N}(\mathcal{S}) \text { and } \\
& {\left[\left[\left(Y-\mathbf{I}_{N} \otimes\left(\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}\right)\right)^{-1}\right]\right] } \\
\leq & \left\{\begin{array}{cl}
c_{0}(1+\llbracket Y \rrbracket)^{c_{1}}(1+1 / \Im z)^{c_{2}} & \text { in general, } \\
\frac{1}{2} \wedge \frac{1}{t-\mathfrak{T}} & \text { if } t>\mathfrak{T},
\end{array}\right.
\end{aligned}
$$

by definition of a SALT block design along with Remark 7.1.2. By the resolvent identity (32), the following crucial (if trivial) bound follows:

The Lipschitz constant of the map

$$
\begin{aligned}
& \left(t \mapsto\left(Y-\mathbf{I}_{N} \otimes\left(\Theta+z e+\mathrm{it} 1_{\mathcal{S}}\right)\right)^{-1}\right):[0, \infty) \rightarrow \operatorname{Mat}_{n}(\mathcal{S}) \\
& \text { does not exceed } c_{0}^{2}(1+\llbracket Y \rrbracket)^{2 c_{1}}(1+1 / \Im z)^{2 c_{2}},
\end{aligned}
$$

for each fixed $z \in \mathfrak{h}$.
7.2. Application of the modified tunnel estimates. We specialize Proposition 3.5.2 to precisely the situation in which it is needed for the proof of Theorem 2.6.4. Remarks 7.1.3 and 7.1.4 do most of the work of checking hypotheses.
7.2.1. The auxiliary random variable $\mathbf{t}$. For the rest of the paper $\mathbf{t}$ denotes a nonnegative random variable independent of $\sigma(\mathcal{F}, \mathbf{z})$ which on the flip of a fair coin is either a unit mass at the origin or a standard exponential random variable. (Recall that the auxiliary random variable $\mathbf{z}$ was introduced in §2.6.3.) Ultimately $\mathbf{t}$ will play a role of importance equal to that of $\mathbf{z}$ in the proof of Theorem 2.6.4. Given any $\sigma(\mathcal{F}, \mathbf{z}, \mathbf{t})$-measurable finite-dimensional-Banach-space-valued random variable $Z$ such that $\mathbb{E} \llbracket Z \rrbracket<\infty$, we define

$$
\left.Z\right|_{\mathbf{t}=0}=2 \mathbb{E}\left(Z \mathbf{1}_{\mathbf{t}=0} \mid \mathcal{F}, \mathbf{z}\right)
$$

We use the yet briefer notation $Z_{0}=\left.Z\right|_{\mathbf{t}=0}$ when context permits. As the notation is meant to suggest, one should think $Z_{0}$ as the value of $Z$ at $\mathbf{t}=0$. For simplicity we assume that $\mathbf{t} \in[0, \infty)$ for all sample points without exception.
7.2.2. Data and assumptions for the application. Beyond the data and assumptions for Theorems 2.3.6 and 2.6.4 we fix the following objects:

- Let $(\mathcal{S}, L, \Theta, e), c_{0}, c_{1}, c_{2}$ and $\mathfrak{T}$ be as in Definition 7.1.1.
- Let $N$ be a positive integer.
- Let $I \subset\{1, \ldots, N\}$ of cardinality $n>0$.

Put

$$
\mathfrak{G}=2 c_{0}(1+\llbracket L(\Xi) \rrbracket)^{c_{1}}(1+1 / \Im \mathbf{z})^{c_{2}}
$$

and assume that

$$
\begin{equation*}
\mathbb{E} \mathfrak{G}^{2}<\infty \tag{74}
\end{equation*}
$$

Note that the latter holds provided that the repulsion of $\mathbf{z}$ from the real axis is sufficiently strong.
7.2.3. Random variables. For each positive integer $\ell$ let $\eta_{\ell} \in \operatorname{Mat}_{n}(\mathcal{S})_{\text {sa }}$ be the random matrix gotten by striking all rows and columns of $\Xi_{\ell}^{N}$ with indices not belonging to the set $I$. Noting that $\eta=\left\{\eta_{\ell}\right\}_{\ell=1}^{\infty} \in \operatorname{Mat}_{n}(\mathcal{S})_{\text {salt }}^{\infty}$, put

$$
Y=\frac{L(\eta)}{\sqrt{N}} \in \operatorname{Mat}_{n}(\mathcal{S})_{\mathrm{sa}}
$$

Note that $\|\llbracket Y \rrbracket\|_{p}<\infty$ for $p \in[1, \infty)$ by assumption (1). Put

$$
\Lambda=\Theta+\mathbf{z e} e \mathrm{i} \mathbf{t} 1_{\mathcal{S}}, \quad \mathfrak{C}=99\left(1+\llbracket \Phi_{L} \rrbracket+\llbracket \Theta \rrbracket+|\mathbf{z}|\right), \quad \mathfrak{L}=(1+\llbracket Y \rrbracket)^{2 c_{1}} \mathfrak{G}^{2}
$$

Note that $\llbracket \Lambda \rrbracket$ and $\mathfrak{C}$ have moments of all orders. Note that $\mathbb{E} \mathfrak{L}<\infty$. Let $\Phi_{L}$ be as in Definition 5.4.2. Put

$$
R=\left(Y-\mathbf{I}_{n} \otimes \Lambda\right)^{-1}, \quad F=\frac{1}{N} \operatorname{tr}_{\mathcal{S}} R, \quad E=1_{\mathcal{S}}+\left(\Lambda+\Phi_{L}(F)\right) F
$$

Using Remark 7.1.4 we have $\llbracket R \rrbracket \leq \sqrt{\mathfrak{L}}$. Thus, a fortiori we have $\mathbb{E} \llbracket F \rrbracket^{2}<\infty$ and $\mathbb{E} \llbracket E \rrbracket<\infty$. In turn we define

$$
\begin{aligned}
\bar{F} & =\mathbb{E}(F \mid \mathbf{z}, \mathbf{t}), \bar{E}=1_{\mathcal{S}}+\left(\Lambda+\Phi_{L}(\bar{F})\right) \bar{F}, \overline{\mathfrak{L}}=\mathbb{E}(\mathfrak{L} \mid \mathbf{z}), \\
\mathfrak{E} & =\mathbb{E}(\llbracket E \rrbracket \mid \mathcal{F}, \mathbf{z}), \overline{\mathfrak{E}}=\mathbb{E}([[\bar{E}]] \mid \mathbf{z}) .
\end{aligned}
$$

(We apologize for all the E's but, alas, the alphabet is finite.) We apply the procedure of evaluation at $\mathbf{t}=0$ introduced in $\S 7.2 .1$ above to define random variables $\Lambda_{0}, F_{0}, E_{0}, \bar{F}_{0}$ and $\bar{E}_{0}$.

Proposition 7.2.4. Notation and assumptions are as above. We have

$$
\begin{align*}
\llbracket F_{0}-G_{L}\left(\Lambda_{0}\right) \rrbracket & \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \mathfrak{L}\right)^{6}\left(\mathfrak{E}+\mathfrak{E}^{2}\right),  \tag{75}\\
{\left[\left[F_{0}+\mathbf{D}\left[G_{L}\right]\left(\Lambda_{0} ; E_{0} G_{L}\left(\Lambda_{0}\right)^{-1}\right)-G_{L}\left(\Lambda_{0}\right)\right]\right] } & \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \mathfrak{L}\right)^{12}\left(\mathfrak{E}^{2}+\mathfrak{E}^{4}\right),  \tag{76}\\
{\left[\left[\bar{F}_{0}-G_{L}\left(\Lambda_{0}\right)\right]\right] } & \leq\left(e^{\mathfrak{T}} \mathfrak{C G} \overline{\mathfrak{L}}\right)^{6}\left(\overline{\mathfrak{E}}+\overline{\mathfrak{E}}^{2}\right),  \tag{77}\\
{\left[\left[\bar{F}_{0}+\mathbf{D}\left[G_{L}\right]\left(\Lambda_{0} ; \bar{E}_{0} G_{L}\left(\Lambda_{0}\right)^{-1}\right)-G_{L}\left(\Lambda_{0}\right)\right]\right] } & \leq\left(e^{\mathfrak{T}} \mathfrak{C G} \overline{\mathfrak{L}}\right)^{12}\left(\overline{\mathfrak{E}}^{2}+\overline{\mathfrak{E}}^{4}\right), \tag{78}
\end{align*}
$$

almost surely.
The reader should notice that this proposition has been phrased entirely in the language of finite-dimensional random vectors and conditional expectations. While stochastic processes with nondiscrete parameters (in quite rudimentary form) do appear in the proof below, they have been purged from the proposition's statementand thus from the rest of the paper-with an attendant gain in simplicity.

Proof. By Lemma 4.1.3, after discarding an $\mathcal{F}$-measurable set of probability zero from the probability space on which we are working if necessary, we may assume that

$$
Y-\mathbf{I}_{n} \otimes\left(\Theta+z e+\mathrm{i} t 1_{\mathcal{S}}\right) \in \mathrm{GL}_{n}(\mathcal{S})
$$

for every sample point, $z \in \mathfrak{h}$ and $t \in[0, \infty)$, without exception. (Recall that, in the same spirit, we always assume that $\mathbf{z} \in \mathfrak{h}$ and $\mathbf{t} \in[0, \infty)$ for all sample points.) After making these adjustments and choosing versions of conditional expectations carefully we will be able to prove the claimed inequalities "on the nose", i.e, for every sample point.

For constructing versions of various conditional expectations it is convenient to define for every $z \in \mathfrak{h}$ and $t \in[0, \infty)$ the following random variables:

$$
\begin{aligned}
\Lambda_{t, z} & =\Theta+z e+\mathrm{i} 1_{\mathcal{S}}, \quad R_{t, z}=\left(Y-\mathbf{I}_{n} \otimes \Lambda_{t, z}\right)^{-1} \\
F_{t, z} & =\frac{1}{N} \operatorname{tr}_{\mathcal{S}} R_{t, z}, \bar{F}_{z, t}=\mathbb{E} F_{z, t}, \\
E_{t, z} & =1_{\mathcal{S}}+\left(\Lambda_{z, t}+\Phi\left(F_{z, t}\right)\right) F_{z, t}, \quad \bar{E}_{t, z}=1_{\mathcal{S}}+\left(\Lambda_{z, t}+\Phi\left(\bar{F}_{z, t}\right)\right) \bar{F}_{z, t} \\
\mathfrak{E}_{z} & =\frac{1}{2} \llbracket E_{0, z} \rrbracket+\frac{1}{2} \int_{0}^{\infty} \llbracket E_{z, t} \rrbracket e^{-t} d t, \quad \overline{\mathfrak{E}}_{z}=\frac{1}{2}\left[\left[\bar{E}_{0, z}\right]\right]+\frac{1}{2} \int_{0}^{\infty}\left[\left[\bar{E}_{z, t}\right]\right] e^{-t} d t .
\end{aligned}
$$

Because of the adjustments we made above, each of these families of random variables is a stochastic process with continuous sample paths. In particular, the last two expressions do in fact define random variables.

For the proof of all three claimed estimates, we note that by Remark 7.1.3, the collection $\left(G_{L}: \mathcal{D}_{L} \rightarrow \mathcal{S}, \Phi_{L}, \Theta+\mathbf{z} e, \mathfrak{T}, \mathfrak{G}\right)$ is a Schwinger-Dyson tunnel, albeit a random one, and that $\mathfrak{C}$ as defined here, when realized at a given sample point, bounds the corresponding constant appearing in Proposition 3.5.2.

Now we turn to the proof of the "non-overlined" estimates. By Remark 7.1.4, for each sample point, the function $\left(t \mapsto F_{t, \mathbf{z}}\right):[0, \infty) \rightarrow \mathcal{S}$ is Lipschitz with Lipschitz constant bounded by $\mathfrak{L}$. Furthermore, $\llbracket F_{\mathfrak{T}, \mathbf{z}} \rrbracket \leq 1 / 2$, $\mathfrak{E}_{\mathbf{z}}$ is a version of $\mathfrak{E}, \Lambda_{0}=\Lambda_{0, \mathbf{z}}$, $F_{0}=F_{0, \mathbf{z}}$ and $E_{0}=E_{0, \mathbf{z}}$. Thus we can apply Proposition 3.5.2 at each sample point to obtain the bounds (75) and (76).

Very similar reasoning proves the "overlined" estimate. By Remark 7.1.4 and Jensen's inequality, $\left(t \mapsto \bar{F}_{t, \mathbf{z}}\right):[0, \infty) \rightarrow \mathcal{S}$ is Lipschitz with Lipschitz constant bounded by $\overline{\mathfrak{L}}$. Furthermore, $\left[\left[\bar{F}_{\mathfrak{T}, \mathbf{z}}\right]\right] \leq 1 / 2, \overline{\mathfrak{E}}_{\mathbf{z}}$ is a version of $\overline{\mathfrak{E}}, \bar{F}_{0}=\bar{F}_{0, \mathbf{z}}$ and $\bar{E}_{0}=\bar{E}_{0, \mathbf{z}}$. Thus, once again, we can apply Proposition 3.5.2 at each sample point to obtain the bounds (77) and (78).

Remark 7.2.5. In the setup of Proposition 7.2.4, for each $\zeta \in \mathcal{S}$, the bound

$$
\left[\left[\mathbf{D}[G]\left(\Lambda_{0} ; \zeta G\left(\Lambda_{0}\right)^{-1}\right)\right]\right] \leq \mathfrak{C G}^{3} \llbracket \zeta \rrbracket
$$

holds almost surely. We will need this estimate for our calculations in the endgame. This estimate is easy to derive using the definition of an SD tunnel and the SD equation itself. A version of this bound was already employed in the proof of Proposition 3.4.2.

## 8. The self-Adjoint linearization trick

We refine the linearization trick of [9] and [8] so as to preserve self-adjointness. We also introduce a secondary trick aimed at calculating correction terms.

### 8.1. Formulation of the self-adjoint linearization trick.

Definition 8.1.1. Let $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ be given. A SALT block design $(\mathcal{S}, L, \Theta, e)$ is called a self-adjoint linearization of $f$ under four conditions. The first two conditions are relatively minor: $\Theta$ should be self-adjoint and one should be able to take $c_{2}=1$ in estimate (72). The third and fourth are the most important and are as follows: For any $C^{*, T}$-probability space $(\mathcal{A}, \phi)$, sequence $\xi \in \mathcal{A}_{\text {salt }}^{\infty}$ and complex number $z \in \mathbb{C}$, we have

$$
\begin{equation*}
z \in \mathbb{C} \backslash \operatorname{Spec}(f(\xi)) \Leftrightarrow L(\xi)-1_{\mathcal{A}} \otimes(\Theta+z e) \in(\mathcal{A} \otimes \mathcal{S})^{\times} \tag{79}
\end{equation*}
$$

and under these equivalent conditions

$$
\begin{equation*}
\frac{1}{n} \phi \circ \operatorname{tr}_{\mathcal{A}}\left(\left(f(\xi)-z \mathbf{I}_{n} \otimes 1_{\mathcal{A}}\right)^{-1}\right)=\tau_{\mathcal{S}, e} \circ \phi_{\mathcal{S}}\left(\left(L(\xi)-1_{\mathcal{A}} \otimes(\Theta+z e)\right)^{-1}\right) \tag{80}
\end{equation*}
$$

where $\phi_{\mathcal{S}}$ is the $\mathcal{S}$-linear extension of the given state $\phi$ and $\tau_{\mathcal{S}, e}$ is the state defined in Remark 5.3.4.

Our main result in this section is as follows.
Proposition 8.1.2. Every $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ has a self-adjoint linearization.
The proof commences in $\S 8.2$ after some further discussion of SALT block designs and the trick itself. The proof will be completed in $\S 8.3 .3$ below.

Remark 8.1.3. Suppose that $(\mathcal{S}, L, \Theta, e)$ is the self-adjoint linearization of some $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$. Then for $z \in \mathfrak{h}$, by definition of a self-adjoint linearization, we have a representation

$$
S_{\mu_{f}^{N}}(z)=\frac{1}{N} \tau_{\mathcal{S}, e} \circ \operatorname{tr}_{\mathcal{S}}\left(\left(L\left(\frac{\Xi^{N}}{\sqrt{N}}\right)-\mathbf{I}_{N} \otimes(\Theta+z e)\right)^{-1}\right)
$$

for the Stieltjes transform $S_{\mu_{f}^{N}}(z)$ figuring in Theorem 2.6.4. Using again the procedure of evaluation at $\mathbf{t}=0$ defined before Proposition 7.2.4, we then have

$$
S_{\mu_{f}^{N}}(\mathbf{z})=\left.\frac{1}{N} \tau_{\mathcal{S}, e} \circ \operatorname{tr}_{\mathcal{S}}\left(\left(L\left(\frac{\Xi^{N}}{\sqrt{N}}\right)-\mathbf{I}_{N} \otimes\left(\Theta+\mathbf{z e} e+\mathbf{i} \mathbf{t} 1_{\mathcal{S}}\right)\right)^{-1}\right)\right|_{\mathbf{t}=0} .
$$

Whatever one may wish to call it (we can't think of a good name), this last way of representing $S_{\mu_{f}^{N}}(z)$ is the centerpiece of our approach to proving Theorem 2.6.4.

Remark 8.1.4. We return to the setup of Remark 7.1.3. If $(\mathcal{S}, L, \Theta, e)$ is a selfadjoint linearization of some $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$, then for $z \in \mathbb{C}$ we have

$$
\begin{equation*}
\Theta+z e \in \mathcal{D}_{L} \Leftrightarrow z \in \mathbb{C} \backslash \operatorname{Spec}(f(\Xi)) \Leftrightarrow z \in \mathbb{C} \backslash \operatorname{supp} \mu_{f} . \tag{81}
\end{equation*}
$$

The first equivalence holds by definition of a self-adjoint linearization while the second holds by the crucially important Proposition 2.3.3. For $z \in \mathbb{C}$ satisfying the equivalent conditions above we then have the representation

$$
\begin{equation*}
S_{\mu_{f}}(z)=\tau_{\mathcal{S}, e} \circ G_{L}(\Theta+z e) \tag{82}
\end{equation*}
$$

for the Stieltjes transform $S_{\mu_{f}}(z)$ figuring in Theorem 2.6.4.
Remark 8.1.5. In the setting of the previous remark, we will use the formula

$$
\operatorname{bias}^{N}(z)=\tau_{\mathcal{S}, e} \circ \operatorname{Bias}_{L}^{N}(\Theta+z e) \text { for } z \in \mathbb{C} \backslash \operatorname{supp} \mu_{f}
$$

to define the heretofore mysterious sequence $\left\{\operatorname{bias}^{N}\right\}$ figuring in Theorem 2.6.4. Note that the right side is well-defined by (81) and the fact that $\operatorname{Bias}_{L}^{N}$ has the same domain of definition $\mathcal{D}_{L}$ as does $G_{L}$. Note that bias ${ }^{N}$ commutes with $*$ because both $\tau_{\mathcal{S}, e}$ and $\operatorname{Bias}_{L}^{N}$ do. Note furthermore that by estimate (73), condition (14) will automatically be satisfied provided that the strength of the repulsion of $\mathbf{z}$ from the real axis is sufficiently high, depending on $p$.
8.2. The self-adjoint linearization trick in raw form. We start proving Proposition 8.1.2. In this subsection we focus on the purely algebraic aspects of the proof.
Proposition 8.2.1. Let $\mathcal{A}$ be $a *$-algebra. Let $s>n>0$ be integers. Let matrices $f \in \operatorname{Mat}_{n}(\mathcal{A})_{\mathrm{sa}}, d \in \mathrm{GL}_{s-n}(\mathcal{A}) \cap \operatorname{Mat}_{s-n}(\mathcal{A})_{\text {sa }}$ and $b \in \operatorname{Mat}_{n \times(s-n)}(\mathcal{A})$ be given such that $f=-b d^{-1} b^{*}$. Then for all $\lambda \in \operatorname{Mat}_{n}(\mathcal{A})$ we have

$$
f-\lambda \in \operatorname{GL}_{n}(\mathcal{A}) \Leftrightarrow\left[\begin{array}{cc}
-\lambda & b  \tag{83}\\
b^{*} & d
\end{array}\right] \in \operatorname{GL}_{s}(\mathcal{A})
$$

and under these equivalent conditions

$$
\left[\begin{array}{cc}
-\lambda & b  \tag{84}\\
b^{*} & d
\end{array}\right]^{-1}=\left[\begin{array}{cc}
(f-\lambda)^{-1} & -(f-\lambda)^{-1} b d^{-1} \\
-d^{-1} b^{*}(f-\lambda)^{-1} & d^{-1}+d^{-1} b^{*}(f-\lambda)^{-1} b d^{-1}
\end{array}\right] .
$$

The proposition is just a specialization of Proposition 4.5.2 and needs no proof. It begs the question of the existence of "nice" $b$ and $d$ for a given $f$. The raw self-adjoint linearization trick gives a precise affirmative answer under "practical" hypotheses.

