Yin's lemma, the quadratic Marcinkiewicz-Zygmund inequality and asymptotic liberation

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Objective of the talk

Our aim is to explain a couple of recent asymptotic freeness results the proofs of which both turn on combinatorial ideas related to the Marcinkiewicz-Zygmund inequality.

We also discuss the general framework of asymptotic liberation and indicate the role there of the quadratic M.-Z. inequality.

We take basic notions of free probability for granted.

Talk outline

(0) Quick preview of main results
(I) Around the Marcinkiewicz-Zygmund inequality
(II) Application: An asymptotic freeness result from EE
(III) Yin's Lemma
(IV) Quadratic inequalities of M.-Z. type
(V) Application: Fake Haar unitaries
(VI) Remarks on asymptotic liberation

Part 0: Quick preview of main results

Preview of preview

We provide statements of the two sample results the talk is focused on.

1. An asymptotic freeness result from EE

2. Fake Haar unitaries

1. An asymptotic freeness result from EE

A. M. Tulino, G. Caire, S. Shamai, S. Verdú, *Capacity of channels with frequency-selective and time-selective fading.* IEEE Trans. Inform. Theory **56** (2010), no. 3, 1187–1215. MR2723670

Setup for the result

$$F_N = \begin{bmatrix} & \vdots & \\ \cdots & \frac{\exp\left(2\pi\sqrt{-1}\,\frac{ij}{N}\right)}{\sqrt{N}} & \cdots \\ & \vdots & \end{bmatrix}$$

(N-by-N discrete Fourier transform matrix)

Setup for result (continued)

X, Y: bounded real random variables

$$X_N = \begin{bmatrix} X_N(1,1) & & \\ & \ddots & \\ & & X_N(N,N) \end{bmatrix}, \text{ (i.i.d. diagonal, } X_N(i,i) \sim X\text{)}$$

$$Y_N = \begin{bmatrix} Y_N(1,1) & & \\ & \ddots & \\ & & Y_N(N,N) \end{bmatrix}, \text{ (i.i.d. diagonal, } Y_N(i,i) \sim Y\text{).}$$

Furthermore, X_N and Y_N are independent.

Statement of the result

Theorem (Special case of Lemma 1, p. 1194 of TCSV)

Let X and Y be bounded real random variables. Let X_N and Y_N be independent N-by-N diagonal random matrices with i.i.d. copies of X and Y, respectively, down the diagonal. Let F_N be the discrete Fourier transform matrix with entries $F_N(i,j) = \zeta_N^{ij}/\sqrt{N}$ ($\zeta_N = \exp(2\pi\sqrt{-1}/N)$). Then X_N and $F_N Y_N F_N^*$ are asymptotically free as $N \to \infty$.

Thus, strikingly, the discrete Fourier transform matrix, in a specialized setting, can do the work of a Haar-distributed unitary. The theorem stated here is only a small part of the work on signal-processing carried out in TCSV.

Remark on proof

The idea of the proof of this result is closely linked with the Marcinkiewicz-Zygmund inequality. We will review the latter, related combinatorics, and then use those ideas to sketch a proof of the theorem stated above.

The proof turns out not to be very delicate. One only need assume that F_N is a random unitary independent of X_N and Y_N such that $\sqrt{N}\sup_N \max_{i,j=1}^N \|F_N(i,j)\|_p < \infty$. Furthermore, less than independence need be assumed of the entries of X_N and Y_N.

2. Fake Haar unitaries

Anderson, G., Farrell, B., *Asymptotically liberating sequences of random unitary matrices*, Advances in Math. **255**(2014), 381–413.

Setup for the result

$$H_N = N\text{-by-}N \text{ complex Hadamard matrix,}$$

i.e.,

- $|H_N(i,j)| = 1$ for $i, j = 1, \ldots, N$.
- $H_N/\sqrt{N}$ is unitary.

E.g., $H_N = F_N$ but there are many more examples.

Setup for the result (continued)

W_N: an N-by-N uniformly distributed

random signed permutation matrix

In other words, W_N factors as a uniformly distributed N-by-N permutation matrix times an independent N-by-N diagonal matrix with i.i.d. diagonal entries, each uniform in $\{\pm 1\}$.

Setup for the result (continued)

$$U_N = \frac{W_N H_N W_N^*}{\sqrt{N}},$$

(a random N-by-N unitary)

The only randomness here is coming from W_N, which has a discrete distribution.

Setup for the result (continued)

X_N, Y_N: N-by-N hermitian matrices

Assume that X_N and Y_N have L^2 operator norms bounded uniformly in N.

Assume that the E.S.D.'s of X_N and Y_N converge weakly to limits.

Statement of the result

Theorem (A.-Farrell, Corollary 3.5)

Let X_N and Y_N be N-by-N hermitian matrices with eigenvalues bounded uniformly in N. Assume that the empirical distributions of eigenvalues of X_N and Y_N converge weakly. Let H_N be an N-by-N complex Hadamard matrix. Let W_N be a uniformly distributed N-by-N signed permutation matrix. Let $U_N = \frac{W_N H_N W_N^*}{\sqrt{N}}$. Then the pair $(X_N, U_N Y_N U_N^*)$ is asymptotically free as $N \to \infty$.

In this situation, with V_N a Haar-distributed random N-by-N unitary, Voiculescu's classical result is that X_N and $V_N Y_N V_N^*$ are asymptotically free. Thus U_N is a "fake" Haar unitary.