### 8.2.2. Valuations. Let $\mathcal{A}$ be a $*$-algebra. A function

$$
\operatorname{deg}: \mathcal{A} \rightarrow\{-\infty\} \cup\{0,1,2,3, \ldots,\}
$$

will be called a valuation if, for all $a, b \in \mathcal{A}$ and scalars $\alpha \in \mathbb{C}$, the following relations hold:
$\operatorname{deg} a^{*}=\operatorname{deg} a, \quad \operatorname{deg} a b \leq \operatorname{deg} a+\operatorname{deg} b, \operatorname{deg}(a+b) \leq(\operatorname{deg} a) \vee(\operatorname{deg} b)$,
$\operatorname{deg} a=-\infty \quad \Leftrightarrow \quad a=0$ and $\operatorname{deg} \alpha 1_{\mathcal{A}} \leq 0$.
In this situation, given any matrix $A \in \operatorname{Mat}_{k \times \ell}(\mathcal{A})$, we define $\operatorname{deg} A=\vee \operatorname{deg} A(i, j)$ and we say that $\operatorname{deg} A$ is defined by entrywise extension. Note that the entrywise extension of a valuation on $\mathcal{A}$ to $\operatorname{Mat}_{n}(\mathcal{A})$ is again a valuation.
8.2.3. Reducing valuations. Let $\mathcal{A}$ be a $*$-algebra equipped with a valuation deg. We call the valuation deg reducing if for every $x \in \mathcal{A}_{\text {sa }}$ there exists a positive integer $k$ and elements $a_{1}, \ldots, a_{2 k} \in \mathcal{A}$ such that

$$
x=\sum_{i=1}^{k}\left(a_{i} a_{k+i}^{*}+a_{k+i} a_{i}^{*}\right) \text { and } \bigvee_{i=1}^{2 k} \operatorname{deg} a_{i} \leq 1 \vee \frac{2}{3} \operatorname{deg} x .
$$

Note that entrywise extension of a reducing valuation on $\mathcal{A}$ to $\operatorname{Mat}_{n}(\mathcal{A})$ is again a reducing valuation.

Remark 8.2.4. For every nonnegative integer $n$ there exist nonnegative integers $n_{1}$ and $n_{2}$ such that $n=n_{1}+n_{2}$ and $n_{1} \vee n_{2} \leq 1 \vee \frac{2 n}{3}$. Thus the usual degree function on the algebra $\mathbb{C}[T]$ of polynomials in one self-adjoint variable $T$ is a reducing valuation. Similarly the total degree function on $\mathbb{C}\langle\mathbf{X}\rangle$ is a reducing valuation, and in turn the entrywise extension of the total degree function to $\operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)$ is a reducing valuation.

Proposition 8.2.5 (The raw self-adjoint linearization trick). Let $\mathcal{A}$ be $a *$-algebra equipped with a reducing valuation deg. For every $f \in \mathcal{A}_{\text {sa }}$ there exist a positive integer $s$, a matrix $d \in \operatorname{GL}_{s}(\mathcal{A}) \cap \operatorname{Mat}_{s}(\mathcal{A})_{\mathrm{sa}}$ and a row vector $b \in \operatorname{Mat}_{1 \times s}(\mathcal{A})$ such that $f=-b d^{-1} b^{*}$ and $(\operatorname{deg} b) \vee(\operatorname{deg} d) \leq 1$.

Proof. By the definition of a reducing valuation, at least we can find a positive integer $k$ and a row vector $g \in \operatorname{Mat}_{1 \times 2 k}(\mathcal{A})$ such that

$$
f=-g h g^{*} \text { and } \operatorname{deg} g \leq 1 \vee \frac{2}{3} \operatorname{deg} f
$$

where

$$
h=-\left[\begin{array}{cc}
0 & \mathbf{I}_{k} \otimes 1_{\mathcal{A}} \\
\mathbf{I}_{k} \otimes 1_{\mathcal{A}} & 0
\end{array}\right]=h^{-1}=h^{*} \in \operatorname{Mat}_{2 k}(\mathcal{A}) .
$$

We may assume $\operatorname{deg} f \geq 3$ because otherwise we are already done. By induction on $\operatorname{deg} f$, working in the $*$-algebra $\operatorname{Mat}_{2 k+1}(\mathcal{A})$ equipped by entrywise extension with the reducing valuation deg, for some positive integer $\ell$, we may write

$$
\left[\begin{array}{cc}
0 & g \\
g^{*} & h
\end{array}\right]=-B D^{-1} B^{*} \text { and }(\operatorname{deg} B) \vee \operatorname{deg} D \leq 1
$$

where

$$
B \in \operatorname{Mat}_{(2 k+1) \times(2 k+1) \ell}(\mathcal{A}) \text { and } D \in \operatorname{GL}_{(2 k+1) \ell}(\mathcal{A})^{\times} \cap \operatorname{Mat}_{(2 k+1) \ell}(\mathcal{A})_{\mathrm{sa}} .
$$

Now we write

$$
\left[\begin{array}{cc}
0 & B \\
B^{*} & D
\end{array}\right]=\left[\begin{array}{cc}
0 & b \\
b^{*} & d
\end{array}\right]=\left[\begin{array}{ccc}
0 & 0 & p \\
0 & 0 & P \\
p^{*} & P^{*} & D
\end{array}\right]
$$

where

$$
\begin{aligned}
& s=(2 k+1)(\ell+1)-1, \quad b \in \operatorname{Mat}_{1 \times s}(\mathcal{A}), \quad d \in \operatorname{Mat}_{s}(\mathcal{A})_{\mathrm{sa}}, \\
& p \in \operatorname{Mat}_{1 \times(2 k+1) \ell}(\mathcal{A}), \quad P \in \operatorname{Mat}_{2 k \times(2 k+1) \ell}(\mathcal{A}) .
\end{aligned}
$$

Note that

$$
h=-P D^{-1} P^{*}, \quad-p D^{-1} P^{*}=g, \quad p D^{-1} p^{*}=0 .
$$

By Proposition 8.2.1 we have $d=\left[\begin{array}{cc}0 & P \\ P^{*} & D\end{array}\right] \in \operatorname{GL}_{s}(\mathcal{A})$ and more precisely

$$
\begin{aligned}
& -b d^{-1} b^{*}=-\left[\begin{array}{ll}
0 & p
\end{array}\right]\left[\begin{array}{cc}
0 & P \\
P^{*} & D
\end{array}\right]^{-1}\left[\begin{array}{c}
0 \\
p^{*}
\end{array}\right] \\
= & -\left[\begin{array}{ll}
0 & p
\end{array}\right]\left[\begin{array}{cc}
h & -h P D^{-1} \\
-D^{-1} P^{*} h & D^{-1}+D^{-1} P^{*} h P D^{-1}
\end{array}\right]\left[\begin{array}{c}
0 \\
p^{*}
\end{array}\right]=-g h g^{*}=f,
\end{aligned}
$$

which finishes the proof.
8.3. Naive application of the raw trick and proof of Proposition 8.1.2. Now we combine the algebra of the previous subsection with some operator theory.

Definition 8.3.1. A naive self-adjoint linearization of $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ is a matrix of the form

$$
\tilde{f}=\left[\begin{array}{cc}
0 & b \\
b^{*} & d
\end{array}\right] \in \operatorname{Mat}_{s}(\mathbb{C}\langle\mathbf{X}\rangle)_{\mathrm{sa}}
$$

where

$$
\begin{aligned}
& \operatorname{deg} \tilde{f} \leq 1, \quad s>n, \quad b \in \operatorname{Mat}_{n \times(s-n)}(\mathbb{C}\langle\mathbf{X}\rangle) \text { and } \\
& d \in \mathrm{GL}_{s-n}(\mathbb{C}\langle\mathbf{X}\rangle) \cap \operatorname{Mat}_{s-n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}
\end{aligned}
$$

such that

$$
f=-b d^{-1} b^{*} .
$$

Existence of such a matrix $\tilde{f}$ for any given $f$ is guaranteed by Proposition 8.2.5.

Lemma 8.3.2. Fix $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ along with a naive self-adjoint linearization $\tilde{f}=\left[\begin{array}{cc}0 & b \\ b^{*} & d\end{array}\right] \in \operatorname{Mat}_{s}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ thereof. Fix a $C^{*}$-algebra $\mathcal{A}$, sequence $\xi \in \mathcal{A}_{\mathrm{sa}}^{\infty}$, point $z \in \mathfrak{h}$ and parameter value $t \geq 0$. Then we have

$$
\begin{align*}
& \tilde{f}(\xi)-\left(\left[\begin{array}{cc}
z \mathbf{I}_{n} & 0 \\
0 & 0
\end{array}\right]+\mathrm{i} t \mathbf{I}_{s}\right) \otimes 1_{\mathcal{A}} \in \mathrm{GL}_{s}(\mathcal{A}) \text { and }  \tag{85}\\
& {\left[\left[\left(\tilde{f}(\xi)-\left(\left[\begin{array}{cc}
z \mathbf{I}_{n} & 0 \\
0 & 0
\end{array}\right]+\mathrm{i} t \mathbf{I}_{s}\right) \otimes 1_{\mathcal{A}}\right)^{-1}\right]\right] }  \tag{86}\\
\leq & \left\{\begin{aligned}
c_{0}(1+[[\tilde{f}(\xi)-\tilde{f}(0)]])^{c_{1}}(1+1 / \Im z) & \text { in general, } \\
\frac{1}{t} & \text { if } t>0,
\end{aligned}\right.
\end{align*}
$$

for finite constants $c_{0} \geq 1$ and $c_{1} \geq 0$ independent of $\mathcal{A}, \xi, z$ and $t$.
Proof. We first consider the case $t=0$. Note that $f(\xi)-z \mathbf{I}_{n} \otimes 1_{\mathcal{A}}$ is invertible since $f(\xi)$ is self-adjoint. Thus (85) holds by (83). Write

$$
\begin{equation*}
\tilde{f}=a_{0} \otimes 1_{\mathbb{C}\langle\mathbf{X}\rangle}+\sum_{i=1}^{\infty} a_{i} \otimes \mathbf{X}_{i} \tag{87}
\end{equation*}
$$

where necessarily the coefficients $a_{i}$ belong to $\operatorname{Mat}_{s}(\mathbb{C})_{\mathrm{sa}}$ and vanish for $i \gg 0$. Let $\left\{\hat{a}_{j}\right\}_{j=1}^{m}$ be a basis over $\mathbb{R}$ for the real linear span of the coefficients $\left\{a_{i}\right\}_{i=1}^{\infty}$. For suitable real linear combinations $\widehat{\mathbf{X}}_{j}$ of the variables $\mathbf{X}_{i}$ we have

$$
\tilde{f}-\tilde{f}(0)=\sum_{i=1}^{\infty} a_{i} \otimes \mathbf{X}_{i}=\sum_{j=1}^{m} \hat{a}_{j} \otimes \widehat{\mathbf{X}}_{j}
$$

Put

$$
\hat{\xi}_{j}=\widehat{\mathbf{X}}_{j}(\xi) \in \mathcal{A}_{\mathrm{sa}} \text { for } j=1, \ldots, m
$$

By Lemma 5.1.7, for a constant $c$ depending only on the family $\left\{\hat{a}_{j}\right\}_{j=1}^{m}$, we have

$$
\bigvee_{j=1}^{m}\left[\left[\hat{\xi}_{j}\right]\right] \leq c\left[\left[\sum_{j=1}^{m} \hat{a}_{j} \otimes \hat{\xi}_{j}\right]\right]=c[[\tilde{f}(\xi)-\tilde{f}(0)]]
$$

Note that the entries of $b$ and $d^{-1}$ can be expanded as noncommutative polynomials in the linear forms $\widehat{\mathbf{X}}_{j}$. Note also that

$$
\left[\left[\left(f(\xi)-z \mathbf{I}_{n} \otimes 1_{\mathcal{A}}\right)^{-1}\right]\right] \leq 1 / \Im z
$$

by Lemma 4.2.6. It follows by (84) that (86) holds for suitable $c_{0}$ and $c_{1}$. This concludes the proof in the case $t=0$. Assume for the rest of the proof that $t>0$. We immediately get (85) and the bound $\frac{1}{t}$ in (86) by Lemma 4.2.6. After replacing $c_{0}$ by $2 c_{0}$, we may suppose that (86) holds for $t=0$ with $\frac{c_{0}}{2}$ in place of $c_{0}$. To finish up we distinguish two cases, namely

$$
t c_{0}\left(1+\left[\left[\sum_{i=1}^{\infty} a_{i} \otimes \xi_{i}\right]\right]\right)^{c_{1}}(1+1 / \Im z)\left\{\begin{array}{l}
>1 \\
\leq 1
\end{array}\right.
$$

In the former case (86) already holds, whereas in the latter case (86) holds by Lemma 4.1.1. The proof is complete.
8.3.3. Proof of Proposition 8.1.2. Given $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ and a naive self-adjoint linearization $\tilde{f} \in \operatorname{Mat}_{s}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$ of $f$, we manufacture a "sophisticated" self-adjoint linearization by the following procedure. Take $\mathcal{S}$ to be a copy of Mat ${ }_{s}(\mathbb{C})$. Use formula (87) again to define coefficients $a_{\ell} \in \operatorname{Mat}_{s}(\mathbb{C})_{\mathrm{sa}}$. Put

$$
L=\sum \mathbf{X}_{\ell} \otimes a_{\ell}, \quad \Theta=-a_{0}, \quad e=\left[\begin{array}{cc}
\mathbf{I}_{n} & 0 \\
0 & 0
\end{array}\right]
$$

Clearly, $L$ and $\Theta$ are self-adjoint and $e$ is a projection. Note that the linear map $\left(A \mapsto \frac{1}{n} \sum_{i=1}^{n} A(i, i)\right)$ coincides with the state $\tau_{e S e}$. The quadruple ( $\left.\mathcal{S}, L, \Theta, e\right)$ satisfies (79) and (80) by Lemma 8.3.2 and thus is a SALT block design. The quadruple $(\mathcal{S}, L, \Theta, e)$ has properties (71) and (72) by Proposition 8.2.1, and thus is a self-adjoint linearization of $f$.
8.4. The secondary trick. We first present the underline construction and then the secondary trick itself. We already saw the idea of the secondary trick in germinal form in the proof of Proposition 6.2.2.
8.4.1. The underline construction for a block algebra. Let $\mathcal{M}_{3}$ denote a block algebra equipped with a standard basis $\left\{e_{i j}\right\}_{i, j=1}^{3}$. Let $\mathcal{S}$ be any block algebra. We define

$$
\underline{\mathcal{S}}=\mathcal{S}^{\otimes 2} \otimes \mathcal{M}_{3}, \quad \nabla_{\mathcal{S}}=1_{\mathcal{S}}^{\otimes 2} \otimes\left(e_{12}+e_{13}\right) \in \underline{\mathcal{S}} .
$$

Furthermore, given $\Lambda \in \mathcal{S}$ we define

$$
\begin{array}{r}
\underline{\Lambda}_{1}=\Lambda \otimes 1_{\mathcal{S}} \in \mathcal{S}^{\otimes 2}, \quad \underline{\Lambda}_{2}=1_{\mathcal{S}} \otimes \Lambda \in \mathcal{S}^{\otimes 2}, \\
\underline{\Lambda}=\underline{\Lambda}_{1} \otimes e_{11}+\underline{\Lambda}_{2} \otimes e_{22}+\underline{\Lambda}_{2}^{\mathrm{T}} \otimes e_{33} \in \underline{\mathcal{S}} .
\end{array}
$$

We also define linear maps

$$
\partial_{1} \in B(\underline{\mathcal{S}}, B(\mathcal{S})) \text { and } \partial_{2} \in B\left(\underline{\mathcal{S}}, \mathcal{S}^{\otimes 2}\right)
$$

by the formulas

$$
\partial_{1}\left(A \otimes e_{i j}\right)=A^{\bullet} \delta_{1 i} \delta_{2 j}, \quad \partial_{2}\left(A \otimes e_{i j}\right)=A^{1 \otimes \mathrm{~T}} \delta_{1 i} \delta_{3 j}
$$

for $A \in \mathcal{S}^{\otimes 2}$ and $i, j=1, \ldots, 3$.
8.4.2. The underline construction for $\mathcal{S}$-linear forms. Let $\mathcal{S}$ be a block algebra and let $\underline{\mathcal{S}}$ be the corresponding "underlined" block algebra as defined in the preceding paragraph. Given an $\mathcal{S}$-linear form $L$ with Hamel expansion $L=\sum \mathbf{X}_{\ell} \otimes a_{\ell}$, we define $\mathcal{S}^{\otimes 2}$-linear forms $\underline{L}_{1}$ and $\underline{L}_{2}$ by the formulas

$$
\underline{L}_{1}=\sum_{\ell} \mathbf{X}_{\ell} \otimes a_{\ell} \otimes 1_{\mathcal{S}}, \quad \underline{L}_{2}=\sum_{\ell} \mathbf{X}_{\ell} \otimes 1_{\mathcal{S}} \otimes a_{\ell}
$$

and in turn define an $\underline{\mathcal{S}}$-linear form $\underline{L}$ by the formula

$$
\underline{L}=\underline{L}_{1} \otimes e_{11}+\underline{L}_{2} \otimes e_{22}+\underline{L}_{2}^{\mathrm{T}} \otimes e_{33} .
$$

8.4.3. The trick itself. Let $\mathcal{S}$ be a block algebra, let $\Lambda \in \mathcal{S}$ be an element and let $L$ be an $\mathcal{S}$-linear form. Let $\underline{\mathcal{S}}, \underline{\Lambda}_{1}, \underline{\Lambda}_{2}, \underline{\Lambda}, \underline{L}_{1}, \underline{L}_{2}$ and $\underline{L}$ be as defined in the preceding two paragraphs. Let $(\mathcal{A}, \phi)$ be a $C^{*, T}$-probability space and fix a sequence $\xi \in \mathcal{A}_{\text {salt }}^{\infty}$. For the rest of this paragraph we abuse notation by writing $x=1_{\mathcal{A}} \otimes x$ for $x \in \mathcal{S}, x \in \mathcal{S}^{\otimes 2}$ or $x \in \underline{\mathcal{S}}$. We assume that $L(\xi)-\Lambda \in(\mathcal{A} \otimes \mathcal{S})^{\times}$, in which case $\underline{L}(\xi)-\underline{\Lambda}-\diamond_{\mathcal{S}} \in(\mathcal{A} \otimes \underline{\mathcal{S}})^{\times}$, via Lemma 5.2.6. Indeed, making further use of the cited lemma, we have

$$
\begin{aligned}
& \left(\underline{L}(\xi)-\underline{\Lambda}-\diamond_{\mathcal{S}}\right)^{-1} \\
= & \left(\underline{L}_{1}(\xi)-\underline{\Lambda}_{1}\right)^{-1} \otimes e_{11}+\left(\underline{L}_{2}(\xi)-\underline{\Lambda}_{2}\right)^{-1} \otimes e_{22}+\left(\left(\underline{L}_{2}(\xi)-\underline{\Lambda}_{2}\right)^{-1}\right)^{\mathrm{T}} \otimes e_{33} \\
& +\left(\left(\underline{L}_{1}(\xi)-\underline{\Lambda}_{1}\right)^{-1}\left(\underline{L}_{2}(\xi)-\underline{\Lambda}_{2}\right)^{-1}\right) \otimes e_{12} \\
& +\left(\left(\underline{L}_{1}(\xi)-\underline{\Lambda}_{1}\right)^{-1}\left(\left(\underline{L}_{2}(\xi)-\underline{\Lambda}_{2}\right)^{-1}\right)^{\mathrm{T}}\right) \otimes e_{13} .
\end{aligned}
$$

It follows that
(88) $\llbracket L(\xi) \rrbracket=\llbracket \underline{L}(\xi) \rrbracket, \quad\left[\left[\left(\underline{L}(\xi)-\underline{\Lambda}-\diamond_{\mathcal{S}}\right)^{-1}\right]\right] \leq 3\left(1 \vee\left[\left[(L(\xi)-\Lambda)^{-1}\right]\right]\right)^{2}$,

$$
\begin{align*}
& \partial_{1} \circ \phi_{\underline{\mathcal{S}}}\left(\left(\underline{L}(\xi)-\underline{\Lambda}-\diamond_{\mathcal{S}}\right)^{-1}\right)=\phi_{\mathcal{S}, \mathcal{S}}\left((L(\xi)-\Lambda)^{-1},(L(\xi)-\Lambda)^{-1}\right)^{\bullet}  \tag{89}\\
& \partial_{2} \circ \phi_{\underline{\mathcal{S}}}\left(\left(\underline{L}(\xi)-\underline{\Lambda}-\diamond_{\mathcal{S}}\right)^{-1}\right)=\phi_{\mathcal{S}, \mathcal{S}}\left((L(\xi)-\Lambda)^{-1},\left((L(\xi)-\Lambda)^{-1}\right)^{\mathrm{T}}\right)^{1 \otimes \mathrm{~T}}
\end{align*}
$$

where to get the last two identities we use the trivial formula (54).
Lemma 8.4.4. For any $S A L T$ block design $(\mathcal{S}, L, \Theta, e)$, again $\left.(\underline{\mathcal{S}}, \underline{L}, \underline{\Theta}+\rangle_{\mathcal{S}}, \underline{e}\right)$ is a SALT block design. More precisely, if $c_{0}, c_{1}$ and $c_{2}$ are constants rendering the estimate (72) valid for $(\mathcal{S}, L, \Theta, e)$, then one can take the corresponding constants $\underline{c}_{0}, \underline{c}_{1}$ and $\underline{c}_{2}$ for $\left(\underline{\mathcal{S}}, \underline{L}, \underline{\Theta}+\nabla_{\mathcal{S}}, \underline{e}\right)$ to be $\underline{c}_{0}=3 c_{0}^{2}, \underline{c}_{1}=2 c_{1}$ and $\underline{c}_{2}=2 c_{2}$.

Proof. One can read off the necessary estimates from (88).
Remark 8.4.5. The proof of Proposition 8.1.2 only generates SALT block designs which are self-adjoint adjoint linearizations, and in particular are such that $\Theta \in \mathcal{S}_{\text {sa }}$ and $c_{2}=1$. But notice that, in the notation used in Lemma 8.4.4, the element $\underline{\Theta}+\nabla_{\mathcal{S}} \in \underline{\mathcal{S}}$ is never self-adjoint and (more significantly) $\underline{c}_{2}=2 c_{2}$. Thus the extra generality in Definition 7.1.1 not used by the "primary" trick described in Proposition 8.1.2 is forced on us by Lemma 8.4.4.

Remark 8.4.6. Let the definitions made in $\S 7.2 .3$ for a given SALT block design ( $\mathcal{S}, L, \Theta, e$ ) and nonempty set $I \subset\{1, \ldots, N\}$ be repeated for the underlined design $\left(\underline{\mathcal{S}}, \underline{L}, \underline{\Theta}+\nabla_{\mathcal{S}}, \underline{e}\right)$. Denote the new set of random variables so arising with underlines. We then have

$$
\begin{equation*}
\partial_{1} \underline{F}=\left(\zeta \mapsto \frac{1}{N} \operatorname{tr}_{\mathcal{S}}\left(R\left(\mathbf{I}_{n} \otimes \zeta\right) R\right)\right) \text { and } \partial_{2} \underline{F}=\frac{1}{N} \sum_{i, j=1}^{n} R(i, j)^{\otimes 2} \tag{91}
\end{equation*}
$$

by $(56),(57),(89)$ and (90). Furthermore, given any point $\Lambda \in \mathcal{D}_{L}$, we automatically have $\underline{\Lambda}+\nabla_{\mathcal{S}} \in \mathcal{D}_{\underline{L}}$ and

$$
\begin{equation*}
\left.\left.\partial_{1} G_{\underline{L}}(\underline{\Lambda}+\rangle_{\mathcal{S}}\right)=\mathbf{D}\left[G_{L}\right](\Lambda) \text { and } \partial_{2} G_{\underline{L}}(\underline{\Lambda}+\rangle_{\mathcal{S}}\right)=\left(\left(G_{L}(\Lambda)^{-1}\right)^{\otimes 2}-\Psi_{L}\right)^{-1} \tag{92}
\end{equation*}
$$

by $(67),(68),(89)$ and (90). Formulas (91) and (92) together are the whole point of the secondary trick.

## 9. TOOLS FOR CONCENTRATION

We begin the long march toward control of the $\mathfrak{E}$-type statistics. In this section we introduce the ensemble of tools we will use to replace the Poincaré-type inequalities used in [9], [8], [19], [4] and [14]. We speak of an ensemble because no one tool seems to contribute more than incrementally.
9.1. Quadratic forms in independent random vectors. Variants of the next result are in common use in RMT. (See, e.g., [2, Lemma 2.7].)

Proposition 9.1.1. Let $Y_{1}, \ldots, Y_{n}$ and $Z_{1}, \ldots, Z_{n}$ be $\mathbb{C}$-valued random variables which for some $p \in[2, \infty)$ all belong to $L^{2 p}$ and have mean zero. Let $A \in \operatorname{Mat}_{n}(\mathbb{C})$ be a (deterministic) matrix. Assume furthermore that the family of $\sigma$-fields $\left\{\sigma\left(Y_{i}, Z_{i}\right)\right\}_{i=1}^{n}$ is independent. Then we have

$$
\left\|\sum_{i, j=1}^{n} A(i, j)\left(Y_{i} Z_{j}-\mathbb{E} Y_{i} Z_{j}\right)\right\|_{p} \leq c\left(\sum_{i, j=1}^{n}|A(i, j)|^{2}\left\|Y_{i}\right\|_{2 p}^{2}\left\|Z_{j}\right\|_{2 p}^{2}\right)^{1 / 2}
$$

for a constant $c$ depending only on $p$.
This result is proved in [21] with an explicit constant $c$ in the special case in which $Y_{i}=Z_{i}=Y_{i}^{*}=Z_{i}^{*}$ and $A$ has real entries. From that special case the general case of the proposition above can be deduced by algebraic manipulation.

We now generalize in an innocuous if superficially complicated way.
Proposition 9.1.2. Fix constants $p \in[2, \infty)$ and $K \in(0, \infty)$. Let $\mathcal{V}$ be a finitedimensional Banach space, let $\mathcal{S}$ be a block algebra and let $\mathcal{G}$ be a $\sigma$-field. Let $Y \in \operatorname{Mat}_{1 \times n}(\mathcal{S})$ and $Z \in \operatorname{Mat}_{n \times 1}(\mathcal{S})$ be random such that

$$
\left(\vee\|\llbracket Y(1, j) \rrbracket\|_{2 p}\right) \vee\left(\vee\|\llbracket Z(i, 1) \rrbracket\|_{2 p}\right) \leq K \text { and } \mathbb{E} Y=0=\mathbb{E} Z
$$

Assume also that the family $\mathcal{G} \cup\{\sigma(Y(1, i), Z(i, 1))\}_{i=1}^{n}$ of $\sigma$-fields is independent. Then for any $\mathcal{G}$-measurable random bilinear map

$$
R \in B\left(\operatorname{Mat}_{1 \times n}(\mathcal{S}), \operatorname{Mat}_{n \times 1}(\mathcal{S}) ; \mathcal{V}\right)
$$

such that $\|\llbracket R \rrbracket\|_{p}<\infty$ we have

$$
\|\llbracket R(Y, Z)-\mathbb{E}(R(Y, Z) \mid \mathcal{G}) \rrbracket\|_{p} \leq C K^{2}\|\llbracket R \rrbracket\|_{p} \sqrt{n}
$$

where the constant $C$ depends only on $p, \mathcal{S}$ and $\mathcal{V}$.
We need two lemmas, the first of which actually proves more than we immediately need but has several further uses later in the paper.

Lemma 9.1.3. Let $\mathcal{S}$ be a block algebra of dimension $s^{2}$. (i) For $X \in \operatorname{Mat}_{k \times \ell}(\mathcal{S})$, we have

$$
\frac{1}{s} \llbracket X \rrbracket^{2} \leq \sum_{i=1}^{k} \sum_{j=1}^{\ell} \llbracket X(i, j) \rrbracket^{2} \leq s(k \wedge \ell) \llbracket X \rrbracket^{2} .
$$

(ii) For $X \in \operatorname{Mat}_{k \times \ell}(\mathcal{S})$ and $Y \in \operatorname{Mat}_{\ell \times k}(\mathcal{S})$, we have

$$
\llbracket \operatorname{tr}_{\mathcal{S}}(X Y) \rrbracket \leq s \ell \llbracket X \rrbracket \llbracket Y \rrbracket .
$$

(iii) For $X \in \operatorname{Mat}_{n}(\mathcal{S})$, we have

$$
\left[\left[\sum_{i, j=1}^{n} X(i, j)^{\otimes 2}\right]\right] \leq s n \llbracket X \rrbracket^{2} .
$$

Proof. Statement (i) is an assertion concerning the Hilbert-Schmidt norm which is easy to verify. Statements (ii) and (iii) follow from statement (i) via the CauchySchwarz inequality.

Lemma 9.1.4. Let $\mathcal{S}$ be a block algebra and let $\left\{e_{i}\right\}_{i=1}^{\ell}$ be a basis of the underlying vector space. Let $\mathcal{V}$ be a finite-dimensional Banach space and let $\left\{v_{k}\right\}_{k=1}^{m}$ be a basis of the underlying vector space. Fix matrices $R_{i j}^{k} \in \operatorname{Mat}_{n}(\mathbb{C})$ for $i, j=1, \ldots, \ell$ and $k=1, \ldots, m$. Define $R \in B\left(\operatorname{Mat}_{1 \times n}(\mathcal{S}), \operatorname{Mat}_{n \times 1}(\mathcal{S}) ; \mathcal{V}\right)$ by requiring that

$$
R\left(x \otimes e_{i}, y \otimes e_{j}\right)=\sum_{k}\left(x R_{i j}^{k} y\right) v_{k}
$$

for $i, j=1, \ldots, \ell, x \in \operatorname{Mat}_{1 \times n}(\mathbb{C})$ and $y \in \operatorname{Mat}_{n \times 1}(\mathbb{C})$. Then

$$
\frac{1}{C} \bigvee_{i, j, k}\left[\left[R_{i j}^{k}\right]\right] \leq \llbracket R \rrbracket \leq C \sum_{i, j, k}\left[\left[R_{i j}^{k}\right]\right]
$$

for a constant $C \geq 1$ which depends only on the data $\left(\mathcal{S},\left\{e_{i}\right\}, \mathcal{V},\left\{v_{k}\right\}\right)$ and in particular is independent of $n$.

Proof. By Lemma 5.1.7 and the fact that the map

$$
(A \mapsto((x, y) \mapsto x A y)): \operatorname{Mat}_{n}(\mathbb{C}) \rightarrow B\left(\operatorname{Mat}_{1 \times n}(\mathbb{C}), \operatorname{Mat}_{n \times 1}(\mathbb{C}) ; \mathbb{C}\right)
$$

is an isometric isomorphism, the proof of the lemma at hand reduces to a straightforward calculation the remaining details of which we can safely omit.
9.1.5. Proof of Proposition 9.1.2. After using standard properties of conditional expectation we may assume that $R$ is deterministic. We may also assume that $\mathcal{S}$ is isomorphic to $\operatorname{Mat}_{s}(\mathbb{C})$ for some $s$ and in turn Lemma 9.1.4 permits us to assume that $\mathcal{S}=\mathbb{C}$. Finally, by Lemma 9.1 .3 , the proposition at hand reduces to Proposition 9.1.1.

Remark 9.1.6. In applications of Proposition 9.1 .2 we will only use two special types of bilinear map $R$. We describe these types and estimate $\llbracket R \rrbracket$ for each. (They conform to the patterns sets by the objects $Q_{I, J, j_{1}, j_{2}}^{N}$ and $P_{I, J, j_{1}, j_{2}}^{N}$ defined in $\S 10$ below, respectively.) (i) In the " $Q$-type" first case of interest, we have $\mathcal{V}=\mathcal{S}$ and for some $A \in \operatorname{Mat}_{n}(\mathcal{S})$ we have $R(y, z)=y A z$, in which case $\llbracket R \rrbracket \leq \llbracket A \rrbracket$. (ii) In the "P-type" second case of interest, we have $\mathcal{V}=B(\mathcal{S})$, and for some $A \in \operatorname{Mat}_{n}(\mathcal{S})$ we have $R(y, z)=\left(B \mapsto \operatorname{tr}_{\mathcal{S}}(A z B y A)\right)$, in which case $\llbracket R \rrbracket \leq s \llbracket A \rrbracket^{2}$ for $s$ equal to the square root of the dimension of $\mathcal{S}$ over the complex numbers by Lemma 9.1.3.
9.2. A conditional variance bound. We present a result which harmlessly generalizes the well-known subadditivity of variance to a situation involving vectorvalued random variables and some dependence.
9.2.1. Setup for the result. Let $\mathcal{V}$ be a finite-dimensional Banach space (either real or complex scalars). Let $\{\mathcal{E}\} \cup\{\mathcal{G}(i, j)\}_{1 \leq i \leq j \leq N}$ be a family of independent $\sigma$-fields and let $\mathcal{G}$ be the $\sigma$-field generated by this family. Let $Z \in \mathcal{V}$ be a $\mathcal{G}$-measurable random vector such that $\|\llbracket Z \rrbracket\|_{p}<\infty$ for $p \in[1, \infty)$. For $k=1, \ldots, N$, let $\widehat{\mathcal{G}}_{k}$ be the $\sigma$-field generated by the subfamily $\{\mathcal{E}\} \cup\{\mathcal{G}(i, j) \mid k \notin\{i, j\}\}$ and let $Z_{k} \in \mathcal{V}$ be a $\widehat{\mathcal{G}}_{k}$-measurable random vector such that $\left\|\llbracket Z_{k} \rrbracket\right\|_{p}<\infty$ for $p \in[1, \infty)$.
Proposition 9.2.2. Notation and assumptions are as above. For every constant $p \in[1, \infty)$ we have

$$
\begin{equation*}
\left\|\mathbb{E}\left(\llbracket Z-\mathbb{E}(Z \mid \mathcal{E}) \rrbracket^{2} \mid \mathcal{E}\right)\right\|_{p} \leq c \sum_{k=1}^{N}\left\|\llbracket Z-Z_{k} \rrbracket\right\|_{2 p}^{2} \tag{93}
\end{equation*}
$$

for a constant $c$ depending only on $\mathcal{V}$ and in particular independent of $p$.
Proof. We may assume that $\mathcal{V}$ is a (finite-dimensional) real Hilbert space, and in this case we will prove the claim with a constant $c=1$. After a routine application of Minkowski and Jensen inequalities, it is enough to prove

$$
\begin{equation*}
\mathbb{E}\left(\llbracket Z-\mathbb{E}(Z \mid \mathcal{E}) \rrbracket^{2} \mid \mathcal{E}\right) \leq \sum_{k=1}^{N} \mathbb{E}\left(\llbracket Z-Z_{k} \rrbracket^{2} \mid \mathcal{E}\right) \tag{94}
\end{equation*}
$$

almost surely. There is also no harm in assuming that $\mathcal{V}=\mathbb{R}$. For $k=0, \ldots, N$, let $\mathcal{G}_{k}$ be the $\sigma$-field generated by the subfamily

$$
\{\mathcal{E}\} \cup\{\mathcal{G}(i, j) \mid 1 \leq i \leq j \leq k\}
$$

In any case, by orthogonality of martingale increments, we have

$$
\mathbb{E}\left(\llbracket Z-\mathbb{E}(Z \mid \mathcal{E}) \rrbracket^{2} \mid \mathcal{E}\right)=\sum_{i=1}^{N} \mathbb{E}\left(\llbracket \mathbb{E}\left(Z \mid \mathcal{G}_{k}\right)-\mathbb{E}\left(Z \mid \mathcal{G}_{k-1}\right) \rrbracket^{2} \mid \mathcal{E}\right)
$$

almost surely. Furthermore, we have

$$
\mathbb{E}\left(\mathbb{E}\left(Z \mid \widehat{\mathcal{G}}_{k}\right) \mid \mathcal{G}_{k}\right)=\mathbb{E}\left(Z \mid \mathcal{G}_{k-1}\right)
$$

almost surely. Finally, we have

$$
\begin{aligned}
& \mathbb{E}\left(\llbracket \mathbb{E}\left(Z \mid \mathcal{G}_{k}\right)-\mathbb{E}\left(Z \mid \mathcal{G}_{k-1}\right) \rrbracket^{2} \mid \mathcal{E}\right)=\mathbb{E}\left(\left[\left[\mathbb{E}\left(Z-\mathbb{E}\left(Z \mid \widehat{\mathcal{G}}_{k}\right) \mid \mathcal{G}_{k}\right)\right]^{2} \mid \mathcal{E}\right)\right. \\
& \leq \mathbb{E}\left(\left[\left[Z-\mathbb{E}\left(Z \mid \widehat{\mathcal{G}}_{k}\right)\right]^{2} \mid \mathcal{E}\right) \leq \mathbb{E}\left(\llbracket Z-Z_{k} \rrbracket^{2} \mid \mathcal{E}\right),\right.
\end{aligned}
$$

almost surely, whence (94).
Definition 9.2.3. The random variable $\mathbb{E}\left(\llbracket Z-\mathbb{E}(Z \mid \mathcal{E}) \rrbracket^{2} \mid \mathcal{E}\right)$ appearing on the left side of (93) will be denoted by $\operatorname{Var}_{\mathcal{V}}(Z \mid \mathcal{E})$ in the sequel.
9.3. Estimates for tensor-cubic forms. We work out a specialized estimate involving three-fold tensor products and partitions of a set of cardinality six. The combinatorial apparatus introduced here will have further uses.
9.3.1. Set partitions and related apparatus. A set partition of $k$ is a disjoint family $\Pi$ of nonempty subsets of the set $\{1, \ldots, k\}$ whose union is $\{1, \ldots, k\}$. Each member of a set partition is called a part. Let $\operatorname{Part}(k)$ be the family of set partitions of $k$. Let $\operatorname{Part}^{*}(2 k)$ be the subset of Part $(2 k)$ consisting of set partitions having no singleton as a part, nor having any of the sets $\{2 i-1,2 i\}$ for $i=1, \ldots, k$ as a part. Let $\operatorname{Part}_{2}^{*}(2 k) \subset \operatorname{Part}^{*}(2 k)$ be the subfamily consisting of partitions all of whose parts have cardinality 2. For each positive integer $k$ let $S_{k}$ be the group of permutation of $\{1, \ldots, k\}$. Let $\Gamma_{k} \subset S_{2 k}$ be the subgroup centralizing the involutive permutation (12) $\cdots(2 k-1,2 k)$. Then $\Gamma_{k}$ acts on the set Part* $(2 k)$. For $\Pi_{1}, \Pi_{2} \in \operatorname{Part}^{*}(2 k)$ belonging to the same $\Gamma_{k}$-orbit we write $\Pi_{1} \sim \Pi_{2}$.
9.3.2. Explicit descriptions of $\operatorname{Part}^{*}(4)$ and Part* $^{*}(6)$. To describe Part* (4) we can easily enumerate it, thus:

$$
\begin{equation*}
\{\{1,2,3,4\}\},\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\} . \tag{95}
\end{equation*}
$$

It can be shown (we omit the tedious details) that for every $\Pi \in \operatorname{Part}{ }^{*}(6)$ there exists exactly one set partition on the list

$$
\begin{align*}
& \{\{1,2,3,4,5,6\}\},\{\{1,6\},\{2,3,4,5\}\},\{\{1,3,5\},\{2,4,6\}\},  \tag{96}\\
& \{\{1,6\},\{2,3\},\{4,5\}\}, \quad\{\{1,2,3\},\{4,5,6\}\}
\end{align*}
$$

belonging to the $\Gamma_{3}$-orbit of $\Pi$.
9.3.3. Sequences and associated partitions. For any finite set $I$ we write

$$
\operatorname{Seq}(k, I)=\{\mathbf{i}:\{1, \ldots, k\} \rightarrow I\}
$$

Given $\mathbf{i} \in \operatorname{Seq}(k, I)$, let $\Pi(\mathbf{i}) \in \operatorname{Part}(k)$ be the set partition generated by i, i.e., the coarsest set partition on the parts of which $\mathbf{i}$ is constant. If $I=\{1, \ldots, n\}$ we write $\operatorname{Seq}(k, I)=\operatorname{Seq}(k, n)$ by abuse of notation. Sometimes we represent elements of $\operatorname{Seq}(k, I)$ as "words" $i_{1} \cdots i_{k}$ spelled with "letters" $i_{1}, \ldots, i_{k} \in I$.
9.3.4. Setup for the main result. Let $\mathcal{S}$ be a block algebra. Let a set partition $\Pi \in \operatorname{Part}^{*}(6)$ and matrices $M_{1}, M_{2}, M_{3} \in \operatorname{Mat}_{n}(\mathcal{S})$ be given. Put

$$
\mathfrak{M}_{\Pi}=\left\{\begin{array}{l}
{\left[\left[\sum_{\substack{\mathbf{i}=i_{1} \ldots i_{6} \\
\in \operatorname{Seq}(6, n) \\
\text { s.t. } \Pi(\mathbf{i})=\Pi}} M_{1}\left(i_{1}, i_{2}\right) \otimes M_{2}\left(i_{3}, i_{4}\right) \otimes M_{3}\left(i_{5}, i_{6}\right)\right]\right.} \\
\sum_{\substack{\mathbf{i}=i_{1} \cdots i_{6} \\
\in \operatorname{Seq}(6, n) \\
\text { s.t. } \Pi(\mathbf{i})=\Pi}} \llbracket M_{1}\left(i_{1}, i_{2}\right) \rrbracket \llbracket M_{2}\left(i_{3}, i_{4}\right) \rrbracket \llbracket M_{3}\left(i_{5}, i_{6}\right) \rrbracket \quad \text { if } \Pi \in \operatorname{Part}_{2}^{*}(6), \\
\end{array} \notin \operatorname{Part}_{2}^{*}(6)\right.
$$

Proposition 9.3.5. Notation and assumptions are as above. For $\Pi \in \operatorname{Part}^{*}(6)$, unless $\Pi \sim\{\{1,2,3\},\{4,5,6\}\}$, we have $\mathfrak{M}_{\Pi} \leq c n \llbracket M_{1} \rrbracket \llbracket M_{2} \rrbracket \llbracket M_{3} \rrbracket$ for a constant $c$ depending only on $\mathcal{S}$.

Proof. We may assume that $\mathcal{S}$ is isomorphic to $\operatorname{Mat}_{s}(\mathbb{C})$ and thus by Lemma 5.1.7 that $\mathcal{S}=\mathbb{C}$. After replacing $\left(M_{1}, M_{2}, M_{3}\right)$ by $\left(M_{\sigma(1)}^{\mathrm{T}^{\nu_{1}}}, M_{\sigma(2)}^{\mathrm{T}^{\nu_{2}}}, M_{\sigma(3)}^{\mathrm{T}^{\nu_{3}}}\right)$ for suitably chosen $\sigma \in S_{3}$ and $\nu_{1}, \nu_{2}, \nu_{3} \in\{0,1\}$, we may assume that $\Pi$ appears on the list (96). We may also assume that each matrix $M_{\alpha}$ is either diagonal or else vanishes
identically on the diagonal. Finally, we may assume that $\mathfrak{M}_{\Pi}>0$. Let $d$ be the number of matrices $M_{\alpha}$ which are diagonal. Consider the following mutually exclusive and exhaustive collection of cases:
(i) $\Pi=\{\{1,6\},\{2,3\},\{4,5\}\}$ and hence $d=0$.
(ii) $\Pi=\{\{1,3,5\},\{2,4,6\}\}$ and hence $d=0$.
(iii) $\Pi=\{\{1,6\},\{2,3,4,5\}\}$ and hence $d=1$.
(iv) $\Pi=\{\{1,2,3,4,5,6\}\}$ and hence $d=3$.

In case (i) we have $\mathfrak{M}_{\Pi}=\left|\operatorname{tr} M_{1} M_{2} M_{3}\right| \leq n \llbracket M_{1} M_{2} M_{3} \rrbracket \leq n \llbracket M_{1} \rrbracket \llbracket M_{2} \rrbracket \llbracket M_{3} \rrbracket$. In case (ii) we have

$$
\begin{aligned}
\mathfrak{M}_{\Pi} & \leq \llbracket M_{1} \rrbracket \sum_{i, j=1}^{n}\left|M_{2}(i, j) M_{3}(i, j)\right| \\
& \leq \llbracket M_{1} \rrbracket \prod_{\alpha \in\{2,3\}}\left(\sum_{i, j=1}^{n}\left|M_{\alpha}(i, j)\right|^{2}\right)^{1 / 2} \leq n \llbracket M_{1} \rrbracket \llbracket M_{2} \rrbracket \llbracket M_{3} \rrbracket .
\end{aligned}
$$

In case (iii), similarly, we have

$$
\mathfrak{M}_{\Pi} \leq \llbracket M_{2} \rrbracket \sum_{i, j=1}^{n}\left|M_{1}(i, j) M_{3}(j, i)\right| \leq n \llbracket M_{1} \rrbracket \llbracket M_{2} \rrbracket \llbracket M_{3} \rrbracket .
$$

Finally, in case (iv) we have $\mathfrak{M}_{\Pi} \leq n \llbracket M_{1} \rrbracket \llbracket M_{2} \rrbracket \llbracket M_{3} \rrbracket$ simply by counting.

## 10. Matrix identities

Throughout this section we fix a block algebra $\mathcal{S}$. Working in a purely algebraic setting, we build up a catalog of identities satisfied by finite chunks of an infinite matrix with entries in $\mathcal{S}$. These identities are a further contribution to our stock of tools for concentration. By and large the identities have a familiar form but noncommutativity of $\mathcal{S}$ leads to some unfamiliar twists.

### 10.1. The setup for studying matrix identities.

10.1.1. An ad hoc infinite matrix formalism. When we write $\operatorname{Mat}_{k \times \ell}(\mathcal{S})$, we now allow $k$ or $\ell$ or both to be infinite, in which case we mean for the corresponding matrix indices to range over all positive integers. Addition, multiplication and adjoints of (possibly) infinite matrices are defined as before, although we never attempt to multiply such matrices unless one of them has only finitely many nonzero entries. For each integer $N>0$, let $\mathcal{I}_{N}$ denote the family of nonempty subsets of the set $\{1, \ldots, N\}$. Given a finite nonempty set $I=\left\{i_{1}<\cdots<i_{k}\right\}$ of positive integers, let $\mathbf{f}_{I} \in \operatorname{Mat}_{k \times \infty}(\mathcal{S})$ and $\mathbf{e}_{I} \in \operatorname{Mat}_{\infty}(\mathcal{S})$ be defined by

$$
\mathbf{f}_{I}(i, j)=\sum_{\alpha=1}^{k} \mathbf{1}_{(i, j)=\left(\alpha, i_{\alpha}\right)} 1_{\mathcal{S}} \text { and } \mathbf{e}_{I}(i, j)=\sum_{\alpha=1}^{k} \mathbf{1}_{(i, j)=\left(i_{\alpha}, i_{\alpha}\right)} 1_{\mathcal{S}},
$$

respectively. Note that $\mathbf{f}_{I} \mathbf{f}_{I}^{*}=\mathbf{I}_{|I|} \otimes 1_{\mathcal{S}}$ and $\mathbf{f}_{I}^{*} \mathbf{f}_{I}=\mathbf{e}_{I}$, where $|I|$ denotes the cardinality of $I$. Note that for all $A \in \operatorname{Mat}_{\infty}(\mathcal{S})$ and finite sets $I$ and $J$ of positive integers, the finite matrix $\mathbf{f}_{I} A \mathbf{f}_{J}^{*} \in \operatorname{Mat}_{|I| \times|J|}(\mathcal{S})$ is the result of striking all rows of $A$ with indices not in $I$ and all columns of $A$ with indices not in $J$. For $A \in \operatorname{Mat}_{\infty}(\mathcal{S})$ with only finitely many nonzero entries, we define $\operatorname{tr}_{\mathcal{S}} A=\sum_{i} A(i, i)$. For such $A$ we also define $\llbracket A \rrbracket=\llbracket \mathbf{f}_{I} A \mathbf{f}_{I}^{*} \rrbracket$ for any finite set $I$ of positive integers such that $\mathbf{e}_{I} A \mathbf{e}_{I}=A$. For each $\zeta \in \mathcal{S}$, let $\mathbf{I}_{\infty} \otimes \zeta \in \operatorname{Mat}_{\infty}(\mathcal{S})$ denote the infinite diagonal matrix with diagonal entries $\zeta$.
10.1.2. Data and assumption. We fix a triple $(X, \Lambda, \Phi)$ where

- $X \in \operatorname{Mat}_{\infty}(\mathcal{S})$,
- $\Lambda \in \mathcal{S}$ and
- $\Phi \in B(\mathcal{S})$,
subject to the condition

$$
\begin{equation*}
\mathbf{f}_{I}\left(\frac{X}{\sqrt{N}}-\mathbf{I}_{\infty} \otimes \Lambda\right) \mathbf{f}_{I}^{*} \in \mathrm{GL}_{|I|}(\mathcal{S}) \text { for } N \text { and } I \in \mathcal{I}_{N} \tag{97}
\end{equation*}
$$

Here and below $N$ is understood to range over the positive integers. Below we will define and analyze various functions of the triple $(X, \Lambda, \Phi)$, calling them recipes.
Remark 10.1.3. Fix a SALT block design $(\mathcal{S}, L, \Theta, e)$ arbitrarily. Let $\bigcup L\left(\Xi^{N}\right) \in \operatorname{Mat}_{\infty}(\mathcal{S})_{\text {sa }}$ denote the matrix gotten by cobbling together the matrices $L\left(\Xi^{N}\right) \in \operatorname{Mat}_{N}(\mathcal{S})$ for varying $N$ using assumption (4). Let $\Phi_{L}$ be as in Definition 5.4.2. Let $\mathbf{z}$ be as in $\S 2.6 .3$ and $\mathbf{t}$ as in $\S 7.2 .1$. Then the triple

$$
\left(\bigcup L\left(\Xi^{N}\right), \Theta+\mathbf{z e} e \mathrm{it} 1_{\mathcal{S}}, \Phi_{L}\right)
$$

satisfies (97). Triples of this type are the ones we need to prove Theorem 2.6.4. We do not immediately move to this specialization because it is already a big job to deduce consequences of (97). There is no point in carrying along the extra dead weight of structure.
10.1.4. The first group of recipes. For $N$ and $I \in \mathcal{I}_{N}$ we define

$$
\begin{aligned}
R_{I}^{N} & =\mathbf{f}_{I}^{*}\left(\mathbf{f}_{I}\left(\frac{X}{\sqrt{N}}-\mathbf{I}_{\infty} \otimes \Lambda\right) \mathbf{f}_{I}^{*}\right)^{-1} \mathbf{f}_{I}, \quad F_{I}^{N}=\frac{1}{N} \operatorname{tr}_{\mathcal{S}} R_{I}^{N} \in \mathcal{S} \\
T_{I}^{N} & =\left(\zeta \mapsto \frac{1}{N} \sum_{i, j \in I} R_{I}^{N}(i, j) \zeta R_{I}^{N}(j, i)\right) \in B(\mathcal{S}) \\
U_{I}^{N} & =\frac{1}{N} \sum_{i, j \in I} R_{I}^{N}(i, j)^{\otimes 2} \in \mathcal{S}^{\otimes 2}
\end{aligned}
$$

Note that $R_{I}^{N}$ is well-defined by assumption (97). For $N$ put

$$
\mathcal{I}_{N}^{(2)}=\left\{(I, J) \in \mathcal{I}_{N} \times \mathcal{I}_{N}\left|J \subset I, I \backslash J \in \mathcal{I}_{N},|J| \leq 2\right\}\right.
$$

For $N$ and $(I, J) \in \mathcal{I}_{N}^{(2)}$ put

$$
R_{I, J}^{N}=\mathbf{f}_{J} R_{I}^{N} \mathbf{f}_{J}^{*} \in \operatorname{Mat}_{|J|}(\mathcal{S})
$$

The recipes in the first group do not depend on $\Phi$, whereas the remaining recipes we are about to define do depend on $\Phi$.
Remark 10.1.5. Note that $R_{I}^{N}$ is the inverse of the matrix

$$
\mathbf{e}_{I}\left(\frac{X}{\sqrt{N}}-\mathbf{I}_{\infty} \otimes \Lambda\right) \mathbf{e}_{I}
$$

as computed in the algebra $\mathbf{e}_{I} \operatorname{Mat}_{\infty}(\mathcal{S}) \mathbf{e}_{I}$ the identity element of which is $\mathbf{e}_{I}$. This observation simplifies calculations below on several occasions.

Remark 10.1.6. The recipe $U_{I}^{N}$ does not figure in any identities stated in $\S 10$ but does become an important random variable later. We therefore include it here so that $\S 10.1$ can serve as a handy catalog of the basic random variables.
10.1.7. Recipes of the second group. For $N$ and $I \in \mathcal{I}_{N}$ put

$$
\begin{aligned}
E_{I}^{N} & =1_{\mathcal{S}}+\left(\Lambda+\Phi\left(F_{I}^{N}\right)\right) F_{I}^{N} \in \mathcal{S} \\
H_{I}^{N} & =\left\{\begin{aligned}
-\left(\Lambda+\Phi\left(F_{I}^{N}\right)\right)^{-1} \in \mathcal{S}^{\times} & \text {if }\left[\left[E_{I}^{N}\right]\right]<1 / 2 \\
0 \in \mathcal{S} & \text { if }\left[\left[E_{I}^{N}\right]\right] \geq 1 / 2
\end{aligned}\right.
\end{aligned}
$$

Note that $H_{I}^{N}$ is well-defined by Lemma 4.1.1. For $N,(I, J) \in \mathcal{I}_{N}^{(2)}$ and $j_{1}, j_{2} \in J$, we define

$$
\begin{aligned}
H_{I, J}^{N}= & \mathbf{I}_{|J|} \otimes H_{I \backslash J}^{N} \in \operatorname{Mat}_{|J|}(\mathcal{S}), \\
\frac{Q_{I, J}^{N}}{\sqrt{N}}= & -\frac{\mathbf{f}_{J} X \mathbf{f}_{J}^{*}}{\sqrt{N}}+\frac{\mathbf{f}_{J} X R_{I \backslash J}^{N} X \mathbf{f}_{J}^{*}}{N}-\mathbf{I}_{|J|} \otimes \Phi\left(F_{I \backslash J}^{N}\right) \in \operatorname{Mat}_{|J|}(\mathcal{S}), \\
Q_{I, J, j_{1}, j_{2}}^{N}= & \mathbf{f}_{j_{1}} \mathbf{f}_{J}^{*} Q_{I, J}^{N} \mathbf{f}_{J} \mathbf{f}_{j_{2}}^{*} \in \mathcal{S}, \\
\frac{P_{I, J}^{N}=}{\sqrt{N}=} & \left(A \mapsto \frac{1}{N} \operatorname{tr}_{\mathcal{S}}\left(R_{I \backslash J}^{N} X \mathbf{f}_{J}^{*} A \mathbf{f}_{J} X R_{I \backslash J}^{N}\right)-T_{I \backslash J}^{N} \circ \Phi \circ \operatorname{tr}_{\mathcal{S}}(A)\right) \\
& \in B\left(\operatorname{Mat}_{|J|}(\mathcal{S}), \mathcal{S}\right), \\
P_{I, J, j_{1}, j_{2}}^{N}= & \left(\zeta \mapsto P_{I, J}^{N}\left(\mathbf{f}_{J} \mathbf{f}_{j_{1}}^{*} \zeta \mathbf{f}_{j_{2}} \mathbf{f}_{J}^{*}\right)\right) \in B(\mathcal{S}), \\
\Delta_{I, J}^{N}= & H_{I, J}^{N} Q_{I, J}^{N}+\sqrt{N} \mathbf{I}_{|J|} \mathbf{1}_{\left[\left[E_{I \backslash J}^{N}\right]\right] \geq 1 / 2} \in \operatorname{Mat}_{|J|}(\mathcal{S}) .
\end{aligned}
$$

10.1.8. Abuses of notation. We write

$$
\Delta^{k} R_{I, J}^{N}=\left(\Delta_{I, J}^{N}\right)^{k} R_{I, J}^{N} \text { and } \Delta R_{I, J}^{N}=\Delta^{1} R_{I, J}^{N} .
$$

We often write $j$ where we should more correctly write $\{j\}$, e.g., we write $Q_{I, j}^{N}$ instead of $Q_{I,\{j\}}^{N}$. Note that

$$
R_{I, j}^{N}=R_{I}^{N}(j, j), \quad H_{I, j}^{N}=H_{I \backslash j}^{N}, \quad Q_{I, j, j, j}^{N}=Q_{I, j}^{N}, \quad P_{I, j, j, j}^{N}=P_{I, j}^{N} .
$$

In the same spirit, we occasionally write $N$ in place of $\{1, \ldots, N\}$.
10.2. Basic identities. We obtain block-type generalizations of matrix identities familiar from the study of resolvents of standard Wigner matrices.
Lemma 10.2.1. For $N$ and $I \in \mathcal{I}_{N}$, along with any positive integer $k$,

$$
\begin{equation*}
R_{I}^{N+1}=R_{I}^{N}+\sum_{\nu=1}^{k-1}\left(\delta_{N} R_{I}^{N} \frac{\mathbf{e}_{I} X \mathbf{e}_{I}^{*}}{\sqrt{N}}\right)^{\nu} R_{I}^{N}+\left(\delta_{N} R_{I}^{N} \frac{\mathbf{e}_{I} X \mathbf{e}_{I}^{*}}{\sqrt{N}}\right)^{k} R_{I}^{N+1} \tag{98}
\end{equation*}
$$

where $\delta_{N}=\sqrt{N}\left(\frac{1}{\sqrt{N}}-\frac{1}{\sqrt{N+1}}\right)$.
Proof. By induction we may assume $k=1$. Then, in view of Remark 10.1.5, formula (98) is merely an instance of the resolvent identity (32).

Lemma 10.2.2. For $N$ and $(I, J) \in \mathcal{I}_{N}^{(2)}$,

$$
\begin{align*}
R_{I, J}^{N} & =\left(\frac{\mathbf{f}_{J} X \mathbf{f}_{J}^{*}}{\sqrt{N}}-\mathbf{I}_{|J|} \otimes \Lambda-\frac{\mathbf{f}_{J} X R_{I \backslash J}^{N} X \mathbf{f}_{J}^{*}}{N}\right)^{-1},  \tag{99}\\
R_{I}^{N}-R_{I \backslash J}^{N} & =\left(\mathbf{f}_{J}^{*}-R_{I \backslash J}^{N} \frac{X}{\sqrt{N}} \mathbf{f}_{J}^{*}\right) R_{I, J}^{N}\left(\mathbf{f}_{J}-\mathbf{f}_{J} \frac{X}{\sqrt{N}} R_{I \backslash J}^{N}\right) . \tag{100}
\end{align*}
$$

In particular, we automatically have $R_{I, J}^{N} \in \mathrm{GL}_{|J|}(\mathcal{S})$.

Proof. In Proposition 4.5.2, let us now take

$$
\mathcal{A}=\mathbf{e}_{I} \operatorname{Mat}_{\infty}(\mathcal{S}) \mathbf{e}_{I}, \quad x=\mathbf{e}_{I}\left(\frac{X}{\sqrt{N}}-\mathbf{I}_{\infty} \otimes \Lambda\right) \mathbf{e}_{I}, \quad \pi=\mathbf{e}_{J}, \quad \pi^{\perp}=\mathbf{e}_{I \backslash J}, \quad \sigma=\mathbf{e}_{I} .
$$

Rewritten in the form

$$
\mathbf{e}_{J} R_{I}^{N} \mathbf{e}_{J}=\mathbf{f}_{J}^{*} R_{I, J}^{N} \mathbf{f}_{J}=\mathbf{f}_{J}^{*}\left(\mathbf{f}_{J}\left(x-x R_{I \backslash J}^{N} x\right) \mathbf{f}_{J}^{*}\right)^{-1} \mathbf{f}_{J},
$$

identity (99) becomes a special case of (40). Similarly, rewritten in the form

$$
R_{I}^{N}-R_{I \backslash J}^{N}=\left(\mathbf{e}_{J}-R_{I \backslash J}^{N} x \mathbf{e}_{J}\right) R_{I}^{N}\left(\mathbf{e}_{J}-\mathbf{e}_{J} x R_{I \backslash J}^{N}\right),
$$

identity (100) becomes a specialization of (41).
Lemma 10.2.3. For $N$ and $(I, J) \in \mathcal{I}_{N}^{(2)}$, along with any positive integer $k$,

$$
\begin{equation*}
R_{I, J}^{N}=H_{I, J}^{N}+\sum_{\nu=1}^{k-1} \frac{\left(H_{I, J}^{N} Q_{I, J}^{N}\right)^{\nu} H_{I, J}^{N}}{N^{\nu / 2}}+\frac{\Delta^{k} R_{I, J}^{N}}{N^{k / 2}} \tag{101}
\end{equation*}
$$

Proof. By induction on $k$ we may assume $k=1$. Rewrite (99) in the form

$$
\begin{equation*}
-\left(\mathbf{I}_{|J|} \otimes\left(\Lambda+\Phi\left(F_{I \backslash J}^{N}\right)\right)\right) R_{I, J}^{N}=\mathbf{I}_{|J|} \otimes 1_{\mathcal{S}}+\frac{Q_{I, J}^{N} R_{I, J}^{N}}{\sqrt{N}} \tag{102}
\end{equation*}
$$

Then left-multiply by $H_{I, J}^{N}$ on both sides and rearrange slightly to get the result.
10.3. More elaborate identities. We specialize and combine the basic identities.
10.3.1. For $N,(I, J) \in \mathcal{I}_{N}^{(2)}$ and $j_{1}, j_{2} \in J$, we have

$$
\begin{align*}
R_{I}^{N}\left(j_{1}, j_{2}\right)-\delta_{j_{1} j_{2}} H_{I \backslash J} & =\mathbf{f}_{j_{1}} \mathbf{f}_{J}^{*} \frac{\Delta R_{I, J}^{N}}{\sqrt{N}} \mathbf{f}_{J} \mathbf{f}_{j_{2}}^{*},  \tag{103}\\
R_{I}^{N}\left(j_{1}, j_{2}\right)-\delta_{j_{1} j_{2}} H_{I \backslash J}-\frac{H_{I \backslash J} Q_{I, J, j_{1}, j_{2}}^{N} H_{I \backslash J}}{\sqrt{N}} & =\mathbf{f}_{j_{1}} \mathbf{f}_{J}^{*} \frac{\Delta^{2} R_{I, J}^{N}}{N} \mathbf{f}_{J} \mathbf{f}_{j_{2}}^{*} \tag{104}
\end{align*}
$$

by merely rewriting (101) in the cases $k=1$ and $k=2$, respectively, at the level of individual matrix entries.
10.3.2. For $N$ and $(I, J) \in \mathcal{I}_{N}^{(2)}$ we have

$$
\begin{align*}
N\left(F_{I}^{N}-F_{I \backslash J}^{N}\right) & =\operatorname{tr}_{\mathcal{S}}\left(R_{I, J}^{N}\right)+\operatorname{tr}_{\mathcal{S}}\left(R_{I \backslash J}^{N} \frac{X}{\sqrt{N}} \mathbf{f}_{J}^{*} R_{I, J}^{N} \mathbf{f}_{J} \frac{X}{\sqrt{N}} R_{I \backslash J}^{N}\right)  \tag{105}\\
& =\left(\operatorname{tr}_{\mathcal{S}}+T_{I \backslash J}^{N} \circ \Phi \circ \operatorname{tr}_{\mathcal{S}}+\frac{P_{I, J}^{N}}{\sqrt{N}}\right)\left(R_{I, J}^{N}\right)
\end{align*}
$$

by applying $\operatorname{tr}_{\mathcal{S}}$ to both sides of (100). We note also the identity (106) $\left.H_{I}^{N}-H_{I \backslash J}^{N}=H_{I}^{N} \mathbf{1}_{\left[\left[E_{I \backslash J}^{N}\right]\right.}\right] \geq 1 / 2-H_{I \backslash J}^{N} \mathbf{1}_{\llbracket E_{I}^{N} \rrbracket \geq 1 / 2}+H_{I}^{N} \Phi\left(F_{I}^{N}-F_{I \backslash J}^{N}\right) H_{I \backslash J}^{N}$
obtained by exploiting the resolvent identity (32) in evident fashion.
10.3.3. For $N$ and $I \in \mathcal{I}_{N}$ such that $|I| \geq 2$ we have

$$
\begin{equation*}
E_{I}^{N}+\frac{|I|-N}{N} 1_{\mathcal{S}}=\frac{1}{N} \sum_{j \in I}\left(\Phi\left(F_{I}^{N}-F_{I \backslash j}^{N}\right) R_{I, j}^{N}-\frac{Q_{I, j}^{N} R_{I, j}^{N}}{\sqrt{N}}\right) \tag{107}
\end{equation*}
$$

after applying $\frac{1}{N} \sum_{j \in I}(\cdot)$ to both sides of (102) in the singleton case $J=\{j\}$ and rearranging.

Remark 10.3.4. Identity (107) is an approximate version of the Schwinger-Dyson equation. Identities of this sort have long been in use for study of Wigner matrices.
10.3.5. For $N$ and $I \in \mathcal{I}_{N}$ such that $|I| \geq 2$ we also have

$$
\begin{align*}
& E_{I}^{N}+\frac{|I|-N}{N} 1_{\mathcal{S}}+\frac{1}{N} \sum_{j \in I} \frac{Q_{I, j}^{N} H_{I \backslash j}^{N}}{\sqrt{N}}  \tag{108}\\
= & \frac{1}{N} \sum_{j \in I}\left(\Phi\left(F_{I}^{N}-F_{I \backslash j}^{N}\right) R_{I, j}^{N}-\frac{Q_{I, j}^{N} \Delta R_{I, j}^{N}}{N}\right),
\end{align*}
$$

by (101) for $k=1$ in the singleton case $J=\{j\}$ and (107), after rearrangement.
10.3.6. For $N$ and $I \in \mathcal{I}_{N}$ we have

$$
\begin{equation*}
H_{I}^{N}-F_{I}^{N}=H_{I}^{N} E_{I}^{N}-F_{I}^{N} \mathbf{1}_{\llbracket E_{I}^{N} \rrbracket \geq 1 / 2} . \tag{109}
\end{equation*}
$$

by direct appeal to the definitions. One then obtains for $|I| \geq 2$ the identity

$$
\text { 110) } \begin{align*}
& H_{I}^{N}-F_{I}^{N}+\frac{1}{N} \sum_{j \in I} \frac{F_{I \backslash j}^{N} Q_{I, j}^{N} H_{I \backslash j}^{N}}{\sqrt{N}}  \tag{110}\\
& =\frac{N-|I|}{N} F_{I}^{N}+H_{I}^{N}\left(E_{I}^{N}\right)^{2}-\left(F_{I}^{N}+F_{I}^{N} E_{I}^{N}\right) \mathbf{1}_{\llbracket E_{I}^{N} \rrbracket \geq 1 / 2} \\
& +\frac{1}{N} \sum_{j \in I}\left(F_{I}^{N} \Phi\left(F_{I}^{N}-F_{I \backslash j}^{N}\right) R_{I, j}^{N}-\frac{F_{I}^{N} Q_{I, j}^{N} \Delta R_{I, j}^{N}}{N}-\frac{\left(F_{I}^{N}-F_{I \backslash j}^{N}\right) Q_{I, j}^{N} H_{I \backslash j}^{N}}{\sqrt{N}}\right)
\end{align*}
$$

by iterating (109) and combining it with (108).
10.3.7. For $N$ and $(I, J) \in \mathcal{I}_{N}^{(2)}$ we have

$$
\begin{align*}
& F_{I}^{N}-F_{I \backslash J}^{N}-|J| \frac{\left(1_{B(\mathcal{S})}+T_{I \backslash J}^{N} \circ \Phi\right)\left(H_{I \backslash J}^{N}\right)}{N}  \tag{111}\\
= & \frac{P_{I, J}^{N}\left(R_{I, J}^{N}\right)+\left(\operatorname{tr}_{\mathcal{S}}+T_{I \backslash J}^{N} \circ \Phi \circ \operatorname{tr}_{\mathcal{S}}\right)\left(\Delta R_{I, J}^{N}\right)}{N^{3 / 2}}
\end{align*}
$$

by rearrangement of (105), using (101) for $k=1$.
10.3.8. For $N$ and $I \in \mathcal{I}_{N}$ we have

$$
\begin{gather*}
(N+1) F_{I}^{N+1}-N F_{I}^{N}-\frac{1}{2}\left(F_{I}^{N}+T_{I}^{N}(\Lambda)\right)  \tag{112}\\
=\frac{1}{N} \operatorname{tr}_{\mathcal{S}}\left(\left(N \delta_{N}-\frac{1}{2}\right)\left(\mathbf{e}_{I}+R_{I}^{N}\left(\mathbf{I}_{\infty} \otimes \Lambda\right)\right) R_{I}^{N}\right. \\
\left.\quad+\frac{\left(N \delta_{N}\right)^{2}}{N}\left(\mathbf{e}_{I}+R_{I}^{N}\left(\mathbf{I}_{\infty} \otimes \Lambda\right)\right)^{2} R_{I}^{N+1}\right)
\end{gather*}
$$

by Lemma 10.2.1 in the case $k=2$ after using Remark 10.1.5, applying $\operatorname{tr}_{\mathcal{S}}$ on both sides and rearranging. Note that $\frac{1}{2}-\frac{1}{2 N} \leq N \delta_{N} \leq \frac{1}{2}$.
10.3.9. Let

$$
\operatorname{Link}^{N}=\frac{1}{2}\left(F_{N}^{N}+T_{N}^{N}(\Lambda)\right)-F_{N+1}^{N+1}+H_{N}^{N+1}+T_{N}^{N+1}\left(\Phi\left(H_{N}^{N+1}\right)\right)
$$

where here and below in similar contexts we abuse notation by writing $N$ where we should more correctly write $\{1, \ldots, N\}$. We then have

$$
\begin{align*}
& N\left(F_{N+1}^{N+1}-F_{N}^{N}\right)-\operatorname{Link}^{N}  \tag{113}\\
= & (N+1) F_{N}^{N+1}-N F_{N}^{N}-\frac{1}{2}\left(F_{N}^{N}+T_{N}^{N}(\Lambda)\right) \\
& +(N+1)\left(F_{N+1}^{N+1}-F_{N}^{N+1}\right)-H_{N}^{N+1}-T_{N}^{N+1}\left(\Phi\left(H_{N}^{N+1}\right)\right)
\end{align*}
$$

by mere rearrangement of terms.
10.4. The bias identity. We derive the most intricate identity used in the paper.
10.4.1. We first need an intermediate result which continues the process of expansion begun in identity (108). For $N$ and $I \in \mathcal{I}_{N}$ such that $|I| \geq 2$, we have

$$
\begin{align*}
& E_{I}^{N}+\frac{|I|-N}{N} 1_{\mathcal{S}}  \tag{114}\\
& +\frac{1}{N} \sum_{j \in I}\left(\frac{\left(Q_{I, j}^{N} H_{I \backslash j}^{N}\right)^{2}-\left(\Phi+\Phi \circ T_{I \backslash j}^{N} \circ \Phi\right)\left(R_{I, j}^{N}\right) R_{I, j}^{N}}{N}+\frac{\left(Q_{I, j}^{N} H_{I \backslash j}^{N}\right)^{3}}{N^{3 / 2}}\right) \\
= & \frac{1}{N} \sum_{j \in I}\left(-\frac{Q_{I, j}^{N} H_{I \backslash j}^{N}}{\sqrt{N}}-\frac{Q_{I, j}^{N} \Delta^{3} R_{I, j}^{N}}{N^{2}}+\frac{\Phi \circ P_{I, j}^{N}\left(R_{I, j}^{N}\right) R_{I, j}^{N}}{N^{3 / 2}}\right)
\end{align*}
$$

by expanding the terms $\frac{Q_{I, j}^{N} R_{I, j}^{N}}{\sqrt{N}}$ in (107) by using (101) for $k=3$ in the singleton case $J=\{j\}$, and furthermore expanding the terms $\Phi\left(F_{I}^{N}-F_{I \backslash j}^{N}\right) R_{I, j}^{N}$ in (107) by using (105) in the singleton case $J=\{j\}$, after suitable rearrangement.
10.4.2. Fix $N \geq 2$ and $j \in N$ arbitrarily. To compactify notation put

$$
\begin{aligned}
\widetilde{T}_{j}^{N}= & \Phi+\Phi \circ T_{N \backslash j}^{N} \circ \Phi, \widetilde{P}_{j}^{N}=\Phi \circ P_{N \backslash j}^{N}, \check{R}_{j}^{N}=H_{N \backslash j}^{N} Q_{N, j}^{N} H_{N \backslash j}^{N}, \\
\operatorname{Err}_{j}^{N}= & \left(Q_{N, j}^{N} H_{N \backslash j}^{N}\right)^{2}-\widetilde{T}_{j}^{N}\left(H_{N \backslash j}^{N}\right) H_{N \backslash j}^{N}+\left(Q_{N, j}^{N} H_{N, j}^{N}\right)^{3} / \sqrt{N}, \\
\operatorname{Err}_{j}^{N, 1}= & \left(\widetilde{T}_{j}^{N}\left(H_{N \backslash j}^{N}\right) \check{R}_{j}^{N}+\widetilde{T}_{j}^{N}\left(\check{R}_{j}^{N}\right) H_{N \backslash j}^{N}+\widetilde{P}_{j}^{N}\left(H_{N \backslash j}^{N}\right) H_{N \backslash j}^{N}\right) / N-Q_{N, j}^{N} H_{N \backslash j}^{N}, \\
\operatorname{Err}_{j}^{N, 2}= & \widetilde{T}_{j}^{N}\left(H_{N \backslash j}^{N}\right) \Delta^{2} R_{N, j}^{N}+\widetilde{T}_{j}^{N}\left(\check{R}_{j}^{N}\right) \Delta R_{N, j}^{N}+\widetilde{T}_{j}^{N}\left(\Delta^{2} R_{N, j}^{N}\right) R_{N, j}^{N} \\
& +\widetilde{P}_{j}^{N}\left(H_{N \backslash j}^{N}\right) \Delta R_{N, j}^{N}+\widetilde{P}_{j}^{N}\left(\Delta R_{N, j}^{N}\right) R_{N, j}^{N}-Q_{N, j}^{N} \Delta^{3} R_{N, j}^{N} .
\end{aligned}
$$

At last, we obtain the bias identity

$$
\begin{equation*}
E_{N}^{N}+\frac{1}{N} \sum_{j \in N} \frac{\operatorname{Err}_{j}^{N}}{N}=\frac{1}{N} \sum_{j \in N}\left(\frac{\operatorname{Err}_{j}^{N, 1}}{\sqrt{N}}+\frac{\operatorname{Err}_{j}^{N, 2}}{N^{2}}\right) \tag{115}
\end{equation*}
$$

by using (101) several times with $k=1,2$ in the singleton case $J=\{j\}$ to expand the terms $\left(\Phi+\Phi \circ T_{N \backslash j}^{N} \circ \Phi\right)\left(R_{N, j}^{N}\right) R_{N, j}^{N}$ and $\Phi \circ P_{N, j}^{N}\left(R_{N, j}^{N}\right) R_{N, j}^{N}$ in (114), after suitable rearrangement.

## 11. $L^{p}$ estimates for THE BLOCK WIGNER MODEL

We introduce a straightforward generalization of the usual Wigner matrix model with matrix entries in a block algebra. Making use of all the tools collected in $\S 9$ and $\S 10$, we investigate how control of moments of "randomized resolvents" propagates to give control of moments of many related random variables. This is a continuation of our development of concentration tools.
11.1. The block Wigner model. The ad hoc infinite matrix formalism of $\S 10.1 .1$ will be the algebraic framework for our discussion of the block Wigner model.
11.1.1. Data. Data for the block Wigner model consist of

- a block algebra $\mathcal{S}$,
- a random matrix $X \in \operatorname{Mat}_{\infty}(\mathcal{S})_{\text {sa }}$,
- a (deterministic) linear map $\Phi \in B(\mathcal{S})$,
- a (deterministic) tensor $\Psi \in \mathcal{S}^{\otimes 2}$,
- a random element $\Lambda \in \mathcal{S}$ and
- a random variable $\mathfrak{G} \in[1, \infty)$.
11.1.2. $\sigma$-fields and auxiliary random variables. For convenience in the "endgame" we keep for use in the present setup the same system $\{\mathcal{F}(i, j)\}_{1 \leq i \leq j<\infty}$ of independent $\sigma$-fields mentioned in $\S 2.2 .1$. As before, let $\mathcal{F}$ denote the $\sigma$-field generated by all the $\mathcal{F}(i, j)$. More generally, for any set $I$ of positive integers let $\mathcal{F}_{I}$ denote the $\sigma$-field generated by $\{\mathcal{F}(i, j) \mid i, j \in I\}$. We also keep the random variables $\mathbf{z}$ and $\mathbf{t}$ on hand. Actually we do not so much care about the variables themselves, but we do need the $\sigma$-fields these random variables generate for bookkeeping purposes.
11.1.3. Assumptions. Of the sextuple $(\mathcal{S}, X, \Phi, \Psi, \Lambda, \mathfrak{G})$ we assume the following:

$$
\begin{align*}
& \sup _{i, j=1}^{\infty}\|\llbracket X(i, j) \rrbracket\|_{p}<\infty \text { for } 1 \leq p<\infty .  \tag{116}\\
& X(i, j) \text { is } \mathcal{F}(i \wedge j, i \vee j) \text {-measurable and of mean zero for all } i \text { and } j .  \tag{117}\\
& \Phi=(\zeta \mapsto \mathbb{E} X(i, j) \zeta X(j, i)) \text { and } \Psi=\mathbb{E}\left(X(i, j)^{\otimes 2}\right) \text { for distinct } i \text { and } j . \tag{118}
\end{align*}
$$

$$
\begin{equation*}
\llbracket \Lambda \rrbracket_{p}<\infty \text { for } p \in[1, \infty) . \tag{119}
\end{equation*}
$$

$\Lambda$ is $\sigma(\mathbf{z}, \mathbf{t})$-measurable and $\mathfrak{G}$ is $\sigma(\mathbf{z})$-measurable.

$$
\begin{equation*}
\mathbf{f}_{I}\left(\frac{X}{\sqrt{N}}-\mathbf{I}_{\infty} \otimes \Lambda\right) \mathbf{f}_{I}^{*} \in \mathrm{GL}_{|I|}(\mathcal{S}) \text { for } N \text { and } I \in \mathcal{I}_{N} \tag{120}
\end{equation*}
$$

For simplicity we assume that (121) holds for every sample point without exception. For $N$ and $I \in \mathcal{I}_{N}$ we then put

$$
R_{I}^{N}=\mathbf{f}_{I}^{*}\left(\mathbf{f}_{I}\left(\frac{X}{\sqrt{N}}-\mathbf{I}_{\infty} \otimes \Lambda\right) \mathbf{f}_{I}^{*}\right)^{-1} \mathbf{f}_{I} \in \operatorname{Mat}_{\infty}(\mathcal{S})
$$

which is a generalized resolvent (Green's function). Finally, we assume that

$$
\begin{equation*}
\left.\sup _{N} \bigvee_{I \in \mathcal{T}_{N}} \|\left[R_{I}^{N} / \mathscr{G}\right]\right] \|_{p}<\infty . \tag{122}
\end{equation*}
$$

We work with a fixed instance $(\mathcal{S}, X, \Phi, \Psi, \Lambda, \mathfrak{G})$ of the block Wigner model over $\mathcal{S}$ for the rest of $\S 11$.
11.1.4. Random variables defined by recipes. Since assumption (121) is a verbatim repetition of assumption (97), all the recipes of $\S 10.1$ define random variables in the present setting. The object $R_{I}^{N}$ figuring in assumption (122) is of course a recipe. We now furthermore have random variables $F_{I}^{N}, H_{I}^{N}, T_{I}^{N}, U_{I}^{N}$, etc. at our disposal. The compound objects Link ${ }^{N}, \operatorname{Err}_{j}^{N}$, etc. figuring in the more elaborate identities also become random variables in the present setting.

### 11.1.5. Partially averaged random variables. For $N$ and $I \in \mathcal{I}_{N}$ we define

$$
\bar{F}_{I}^{N}=\mathfrak{G} \mathbb{E}\left(F_{I}^{N} / \mathfrak{G} \mid \mathbf{z}, \mathbf{t}\right) \in \mathcal{S} \text { and } \bar{E}_{I}^{N}=1_{\mathcal{S}}+\left(\Lambda+\Phi\left(\bar{F}_{I}^{N}\right)\right) \bar{F}_{I}^{N} \in \mathcal{S} .
$$

Since $\left[\left[F_{I}^{N}\right]\right] / \mathfrak{G}$ is integrable by assumption (122), in fact $\bar{F}_{I}^{N}$ and $\bar{E}_{I}^{N}$ are welldefined, almost surely. In $\S 11.4$ below we will work out a delicate approximation to $\bar{E}_{N}^{N}$. (Recall our abuse of notation $N=\{1, \ldots, N\}$.)

Remark 11.1.6. Let $(\mathcal{S}, L, \Theta, e), c_{0}, c_{1}, c_{2}$ and $\mathfrak{T}$ be as in Definition 7.1.1. We keep the notation of Remark 10.1.3; also let $\Psi_{L}$ be as in Definition 5.4.2. Then, using assumptions (1)-(8) along with Remark 7.1.4, it is easy to verify that

$$
\begin{aligned}
& (\mathcal{S}, X, \Phi, \Psi, \Lambda, \mathfrak{G}) \\
= & \left(\mathcal{S}, \bigcup_{N} L\left(\Xi^{N}\right), \Phi_{L}, \Psi_{L}, \Theta+\mathbf{z e} e+\mathrm{it} 1_{\mathcal{S}}, 2 c_{0}(1+\llbracket L(\Xi) \rrbracket)^{c_{1}}(1+1 / \Im \mathbf{z})^{c_{2}}\right)
\end{aligned}
$$

satisfies assumptions (116)-(122) and thus is an instance of the block Wigner model. We emphasize that assumption (2) is the key to verifying property (122). We note also that (122) can be considerably refined in this specialization. Namely, for each $N$ and $I \in \mathcal{I}_{N}$, we have bounds

$$
\begin{align*}
{\left[\left[R_{I}^{N}\right]\right] } & \leq\left(1+\left[\left[\frac{\mathbf{f}_{I} X \mathbf{f}_{I}^{*}}{\sqrt{N}}\right]\right]\right)^{c_{1}} \mathfrak{G}  \tag{123}\\
{\left[\left[R_{I}^{N}\right]\right] \mathbf{1}_{\mathbf{t} \geq \mathfrak{T}} } & \leq \frac{1}{2} \wedge \frac{1}{\mathbf{t}-\mathfrak{T}} \tag{124}
\end{align*}
$$

One also checks easily that

$$
\left(G_{L}: \mathcal{D}_{L} \rightarrow \mathcal{S}, \Phi_{L},\left.\Lambda\right|_{\mathbf{t}=0}, \mathfrak{T}, \mathfrak{G}\right)
$$

is an SD tunnel, albeit a random one. In short we have an overwhelming amount of structure to work with in this specialization. In fact, it is too much for us to handle all at once. That is why we retreat to the relatively austere setting of the block Wigner model for now. It is enough work just to draw consequences of (122).

Remark 11.1.7. We continue in the setup of the previous remark. If $(\mathcal{S}, L, \Theta, e)$ is a self-adjoint linearization of some $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)_{\text {sa }}$, then we have a representation

$$
\begin{equation*}
S_{\mu_{f}^{N}}(\mathbf{z})=\left.\tau_{\mathcal{S}, e} \circ F_{N}^{N}\right|_{\mathbf{t}=0} \tag{125}
\end{equation*}
$$

for the random variable $S_{\mu_{f}^{N}}(\mathbf{z})$ figuring in Theorem 2.6.4. (Recall that the notation for evaluation at $\mathbf{t}=0$ is defined before Proposition 7.2.4.)

### 11.2. Basic estimates.

11.2.1. The norms $\|\cdot\|_{p, k}$. Given a constant $p \in[1, \infty)$, a positive integer $k$ and a finite-dimensional-Banach-space-valued random variable $Z$ defined on the same probability space as $\mathfrak{G}$, we write $\|Z\|_{p, k}=\left\|\llbracket Z \rrbracket / \mathfrak{G}^{k}\right\|_{p}$ to compress notation.
Proposition 11.2.2. For each constant $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N} \bigvee_{I \in \mathcal{I}_{N}}\| \| R_{I}^{N}\| \|_{p, 1} \vee\left\|F_{I}^{N} \mid\right\|_{p, 1} \vee\left\|H_{I}^{N}\right\|\left\|_{p, 1} \vee\right\| T_{I}^{N}\| \|_{p, 2} \vee\left\|U_{I}^{N}\right\| \|_{p, 2}<\infty  \tag{126}\\
& \sup _{N} \bigvee_{(I, J) \in \mathcal{I}_{N}^{(2)}}\| \| R_{I, J}^{N}\| \|_{p, 1} \vee\left\|H_{I, J}^{N}\right\| \|_{p, 1}<\infty  \tag{127}\\
& \sup _{N} \bigvee_{(I, J) \in \mathcal{I}_{N}^{(2)}} N\left\|F_{I}^{N}-F_{I \backslash J}^{N}\right\| \|_{p, 3}<\infty \tag{128}
\end{align*}
$$

Proof. The claim made in (126) for $R_{I}^{N}$ just repeats the hypothesis (122) in different notation. We have

$$
\left.\left[\left[R_{I}^{N}\right]\right] \geq\left[\left[F_{I}^{N}\right]\right] \vee\left[\left[T_{I}^{N}\right]\right]\right]^{1 / 2} \vee \frac{1}{2}\left[\left[H_{I}^{N}\right]\right] \vee \frac{1}{\sqrt{s}}\left[\left[U_{I}^{N}\right]\right]^{1 / 2}
$$

obviously in the first two cases, by Lemma 4.1.1 in the penultimate case and Lemma 9.1.3 in the last, where $s^{2}$ is the dimension of $\mathcal{S}$ over the complex numbers. Thus (126) holds in general. Clearly, we have

$$
\left[\left[R_{I, J}^{N}\right]\right] \leq\left[\left[R_{I}^{N}\right]\right] \quad \text { and } \quad\left[\left[H_{I, J}^{N}\right]\right]=\left[\left[H_{I \backslash J}^{N}\right]\right]
$$

whence (127) via (126). By Lemma 9.1.3 and identity (105) we have

$$
\frac{N}{|J|^{2}}\left[\left[F_{I}^{N}-F_{I \backslash J}^{N}\right]\right] \leq\left[\left[R_{I}^{N}\right]\right]+\frac{s}{N}\left[\left[R_{I}^{N}\right]\right]\left[\left[R_{I \backslash J}^{N}\right]\right]^{2} \sum_{(i, j) \in(I \backslash J) \times J} \llbracket X(i, j) \rrbracket^{2}
$$

From this, estimate (128) follows by assumption (116), the Minkowski inequality and estimate (126).
11.2.3. The seminorms $\|\cdot\|_{p, k, I}$. Given a constant $p \in[1, \infty)$, a positive integer $k$, a set $I$ of positive integers and a finite-dimensional-Banach-space-valued random variable $Z$ defined on the same probability space as $\mathfrak{G}$ such that $\|Z\|_{p, k}<\infty$, we define

$$
\|Z\|_{p, k, I}=\left\|\left[\left[\mathbb{E}\left(Z / \mathfrak{G}^{k} \mid \mathcal{F}_{I}, \mathbf{z}, \mathbf{t}\right)\right]\right]\right\|_{p} .
$$

Since the random variable $\left[\left[Z / \mathfrak{G}^{k}\right]\right]$ is assumed to be in $L^{p} \subset L^{1}$, the conditional expectation appearing on the right is well-defined, almost surely, and moreover

$$
\|Z\|_{p, k} \geq\|Z\|_{p, k, I} \geq\|Z\|_{p, k, J}
$$

for any set $J \subset I$ by Jensen's inequality. In particular,

$$
\left\|\mathbb{E}\left(Z / \mathfrak{G}^{k} \mid \mathbf{z}, \mathbf{t}\right)\right\|_{p}=\|Z\|_{p, k, \emptyset}
$$

whenever $\|Z\|_{p, k}<\infty$.
Proposition 11.2.4. For each constant $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N} \bigvee_{(I, J) \in \mathcal{I}_{N}^{(2)}}\left\|Q_{I, J}^{N}\right\|_{p, 1} \vee\left\|P_{I, J}^{N}\right\|_{p, 2}<\infty,  \tag{129}\\
& \sup _{N} \bigvee_{(I, J) \in \mathcal{I}_{N}^{(2)}}\left\|Q_{I, J}^{N}\right\|_{p, 1, I \backslash J} \vee\left\|P_{I, J}^{N}\right\|_{p, 2, I \backslash J}=0 . \tag{130}
\end{align*}
$$

It is hard to overestimate the importance of this proposition. Our exploitation of it is of course an imitation of the procedure of [2].

Proof. Fix $N,(I, J) \in \mathcal{I}_{N}^{(2)}$ and $j_{1}, j_{2} \in J$ arbitrarily. By definition we have

$$
\begin{align*}
Q_{I, J, j_{1}, j_{2}}^{N} & =-X\left(j_{1}, j_{2}\right)+\frac{1}{\sqrt{N}}\left(\mathbf{f}_{j_{1}} X R_{I \backslash J}^{N} X \mathbf{f}_{j_{2}}^{*}-N \delta_{j_{1} j_{2}} \Phi\left(F_{I \backslash J}^{N}\right)\right)  \tag{131}\\
P_{I, J, j_{1}, j_{2}}^{N} & =\left(\zeta \mapsto \frac{1}{\sqrt{N}}\left(\operatorname{tr}_{\mathcal{S}}\left(R_{I \backslash J}^{N} X \mathbf{f}_{j_{1}}^{*} \zeta \mathbf{f}_{j_{2}} X R_{I \backslash J}^{N}\right)-N \delta_{j_{1} j_{2}} T_{I \backslash J}^{N}(\Phi(\zeta))\right)\right)
\end{align*}
$$

By (116) and (126), the random variables $\left[\left[Q_{I, J, j_{1}, j_{2}}^{N}\right]\right]$ and $\left[\left[P_{I, J, j_{1}, j_{2}}^{N}\right]\right]$ are integrable, hence the conditional expectations

$$
\mathbb{E}\left(Q_{I, J, j_{1}, j_{2}}^{N} / \mathfrak{G} \mid \mathcal{F}_{I \backslash J}, \mathbf{z}, \mathbf{t}\right) \text { and } \mathbb{E}\left(P_{I, J, j_{1}, j_{2}}^{N} / \mathfrak{G}^{2} \mid \mathcal{F}_{I \backslash J}, \mathbf{z}, \mathbf{t}\right)
$$

are well-defined and vanish almost surely by assumptions (116), (117) and (118). By Proposition 9.1.2, Remark 9.1.6, estimate (126) and the hypotheses of the block Wigner model, the quantities

$$
\left\|Q_{I, J, j_{1}, j_{2}}^{N}+X\left(j_{1}, j_{2}\right)\right\|_{p, 1},\left\|X\left(j_{1}, j_{2}\right)\right\|_{p, 1} \text { and }\left\|P_{I, J, j_{1}, j_{2}}^{N}\right\|_{p, 2}
$$

are bounded uniformly in $N, I, J, j_{1}$ and $j_{2}$. Thus claims (129) and (130) hold.
11.3. More elaborate estimates. We combine and specialize the basic estimates.

Proposition 11.3.1. For each constant $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \\
\text { s.t. }|I| \geq 2}} \sqrt{N}\left\|E_{I}^{N}+\frac{|I|-N}{N} 1_{\mathcal{S}}\right\| \|_{p, 4}<\infty  \tag{132}\\
& \sup _{N} \bigvee_{\substack{(I, J) \in \mathcal{I}_{N}^{(2)} \text { s.t. } \\
I I \mid \geq N-\sqrt{N}}}\left\|\Delta_{I, J}^{N}\right\|_{p, 4}<\infty  \tag{133}\\
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\
|I| \geq N-99}} N\left\|E_{I}^{N}\right\|_{p, 6, \emptyset}<\infty \tag{134}
\end{align*}
$$

Proof. We take Propositions 11.2.2 and 11.2.4 for granted at every step. Identity (107) implies the estimate (132). Estimate (132) and the Chebychev bound

$$
\begin{equation*}
\mathbf{1}_{\llbracket E_{I}^{N} \rrbracket \geq 1 / 2} \leq\left(2\left[\left[E_{I}^{N}\right]\right]\right)^{c} \quad(c \geq 0) \tag{135}
\end{equation*}
$$

imply estimate (133). Identity (108) and estimate (133) imply the estimate

$$
\sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \\ \text { s.t. }|I| \geq N-\sqrt{N}}} N\| \| E_{I}^{N}+\frac{|I|-N}{N} 1_{\mathcal{S}}+\frac{1}{N} \sum_{j \in I} \frac{Q_{I, j}^{N} H_{I \backslash j}^{N}}{\sqrt{N}}\| \|_{p, 6}<\infty
$$

Estimate (134) follows via (130).

Proposition 11.3.2. For each constant $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N} N^{3 / 2}\left\|\left.\right|_{N+1} ^{N+1}-F_{N}^{N}-\frac{\operatorname{Link}^{N}}{N}\right\| \|_{p, 7}<\infty,  \tag{136}\\
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\
|I| \geq N-99}} N^{2}\left\|\bar{E}_{I}^{N}-E_{I}^{N}\right\| \|_{p, 14, \emptyset}<\infty . \tag{137}
\end{align*}
$$

Proof. We take Propositions 11.2.2, 11.2.4 and 11.3.1 for granted at every step. We have

$$
\sup _{N} N\left\|(N+1) F_{N}^{N+1}-N F_{N}^{N}-\frac{1}{2}\left(F_{N}^{N}+T_{N}^{N}(\Lambda)\right)\right\| \|_{p, 3}<\infty
$$

by identity (112) along with assumption (119). The estimate

$$
\sup _{N} \bigvee_{\substack{(I, J) \in \mathcal{I}_{N}^{(2)} \\|I| \geq N-\sqrt{N}}} N^{3 / 2}\left\|F_{I}^{N}-F_{I \backslash J}^{N}-|J| \frac{\left(1_{B(\mathcal{S})}+T_{I \backslash J}^{N} \circ \Phi\right)\left(H_{I \backslash J}^{N}\right)}{N}\right\|_{p, 7}<\infty
$$

follows from identity (111). Estimate (136) then follows via the definition of Link ${ }^{N}$. From the last estimate above it also follows that

$$
\sup _{N} \sup _{\substack{I \in \mathcal{I}_{N} \\|I| \geq N-99}} N^{2}\left\|\operatorname{Var}_{\mathcal{S}}\left(F_{I}^{N} / \mathfrak{G}^{7} \mid \mathbf{z}, \mathbf{t}\right)\right\|_{p}<\infty
$$

via Proposition 9.2.2, whence estimate (137).
Proposition 11.3.3. For each constant $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N} \bigvee_{\substack{(I, J) \in \mathcal{I}_{N}^{(2)} \\
|I| \geq N-\sqrt{N}}} \bigvee_{\substack{\text { s.t. }}} \sqrt{N}\left\|R_{1}, j_{2} \in J, N\left(j_{1}, j_{2}\right)-\delta_{j_{1} j_{2}} H_{I \backslash J}^{N}\right\| \|_{p, 5}<\infty,  \tag{138}\\
& \sup _{N} \bigvee_{\substack{(I, J) \in \mathcal{I}_{N}^{(2)} \\
|I| \geq N-\sqrt{N}}} \bigvee_{\substack{N}} N\| \| R_{I}^{N}\left(j_{1}, j_{2}\right)-\delta_{j_{1}, j_{2}} H_{I \backslash J}^{N}\| \|_{p, 9, I \backslash J}<\infty,  \tag{139}\\
& \sup _{N} \bigvee_{\substack{(I, J) \in \mathcal{I}_{N}^{(2)} \\
|I| \geq N-\sqrt{N}}} N\left\|H_{I}^{N}-H_{I \backslash J}^{N}\right\| \|_{p, 9}<\infty,  \tag{140}\\
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\
|I| \geq N-\sqrt{N}}} \sqrt{N}\left\|H_{I}^{N}-F_{I}^{N}\right\|_{p, 5}<\infty,  \tag{141}\\
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \\
|I| \geq N-\sqrt{N}}} N\left\|H_{I}^{N}-F_{I}^{N}\right\|_{p, 9, I \backslash J}<\infty . \tag{142}
\end{align*}
$$

Proof. Taking Propositions 11.2.2, 11.2.4 and 11.3.1 for granted and using again the Chebychev bound (135), one derives the estimates in question from identities (103), (104), (106), (109) and (110), respectively.
11.4. The bias theorem. We work out a delicate approximation to $\bar{E}_{N}^{N}$. We use again the cumulant and shuffle notation introduced in $\S 6.3$. We also use again the apparatus introduced to state and prove Proposition 9.3.5.
11.4.1. Correction terms. For $N \geq 2$ and $j=1, \ldots, N$ we define

$$
\begin{aligned}
\operatorname{Corr}_{j}^{N}= & \left\langle[\Psi, \Psi]_{2},\left[U_{N \backslash j}^{N},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4}-\Phi\left(H_{N \backslash j}^{N}\right) H_{N \backslash j}^{N} \\
& +\left\langle\mathbb{E} X(j, j)^{\otimes 2},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right\rangle_{2}-\frac{1}{\sqrt{N}}\left\langle\mathbb{E} X(j, j)^{\otimes 3},\left(H_{N \backslash j}^{N}\right)^{\otimes 3}\right\rangle_{3} \\
& +\frac{1}{N} \sum_{i \in N \backslash j}\left\langle\mathbf{C}^{(4)}(X(i, j)),\left[\left(R_{I \backslash j, i}^{N}\right)^{\otimes 2},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4} .
\end{aligned}
$$

Theorem 11.4.2. For each constant $p \in[1, \infty)$ we have

$$
\begin{equation*}
\sup _{N \geq 2} N^{2}\| \|_{\bar{E}}=\frac{1}{N} \sum_{j=1}^{N} \frac{\operatorname{Corr}_{j}^{N}}{N} \|_{p, 14, \emptyset}<\infty \tag{143}
\end{equation*}
$$

The proof of the theorem takes up the rest of $\S 11.4$. We need several lemmas.
Lemma 11.4.3. For each constant $p \in[1, \infty)$ we have

$$
\begin{align*}
& \sup _{N \geq 2} \bigvee_{j=1}^{N}\| \| \operatorname{Err}_{j}^{N}\| \|_{p, 6} \vee\left\|\operatorname{Err}_{j}^{N, 1}\right\|\left\|_{p, 6} \vee\right\| \mid \operatorname{Err}_{j}^{N, 2} \|_{p, 14}<\infty,  \tag{144}\\
& \sup _{N \geq 2} \bigvee_{j=1}^{N}\left\|\operatorname{Err}_{j}^{N, 1}\right\| \|_{p, 6, N \backslash j}=0 . \tag{145}
\end{align*}
$$

Proof. Taking Propositions 11.2.2, 11.2.4 and 11.3.1 for granted, these facts can be read off from the definitions presented in $\S 10.4$.
11.4.4. Moment notation. For any sequence $\mathbf{i}=i_{1} \cdots i_{2 k}$ of positive integers and positive integer $j$ not appearing in $\mathbf{i}$ put

$$
\begin{aligned}
\mathbf{M}_{j}(\mathbf{i})= & \mathbb{E}\left[\left(X\left(j, i_{1}\right) \otimes X\left(i_{2}, j\right)-\mathbb{E}\left(X\left(j, i_{1}\right) \otimes X\left(i_{2}, j\right)\right)\right) \otimes \cdots\right. \\
& \left.\cdots \otimes\left(X\left(j, i_{2 k-1}\right) \otimes X\left(i_{2 k}, j\right)-\mathbb{E}\left(X\left(j, i_{2 k-1}\right) \otimes X\left(i_{2 k}, j\right)\right)\right)\right] \in \mathcal{S}^{\otimes 2 k} .
\end{aligned}
$$

Lemma 11.4.5. For sequences $\mathbf{i}=i_{1} \cdots i_{2 k}$ of positive integers, and positive integers $j$ not appearing in $\mathbf{i}$, the following statements hold:
(I) For each fixed $k, \llbracket \mathbf{M}_{j}(\mathbf{i}) \rrbracket$ is bounded uniformly in $\mathbf{i}$ and $j$.
(II) $\mathbf{M}_{j}(\mathbf{i})$ vanishes unless $\Pi(\mathbf{i}) \in \operatorname{Part}^{*}(2 k)$.
(III) If $\Pi(\mathbf{i}) \in \operatorname{Part}_{2}^{*}(2 k)$, then $\mathbf{M}_{j}(\mathbf{i})$ depends only on $\Pi(\mathbf{i})$.

Proof. Assumption (116) implies statement (I). Assumptions (117) implies statement (II). Assumptions (117) and (118) imply statement (III).
11.4.6. Tensor products of resolvent entries. For $N, I \in \mathcal{I}_{N}$ and sequences $\mathbf{i}=i_{1} \cdots i_{2 k} \in \operatorname{Seq}(2 k, I)$ put

$$
R_{I}^{N}(\mathbf{i})=R_{I}^{N}\left(i_{1}, i_{2}\right) \otimes \cdots \otimes R_{I}^{N}\left(i_{2 k-1}, i_{2 k}\right) \in \mathcal{S}^{\otimes k}
$$

11.4.7. The random variable $\operatorname{Rub}_{j}^{N}$. For $N \geq 2$ and $j=1, \ldots, N$ put

$$
\operatorname{Rub}_{j}^{N}=\frac{1}{N^{2}} \sum_{\substack{\mathbf{i} \in \operatorname{Seq}(6, N \backslash j) \text { s.t. } \\ \Pi(\mathbf{i}) \in \operatorname{Part}^{*}(6) \text { and } \\ \Pi(\mathbf{i}) \sim\{\{1,2,3\},\{4,5,6\}\}}}\left\langle\mathbf{M}_{j}(\mathbf{i}),\left[R_{N \backslash j}^{N}(\mathbf{i}),\left(H_{N \backslash j}^{N}\right)^{\otimes 3}\right]_{3}\right\rangle_{6} .
$$

Here we employ again the notation $\sim$ for $\Gamma_{3}$-orbit equivalence previously introduced in connection with the list (96).

Lemma 11.4.8. For $N \geq 3$ and $j=1, \ldots, N$ we have

$$
\left[\left[\mathfrak{G}^{6} \mathbb{E}\left(\operatorname{Err}_{j}^{N} / \mathfrak{G}^{6} \mid \mathcal{F}_{N \backslash j}, \mathbf{z}, \mathbf{t}\right)-\operatorname{Corr}_{j}^{N}-\operatorname{Rub}_{j}^{N}\right]\right] \leq \frac{c}{N}\left[\left[R_{N \backslash j}^{N}\right]\right]^{6}
$$

almost surely, for a constant $c$ independent of $N$ and $j$.
Proof. In the case $(I, J)=(N,\{j\})$, formula (131) above simplifies to

$$
Q_{N, j}^{N}+X(j, j)=\frac{1}{\sqrt{N}}\left(\mathbf{f}_{j} X R_{N \backslash j}^{N} X \mathbf{f}_{j}^{*}-\mathfrak{G} \mathbb{E}\left(\mathbf{f}_{j} X R_{N \backslash j}^{N} X \mathbf{f}_{j}^{*} / \mathfrak{G} \mid \mathcal{F}_{N \backslash j}, \mathbf{z}, \mathbf{t}\right)\right)
$$

Note that the right side is independent of $X(j, j)$. along with Lemma 11.4.5(II) that for $k \in\{2,3\}$,

$$
\begin{aligned}
& \mathfrak{G}^{2 k} \mathbb{E}\left(\left(Q_{N, j}^{N} H_{N \backslash j}^{N}\right)^{k} / \mathfrak{G}^{2 k} \mid \mathcal{F}_{N \backslash j}, \mathbf{z}, \mathbf{t}\right)-(-1)^{k}\left\langle\mathbb{E} X(j, j)^{\otimes k},\left(H_{N \backslash j}^{N}\right)^{\otimes 3}\right\rangle_{3} \\
= & \frac{1}{N^{k / 2}} \sum_{\substack{\mathbf{i} \in \operatorname{Seq}(2 k, N \backslash) \\
\text { s.t. } \Pi(\mathbf{i}) \in \operatorname{Part}^{*}(2 k)}}\left\langle\mathbf{M}_{j}(\mathbf{i}),\left[R_{N \backslash j}^{N}(\mathbf{i}),\left(H_{N \backslash j}^{N}\right)^{\otimes k}\right]_{k}\right\rangle_{2 k} .
\end{aligned}
$$

By a calculation using Lemma 11.4.5(II,III) and enumeration (95), with $\alpha, \beta \in N \backslash j$ arbitrarily chosen distinct elements, we have

$$
\begin{aligned}
& \mathfrak{G}^{4} \mathbb{E}\left(\left(Q_{N, j}^{N} H_{N \backslash j}^{N}\right)^{2} / \mathfrak{G}^{4} \mid \mathcal{F}_{N \backslash j}, \mathbf{z}, \mathbf{t}\right)-\left\langle\mathbb{E}\left(X(j, j)^{\otimes 2}\right),\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right\rangle_{2} \\
= & \frac{1}{N} \sum_{i_{1}, i_{2} \in N \backslash j}\left\langle\mathbf{M}_{j}(\alpha \beta \alpha \beta),\left[R_{N \backslash j}^{N}\left(i_{1}, i_{2}\right)^{\otimes 2},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4} \\
& +\frac{1}{N} \sum_{i_{1}, i_{2} \in N \backslash j}\left\langle\mathbf{M}_{j}(\alpha \beta \beta \alpha),\left[R_{N \backslash j}^{N}\left(i_{1}, i_{2}\right) \otimes R_{N \backslash j}^{N}\left(i_{2}, i_{1}\right),\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4} \\
& +\frac{1}{N} \sum_{i \in N \backslash j}\left\langle\mathbf{M}_{j}(i i i i i)-\mathbf{M}_{j}(\alpha \beta \beta \alpha)-\mathbf{M}_{j}(\alpha \beta \alpha \beta),\left[\left(R_{N \backslash j, i}^{N}\right)^{\otimes 2},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4} \\
= & \left\langle[\Psi, \Psi]_{2},\left[U_{N \backslash j}^{N},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4}+\Phi \circ T_{N \backslash j}^{N} \circ \Phi\left(H_{N \backslash j}^{N}\right) \\
& +\frac{1}{N} \sum_{i \in N \backslash j}\left\langle\mathbf{C}^{(4)}(X(i, j)),\left[\left(R_{N \backslash j, i}^{N}\right)^{\otimes 2},\left(H_{N \backslash j}^{N}\right)^{\otimes 2}\right]_{2}\right\rangle_{4} .
\end{aligned}
$$

It follows that

$$
=\begin{aligned}
& \mathfrak{G}^{6} \mathbb{E}\left(\operatorname{Err}_{j}^{N} / \mathfrak{G}^{6} \mid \mathcal{F}_{N \backslash j}, \mathbf{z}, \mathbf{t}\right)-\operatorname{Corr}_{j}^{N}-\operatorname{Rub}_{j}^{N} \\
&= \frac{1}{N^{2}} \sum_{\substack{\mathbf{i} \in \operatorname{Seq}(6, N \backslash j) \text { s.t. } \\
\Pi(\mathbf{i}) \in \operatorname{Partr}^{*}(6) \text { and } \\
\Pi(\mathbf{i}) \nsucc\{\{1,2,3\},\{4,5,6\}\}}}\left\langle\mathbf{M}_{j}(\mathbf{i}),\left[R_{N \backslash j}^{N}(\mathbf{i}),\left(H_{N \backslash j}^{N}\right)^{\otimes 3}\right]_{3}\right\rangle_{6}, \\
&
\end{aligned}
$$

whence the result by Proposition 9.3.5 and Lemma 11.4.5(I,III).

Lemma 11.4.9. Fix $p \in[1, \infty)$ arbitrarily. For $N \geq 3$ and distinct $j, j_{1}, j_{2} \in N$, the quantity

$$
N\left\|\left(H_{N \backslash j}^{N}\right)^{\otimes 3} \otimes R_{N \backslash j}^{N}\left(j_{1} j_{1} j_{2} j_{2} j_{1} j_{2}\right)\right\|_{p, 14, N \backslash\left\{j, j_{1}, j_{2}\right\}}
$$

is bounded uniformly in $N, j, j_{1}$ and $j_{2}$.
Proof. Put $J=\left\{j, j_{1}, j_{2}\right\}$. The quantity

$$
N\left\|\left(H_{N \backslash J}^{N}\right)^{\otimes 5} \otimes R_{N \backslash j}^{N}\left(j_{1}, j_{2}\right)\right\|_{p, 14, N \backslash J}
$$

is bounded uniformly in $N, j, j_{1}$ and $j_{2}$ by (126) and (139). The quantity

$$
N\left\|\left(H_{N \backslash j}^{N}\right)^{\otimes 3} \otimes R_{N \backslash j}^{N}\left(j_{1} j_{1} j_{2} j_{2} j_{1} j_{2}\right)-\left(H_{N \backslash J}^{N}\right)^{\otimes 5} \otimes R_{N \backslash j}^{N}\left(j_{1}, j_{2}\right)\right\|_{p, 14}
$$

is bounded uniformly in $N, j, j_{1}$ and $j_{2}$ by (138) and (140).
11.4.10. Completion of the proof of Theorem 11.4.2. We have

$$
\sup _{N \geq 2} N^{2}\left\|\bar{E}_{N}^{N}-E_{N}^{N}\right\|_{p, 14, \emptyset}<\infty
$$

by estimate (137). We have

$$
\sup _{N \geq 2} N^{2}\| \| E_{N}^{N}+\frac{1}{N} \sum_{j=1}^{N} \frac{\operatorname{Err}_{j}^{N}}{N}\| \|_{p, 14, \emptyset}<\infty
$$

by the bias identity and Lemma 11.4.3. We have

$$
\sup _{N \geq 2} \bigvee_{j=1}^{N} N\left\|\operatorname{Err}_{j}^{N}-\operatorname{Corr}_{j}^{N}-\operatorname{Rub}_{j}^{N}\right\| \|_{p, 14}<\infty
$$

by Proposition 11.2.2 and Lemma 11.4.8. Finally, we have

$$
\sup _{N \geq 2} \bigvee_{j=1}^{N} N\left\|\operatorname{Rub}_{j}^{N}\right\| \|_{p, 14, \emptyset}<\infty
$$

by Lemma 11.4.5(I) and Lemma 11.4.9, which finishes the proof.

## 12. Endgame

We finish the proof of Theorem 2.6.4.
12.1. Setup for the endgame. Throughout $\S 12$ we fix

$$
(\mathcal{S}, L, \Theta, e), c_{0}, c_{1}, c_{2}, \mathfrak{T}
$$

as in Definition 7.1.1. We emphasize that $(\mathcal{S}, L, \Theta, e)$ may or may not be a selfadjoint linearization. We work with the instance

$$
\begin{aligned}
& (\mathcal{S}, X, \Phi, \Psi, \Lambda, \mathfrak{G}) \\
= & \left(\mathcal{S}, \bigcup_{N} L\left(\Xi^{N}\right), \Phi_{L}, \Psi_{L}, \Theta+\mathbf{z e}+\mathrm{it} 1_{\mathcal{S}}, 2 c_{0}(1+\llbracket L(\Xi) \rrbracket)^{c_{1}}(1+1 / \Im \mathbf{z})^{c_{2}}\right)
\end{aligned}
$$

of the block Wigner model introduced above in Remark 11.1.6. Without loss of generality we impose the integrability condition

$$
\begin{equation*}
\mathbb{E} \mathfrak{G}^{4}<\infty \tag{146}
\end{equation*}
$$

We will employ the abbreviated notation

$$
G: \mathcal{D} \rightarrow \mathcal{S}
$$

in place of the more heavily subscripted notation $G_{L}: \mathcal{D}_{L} \rightarrow \mathcal{S}$. In a similar spirit of lesser adornment, we write

$$
G^{\prime}=\mathbf{D}[G], \quad \check{G}=\left(\left(G^{-1}\right)^{\otimes 2}-\Psi\right)^{-1}, \quad \operatorname{Bias}^{N}=\operatorname{Bias}_{L}^{N} .
$$

Note also that for every $p \in[1, \infty)$ the bounds

$$
\begin{equation*}
\|G(\Lambda)\|_{p, 1}<\infty, \quad\left\|G^{\prime}(\Lambda)\right\|_{p, 2}<\infty, \quad\|\check{G}(\Lambda)\|_{p, 2}, \sup _{N}\left\|\operatorname{Bias}^{N}(\Lambda)\right\|_{p, 5}<\infty \tag{147}
\end{equation*}
$$

hold, as one checks by Remark 7.1.3.
12.2. Application of Proposition 7.2.4. We write out the estimates of Proposition 7.2.4 in the present setup. This is just a matter of inserting sub- and superscripts. Then we draw some immediate consequences via the estimates of $\S 11$.

Recall that

$$
\Lambda_{0}=\left.\Lambda\right|_{\mathbf{t}=0}=\Theta+\mathbf{z e}
$$

and put

$$
\mathfrak{C}=99(1+\llbracket \Phi \rrbracket+\llbracket \Theta \rrbracket+|\mathbf{z}|) .
$$

For $N$ and $I \in \mathcal{I}_{N}$ put

$$
\begin{aligned}
\mathfrak{L}_{I}^{N} & =\mathfrak{G}^{2}\left(1+\left[\left[\frac{\mathbf{f}_{I} X \mathbf{f}_{I}^{*}}{\sqrt{N}}\right]\right]\right)^{2 c_{1}}, \overline{\mathfrak{L}}_{I}^{N}=\mathfrak{G}^{2} \mathbb{E}\left(1+\left[\left[\frac{\mathbf{f}_{I} X \mathbf{f}_{I}^{*}}{\sqrt{N}}\right]\right]\right)^{2 c_{1}} \\
\mathfrak{E}_{I}^{N} & =\mathbb{E}\left(\left[\left[E_{I}^{N}\right]\right] \mid \mathcal{F}, \mathbf{z}\right), \overline{\mathfrak{E}}^{N}=\mathbb{E}\left(\left[\left[\bar{E}_{N}^{N}\right]\right] \mid \mathcal{F}, \mathbf{z}\right)
\end{aligned}
$$

By directly plugging into Proposition 7.2 .4 (note that our assumption (146) makes hypothesis (74) hold) we have

$$
\begin{aligned}
& {\left[\left[\left.F_{I}^{N}\right|_{\mathbf{t}=0}-G\left(\Lambda_{0}\right)\right]\right] \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \mathfrak{L}_{I}^{N}\right)^{6}\left(\mathfrak{E}_{I}^{N}+\left(\mathfrak{E}_{I}^{N}\right)^{2}\right),} \\
& {\left[\left[\left.F_{I}^{N}\right|_{\mathbf{t}=0}+G^{\prime}\left(\Lambda_{0} ;\left(\left.E_{I}^{N}\right|_{\mathbf{t}=0}\right) G\left(\Lambda_{0}\right)^{-1}\right)-G\left(\Lambda_{0}\right)\right]\right]} \\
& \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \mathfrak{L}_{I}^{N}\right)^{12}\left(\left(\mathfrak{E}_{I}^{N}\right)^{2}+\left(\mathfrak{E}_{I}^{N}\right)^{4}\right), \\
& {\left[\left[\left.\bar{F}_{N}^{N}\right|_{\mathbf{t}=0}+G^{\prime}\left(\Lambda_{0} ;\left(\left.\bar{E}_{N}^{N}\right|_{\mathbf{t}=0}\right) G\left(\Lambda_{0}\right)^{-1}\right)-G\left(\Lambda_{0}\right)\right]\right]} \\
& \leq\left(e^{\mathfrak{T}} \mathfrak{C} \mathfrak{G} \overline{\mathfrak{L}}^{N}\right)^{12}\left(\left(\overline{\mathfrak{E}}^{N}\right)^{2}+\left(\overline{\mathfrak{E}}^{N}\right)^{4}\right) .
\end{aligned}
$$

(We also have an "overlined version" of the first estimate but we do not need it.)
Now fix $p \in[1, \infty)$ arbitrarily. The right sides above can be bounded in terms of the (semi)norms $\left\|\|\cdot\|_{p, k}\right.$ and $\| \cdot \cdot \|_{p, k, \emptyset}$ for suitably chosen $k$, as follows. Firstly, $e^{\mathfrak{T}}$ is a constant and $\mathfrak{C}$ has moments of all orders. Secondly, we are in effect allowed to ignore factors of $\mathfrak{G}$ on the right side at the expense of increasing $k$. Thirdly, we have

$$
\sup _{N} \bigvee_{I \in \mathcal{I}_{N}}\| \| \mathfrak{L}_{I}^{N}\| \|_{p, 2}<\infty, \sup _{N}\left\|\overline{\mathfrak{L}}^{N}\right\| \|_{p, 2}<\infty
$$

by the vitally important assumption (2). Now by Jensen's inequality, for $N$ and $I \in \mathcal{I}_{N}$ we have

$$
\left\|\mathfrak{E}_{I}^{N}\right\|_{p, 4} \leq\left\|E_{I}^{N}\right\|_{p, 4},\left\|\overline{\mathfrak{E}}^{N}\right\|\left\|_{p, 14} \leq\right\| \bar{E}_{N}^{N} \|_{p, 14}
$$

Thus, fourthly, we have

$$
\sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\|I| \geq N-\sqrt{N}}} \sqrt{N}\left\|\mathfrak{E}_{I}^{N}\right\|_{p, 4}<\infty, \sup _{N} N\left\|\overline{\mathfrak{G}}^{N}\right\| \|_{p, 14}<\infty
$$

via (132), (134) and (137). Fifthly and finally, note that by Remark 7.2.5 and the bound (134) we have

$$
\sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\|I| \geq N-99}} N\left\|G^{\prime}\left(\Lambda_{0} ;\left(\left.E_{I}^{N}\right|_{\mathbf{t}=0}\right) G\left(\Lambda_{0}\right)^{-1}\right)\right\|_{p, 9, \emptyset}<\infty
$$

We conclude that

$$
\begin{align*}
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\
|I| \geq N-99}} \sqrt{N}\left\|\left|F_{I}^{N}\right|_{\mathbf{t}=0}-G\left(\Lambda_{0}\right)\right\|_{p, 99}<\infty  \tag{148}\\
& \sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\
|I| \geq N-99}} N\left\|\left.| | F_{I}^{N}\right|_{\mathbf{t}=0}-G\left(\Lambda_{0}\right)\right\| \|_{p, 99, \emptyset}<\infty  \tag{149}\\
& \sup _{N} N^{2}\left\|\left.\bar{F}_{N}^{N}\right|_{\mathbf{t}=0}+G^{\prime}\left(\Lambda_{0} ;\left(\left.\bar{E}_{N}^{N}\right|_{\mathbf{t}=0}\right) G\left(\Lambda_{0}\right)^{-1}\right)-G\left(\Lambda_{0}\right)\right\| \|_{p, 99}<\infty . \tag{150}
\end{align*}
$$

We could have written $N-\sqrt{N}$ in (148) instead of $N-99$ but we do not need such refinement. And for that matter $N-99$ is extravagantly better than we need.
12.3. Proof of statement (12) of Theorem 2.6.4. Suppose now that $(\mathcal{S}, L, \Theta, e)$ is a self-adjoint linearization of $f \in \operatorname{Mat}_{n}(\mathbb{C}\langle\mathbf{X}\rangle)$ in which case (recall) that we have formulas

$$
\begin{equation*}
\tau_{\mathcal{S}, e} G\left(\Lambda_{0}\right)=S_{\mu_{f}}(\mathbf{z}) \text { and }\left.\tau_{\mathcal{S}, e} F_{N}^{N}\right|_{\mathbf{t}=0}=S_{\mu_{f}^{N}}(\mathbf{z}) \tag{151}
\end{equation*}
$$

by (82) and (125), respectively. Thus, obviously, we have

$$
\left[\left[S_{\mu_{f}^{N}}(\mathbf{z})-S_{\mu_{f}}(\mathbf{z})\right]\right] \leq\left[\left[\left.F_{N}^{N}\right|_{\mathbf{t}=0}-G\left(\Lambda_{0}\right)\right]\right] .
$$

Now fix $p \in[1, \infty)$ arbitrarily. By (148) it follows that

$$
\sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\|I| \geq N-99}} \sqrt{N}\left\|S_{\mu_{f}^{N}}(\mathbf{z})-S_{\mu_{f}}(\mathbf{z})\right\| \|_{2 p, 99}<\infty
$$

Now this last bound holds no matter what strength of repulsion of $\mathbf{z}$ from the real axis we choose, so long as assumption (146) is satisfied. If we choose the repulsion strength strong enough so that $\left\|\mathfrak{G}^{99}\right\|_{2 p}<\infty$, we then reach the desired conclusion

$$
\sup _{N} \bigvee_{\substack{I \in \mathcal{I}_{N} \text { s.t. } \\|I| \geq N-99}} \sqrt{N}\left\|S_{\mu_{f}^{N}}(\mathbf{z})-S_{\mu_{f}}(\mathbf{z})\right\|_{p}<\infty
$$

This finishes the proof of statement (12) and also explains by example how bounds in the norm $\|\cdot\|_{p, k}$ with $k$ independent of $p$ translate to bounds in the standard $L^{p}$ norm $\|\cdot\|_{p}$ provided that the strength of repulsion of $\mathbf{z}$ from the real axis is sufficiently strong, depending on $p$. In the remainder of the proof of Theorem 2.6.4 we will skip over similar details of translation.
12.4. Easy consequences of (148) and (149). Estimates (148) and (149) along with Propositions 11.2.4 and 11.3.3 yield the following bounds:

$$
\begin{align*}
& \sup _{N} \sup _{\substack{I \in \mathcal{I}_{N} \\
|I| \geq N-99}} \sqrt{N}\left|\left\|\left.H_{I}^{N}\right|_{\mathbf{t}=0}-G\left(\Lambda_{0}\right) \mid\right\|_{p, 99}<\infty,\right.  \tag{152}\\
& \left.\sup _{N} \sup _{\substack{I \in \mathcal{I}_{N} \\
|I| \geq N-99}} N\| \| H_{I}^{N}\right|_{\mathbf{t}=0}-G\left(\Lambda_{0}\right)\| \|_{p, 99, \emptyset}<\infty,  \tag{153}\\
& \sup _{N} \sup _{\substack{(I, J) \in \mathcal{I}_{N}^{(2)}}} \bigvee_{j_{1}, j_{2} \in J} \sqrt{N}\left|\left\|\left.R_{I}^{N}\left(j_{1}, j_{2}\right)\right|_{\mathbf{t}=0}-\delta_{j_{1}, j_{2}} G\left(\Lambda_{0}\right)\right\| \|_{p, 99}<\infty\right.  \tag{154}\\
& \left.\sup _{N} \sup _{\substack{(I, J) \in \mathcal{I}_{N}^{(2)}}} \bigvee_{j_{1}, j_{2} \in J} N\| \| R_{I}^{N}\left(j_{1}, j_{2}\right)\right|_{\mathbf{t}=0}-\delta_{j_{1}, j_{2}} G\left(\Lambda_{0}\right) \mid \|_{p, 99, \emptyset}<\infty \tag{155}
\end{align*}
$$

12.5. Application of the secondary trick. Consider the instance of the block Wigner model

$$
\begin{aligned}
& (\underline{\mathcal{S}}, \underline{X}, \underline{\Phi}, \underline{\Psi}, \underline{\Lambda}, \underline{\mathfrak{G}}) \\
= & \left(\underline{\mathcal{S}}, \bigcup_{N} \underline{L}\left(\Xi^{N}\right), \Phi_{\underline{L}}, \Psi_{\underline{L}}, \underline{\Theta}+\diamond_{\mathcal{S}}+\mathbf{z} \underline{e}+\mathrm{it} 1_{\underline{\mathcal{S}}}, 2 \underline{c}_{0}^{2}(1+\llbracket \underline{L}(\Xi) \rrbracket)^{\underline{c}_{1}}(1+1 / \Im \mathbf{z})^{c_{2}}\right)
\end{aligned}
$$

gotten from the underlined SALT block design $\left(\underline{\mathcal{S}}, \underline{L}, \underline{\Theta}+\diamond_{\mathcal{S}}, \underline{e}\right)$. Consider also the ancillary data $\underline{c}_{0}, \underline{c}_{1}, \underline{c}_{2}$ and $\underline{T}$ for the design. By Lemma 8.4.4 we can take $\underline{c}_{0}=3 c_{0}^{2}$, $\underline{c}_{1}=2 c_{1}$ and $\underline{c}_{2}=2 c_{2}$. In particular, we can take $\mathfrak{G}=\frac{3}{2} \mathfrak{G}^{2}$. (Note that (146) checks hypothesis (74) in the underline case.) By (91) and (92) in combination with (148) and (149) we thus obtain estimates

$$
\begin{align*}
& \sup _{N} \sup _{\substack{I \in \mathcal{I}_{N} \\
|I| \geq N-99}} \sqrt{N}\left\|\left|T_{I}^{N}\right|_{\mathbf{t}=0}-G^{\prime}\left(\Lambda_{0}\right)\right\|_{p, 199}<\infty,  \tag{156}\\
& \sup _{N} \sup _{\substack{I \in \mathcal{I}_{N} \\
|I| \geq N-99}} N\left\|\left.| | T_{I}^{N}\right|_{\mathbf{t}=0}-G^{\prime}\left(\Lambda_{0}\right)\right\|_{p, 199, \emptyset}<\infty  \tag{157}\\
& \left.\sup _{N} \sup _{\substack{I \in \mathcal{I}_{N}}} \sqrt{N}\| \| U_{I}^{N}\right|_{\mathbf{t}=0}-\check{G}\left(\Lambda_{0}\right)\| \|_{p, 199}<\infty  \tag{158}\\
& |I| \geq N-99  \tag{159}\\
& \left.\sup _{N} \sup _{\substack{I \in \mathcal{I}_{N} \\
|I| \geq N-99}} N\| \| U_{I}^{N}\right|_{\mathbf{t}=0}-\check{G}\left(\Lambda_{0}\right) \|_{p, 199, \emptyset}<\infty
\end{align*}
$$

We can dispense now with the underlined SALT block design. We just needed it to get the estimates above.
12.6. Proof of statement (13) of Theorem 2.6.4. Using again (151), we see that it is enough to prove for every $p \in[1, \infty)$ that

$$
\sup _{N} N^{3 / 2}\left\|\left.F_{N+1}^{N+1}\right|_{\mathbf{t}=0}-\left.F_{N}^{N}\right|_{\mathbf{t}=0}\right\|_{p, 999}<\infty .
$$

In turn, by estimate (136), it is enough to prove that

$$
\sup _{N} N^{1 / 2}\left\|\left.\operatorname{Link}^{N}\right|_{\mathbf{t}=0}\right\| \|_{p, 999}<\infty .
$$

But the latter follows in a straightforward way from the functional equation (19) along with (148), (152) and (156).
12.7. The last estimate. We pause to explain how to estimate the seminorm $\left\|\|\cdot\|_{p, k, \emptyset}\right.$ applied to the difference between a tensor product of random variables of the form

$$
\left.F_{I}^{N}\right|_{\mathbf{t}=0},\left.\quad T_{I}^{N}\right|_{\mathbf{t}=0},\left.\quad H_{I}^{N}\right|_{\mathbf{t}=0},\left.\quad U_{I}^{N}\right|_{\mathbf{t}=0},\left.\quad R_{I}^{N}(j, j)\right|_{\mathbf{t}=0}
$$

and a corresponding tensor product of random variables of the form

$$
G\left(\Lambda_{0}\right), \quad G^{\prime}\left(\Lambda_{0}\right), \quad \check{G}\left(\Lambda_{0}\right)
$$

It is worthwhile to have a preparatory abstract discussion of the method so that we can skip an unpleasant proliferation of indices later.

Let $A_{1}, \ldots, A_{m} \in \mathcal{S}$ be random and $\sigma(\mathcal{F}, \mathbf{z})$-measurable. Let $B_{1}, \ldots, B_{m} \in \mathcal{S}$ be random and $\sigma(\mathbf{z})$-measurable. Let $k_{1}, \ldots, k_{m}$ be positive integers and put $k=$ $k_{1}+\cdots+k_{m}$. Assume that for every $p \in[1, \infty)$ we have

$$
\bigvee_{i=1}^{m}\left\|A_{i}\right\|_{p, k_{i}} \vee \bigvee_{i=1}^{m}\left\|B_{i}\right\|_{p, k_{i}}<\infty
$$

thus sidestepping all issues of integrability. We now claim that for every $p \in[1, \infty)$, the following holds:

$$
\begin{align*}
& \left\|A_{1} \otimes \cdots \otimes A_{m}-B_{1} \otimes \cdots \otimes B_{m}\right\|_{p, k}  \tag{160}\\
\leq & \prod_{i=1}^{m}\left(\left\|A_{i}\right\|_{m p, k_{i}}+2\left\|A_{i}-B_{i}\right\|_{m p, k_{i}}+\left\|A_{i}-B_{i}\right\|_{m p, k_{i}, \emptyset}\right) \\
& -\prod_{i=1}^{m}\left\|A_{i}\right\|_{m p, k_{i}}-2 \sum_{j=1}^{m} \prod_{i=1}^{m}\left\{\begin{aligned}
\left\|A_{j}-B_{j}\right\|_{m p, k_{j}} & \text { if } i=j, \\
\left\|A_{i}\right\|_{m p, k_{i}} & \text { if } i \neq j .
\end{aligned}\right.
\end{align*}
$$

To prove the claim, we write

$$
A_{i}=A_{i}^{(0)}+A_{i}^{(1)}+A_{i}^{(2)}
$$

where

$$
A_{i}^{(0)}=B_{i} \quad \text { and } \quad A_{i}^{(2)}=\mathfrak{G}^{k_{i}} \mathbb{E}\left(\left(A_{i}-B_{i}\right) / \mathfrak{G}^{k_{i}} \mid \mathbf{z}\right)
$$

Note that by construction

$$
\mathbb{E}\left(A_{i}^{(1)} / \mathfrak{G}^{k_{i}} \mid \mathbf{z}\right)=0
$$

We then have for every $p \in[1, \infty)$ that

$$
\left\|A_{1} \otimes \cdots A_{m}-B_{1} \otimes \cdots \otimes B_{m}\right\|_{p, k, \emptyset} \leq \sum_{\substack{\left(\nu_{1}, \ldots, \nu_{k}\right) \in\{0,1,2\}^{m} \\ \nu_{1}+\cdots+\nu_{m} \geq 2}} \prod_{i=1}^{m}\left\|A_{i}^{\left(\nu_{i}\right)}\right\|_{m p, k_{i}}
$$

after taking into account the most obvious cancellations and applying the Hölder inequality. The claim follows after estimating each term on the right side in evident fashion.
12.8. Proof of statements (14) and (15) of Theorem 2.6.4. Deploying yet again observation (151) and taking into account the integrability conditions (147) above, it suffices to prove that

$$
\sup _{N} N^{2}\left\|\left|\bar{F}_{N}^{N}\right|_{\mathbf{t}=0}-\frac{\operatorname{Bias}^{N}\left(\Lambda_{0}\right)}{N}-G\left(\Lambda_{0}\right)\right\| \|_{p, 9999}<\infty .
$$

Using both Theorem 11.4.2 and (150) above, along with Remark 7.2.5, it suffices to prove

$$
\sup _{N} \bigvee_{j=1}^{N} N\left\|\widehat{\operatorname{Bias}}_{L}^{N}\left(\Lambda_{0}\right)-\left.\operatorname{Corr}_{j}^{N}\right|_{\mathbf{t}=0}\right\| \|_{p, 999}<\infty
$$

Finally, this last bound is obtained by using the general observation (160) in conjunction with the estimates (152), (153), (154), (155), (158) and (159). The proof of Theorem 2.6.4 is complete.

Acknowledgements I thank K. Dykema for teaching me Lemma 4.3.6 and its proof, as well as explaining its application to free semicircular variables. I thank O. Zeitouni for suggesting the use of an interpolation argument (as formalized in Lemma 2.6.7) to reduce Theorem 2.3.6 to Theorem 2.6.4. I thank both of my coauthors A. Guionnet and O. Zeitouni for teaching me much of direct relevance to this paper in the process of writing the book [1]. Especially Guionnet's forceful advocacy of the Schwinger-Dyson equation was influential.

## References

[1] G. Anderson, A. Guionnet and O. Zeitouni. An Introduction to Random Matrices. Cambridge University Press, to appear.
[2] Z. D. Bai and J. W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. Annals Probab. 26(1998), 316-345.
[3] Z. D. Bai and Y. Q. Yin. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. Annals Probab. 16(1988), 1729-1741.
[4] M. Capitaine and C. Donati-Martin. Strong asymptotic freeness for Wigner and Wishart matrices. Indiana Univ. Math. J., 56(2007), 767-803.
[5] J. Cuntz, Simple $C^{*}$-algebras generated by isometries. Comm. Math. Phys. 57(1977), 173-185.
[6] O. N. Feldheim, S. Sodin, A universality result for the smallest eigenvalues of certain sample covariance matrices arXiv:0812.1961v4 [math-ph] 20 Jun 2009
[7] Z. Füredi and J. Komlós, The eigenvalues of random symmetric matrices. Combinatorica 1(1981), 233-241.
[8] U. Haagerup, H. Schultz and S. Thorbjørnsen. A random matrix approach to the lack of projections in $C_{\mathrm{red}}^{*}\left(\mathbb{F}_{2}\right)$. Adv. Math. 204(2006), 1-83.
[9] U. Haagerup and S . Thorbjørnsen. A new application of random matrices: $\operatorname{Ext}\left(C^{*}\left(\mathbb{F}_{2}\right)\right)$ is not a group. Annals Math. 162(2005), 711-775.
[10] W. Hachem, P. Loubaton, J. Najim, Deterministic equivalents for certain functionals of large random matrices. Ann. Appl. Probab. 17(2007), no. 3, 875-930.
[11] W. Hachem, P. Loubaton, J. Najim, A CLT for information-theoretic statistics of gram random matrices with a given variance profile. Ann. Appl. Probab. 18(2008), no. 6, 2071-2130.
[12] J. Helton, R. Rashidi Far, R. Speicher, Operator-valued semicircular elements: solving a quadratic matrix equation with positivity constraints. Int. Math. Res. Not. IMRN 2007, no. 22, Art. ID rnm086.
[13] R. Horn and C. Johnson, Matrix analysis. Corrected reprint of the 1985 original. Cambridge University Press, Cambridge, 1990
[14] C. Male, The norm of polynomials in large random and deterministic matrices. arXiv:1004.4155v4 [math.PR] 3 Mar2011
[15] M. W. Meckes and S. J. Szarek, Concentration for noncommutative polynomials in random matrices. arXiv:1101.1923v1 [math.PR] 10 Jan 2011
[16] G. J. Murphy, $C^{*}$-algebras and operator theory. Academic Press, San Diego, CA, 1990.
[17] A. Nica, R. Speicher, Lectures on the combinatorics of free probability. London Mathematical Society Lecture Note Series, 335. Cambridge University Press, Cambridge, 2006.
[18] B. Simon. Trace Ideals and Their Applications. Second Edition. Mathematical Surveys and Monographs, Volume 120, Amer. Math. Soc., Providence, RI, 2005.
[19] H. Schultz, Non-commutative polynomials of independent Gaussian random matrices. The real and symplectic cases. Probab. Theory Related Fields 131(2005), 261-309.
[20] D. V. Voiculescu, K. J. Dykema, A. Nica. Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. CRM Monograph Series, 1. American Mathematical Society, Providence, RI, 1992.
[21] P. Whittle. Bounds for the moments of linear and quadratic forms in independent variables. Theory Probab. Appl. 5(1960), 303-305.

University of Minnesota, MplS., MN 55455 U.S.A.
E-mail address: gwanders@umn.edu