Remark on proof

The preceding result is closely linked with a quadratic variant of the Marcinkiewicz-Zygmund inequality. The main point of the combinatorics is Yin's Lemma.

As remarked above in connection with the TCSV result, the proof is not too delicate, and it is similarly possible to relax assumptions on U_N considerably.

Remark on generalizations

In the classical setup of Voiculescu, one can "asymptotically liberate" several hermitian matrices

$$X_N^{(1)}, X_N^{(2)}, X_N^{(3)} \dots, X_N^{(n)}$$

with eigenvalues bounded uniformly and each with weakly converging E.S.D. by using independent Haar-distributed N-by-N random unitaries

$$V_N^{(1)} = I_N, V_N^{(2)}, V_N^{(3)}, \dots, V_N^{(n)},$$

i.e.,the family

$$\{V_N^{(i)} X_N^{(i)} V_N^{(i)*}\}_{i=1}^n$$

is asymptotically free.

Remark on generalizations (concluded)

Similarly, in the A.-Farrell setup (see Cor. 3.2) it is shown, for example, that with

- H_N: an N-by-N deterministic complex Hadamard matrix,
- W_N: a uniformly distributed N-by-N signed permutation matrix, and
- $D_N^{(3)}, \ldots, D_N^{(n)}$: independent diagonal matrices independent of W_N with i.i.d. $\{\pm 1\}$-Bernoulli diagonal entries,

the family

$$I_N, \frac{H_N}{\sqrt{N}} W_N, D_N^{(3)} \frac{H_N}{\sqrt{N}} W_N, \ldots, D^{(n)} \frac{H_N}{\sqrt{N}} W_N$$

is "asymptotically liberating."

The general framework of AF provides many more examples like this but also is far from exhausting the possibilities.

Part I: Around the Marcinkiewicz-Zygmund inequality

The Khinchin inequality

Notation: $\|Z\|_p = (\mathbf{E}|Z|^p)^{1/p}$ for $p \in [1, \infty)$, $\|Z\|_\infty = \text{ess. sup}|Z|$.

$$\epsilon_1, \ldots, \epsilon_N \in \{\pm 1\} \text{ (i.i.d. uniform signs)}$$

$$a_1, \ldots, a_N \text{ (real or complex constants)}$$

Theorem (Khinchin inequality (1923))

Assumptions and notation are as above. For $p \in [1, \infty)$ one has

$$\left(\sum_{i=1}^{N} |a_i|^2\right)^{1/2} A_p \leq \left\| \sum_{i=1}^{N} a_i \epsilon_i \right\|_p \leq B_p \left(\sum_{i=1}^{N} |a_i|^2\right)^{1/2}$$

where the constants A_p and B_p depend only on p.

Littlewood may also be implicated here. Szarek and Haagerup found best constants in the real case.

The Marcinkiewicz-Zygmund inequality

The following is a far-reaching generalization of the relation

$$\mathbf{Var}(X_1 + \cdots + X_N) = \sum_{i=1}^{N} \mathbf{Var}(X_i)$$

satisfied by independent square-integrable random variables.

Theorem (Marcinkiewicz-Zygmund inequality (1937))

Fix $p \in [1, \infty)$. For independent $\mathbb{C}$-valued random variables $X_1, \dots, X_N$, each with finite L^p-norm and of mean zero, one has

$$A_p \left\| \left(\sum_{i=1}^{N} |X_i|^2 \right)^{1/2} \right\|_p \leq \left\| \sum_{i=1}^{N} X_i \right\|_p \leq B_p \left\| \left(\sum_{i=1}^{N} |X_i|^2 \right)^{1/2} \right\|_p$$

for positive constants A_p and B_p depending only on p.

Remark

A typical strategy for proving M.-Z. is to derive it from the Khinchin inequality, by reducing to the case of symmetrically distributed random variables and conditioning on absolute value, e.g., see the probability text by Chow and Teicher (MR1476912).

M.-Z. simplified upper bound

Actually, we will not need the M.-Z. inequality at full strength.

Corollary (M.-Z. simplified upper bound)

Fix $p \in [2, \infty)$. For independent $\mathbb{C}$-valued random variables $X_1, \ldots, X_N$, each with finite L^p-norm and of mean zero, one has

$$\left\| \sum_{i=1}^{N} X_i \right\|_p \leq B_p \left(\sum_{i=1}^{N} \|X_i\|_p^2 \right)^{1/2}$$

with B_p as in the theorem.

Proof, modulo the theorem + Minkowski

$$\left\| \left(\sum_{i=1}^{N} |X_i|^2 \right)^{1/2} \right\|_p^2 = \left\| \sum_{i=1}^{N} |X_i|^2 \right\|_{p/2} \leq \sum_{i=1}^{N} \left\| |X_i|^2 \right\|_{p/2} = \sum_{i=1}^{N} \|X_i\|_p^2.$$

Combinatorial proof of M.-Z. simplified upper bound

We are more interested in the folkloric combinatorial proof one can give of the corollary for $p = 2k$ than the corollary itself.

This proof and its setup are what we are going to generalize in various ways and relate to asymptotic freeness.

Set partitions (interlude)

$\mathrm{Part}_n =$ family of partitions of the set $\langle n \rangle$.

For example, $\{\{1,2\},\{3,4,7\},\{6\},\{5,8,9,10\}\} \in \mathrm{Part}_{10}$.

$\mathrm{Part}_n^\chi = \{\mathcal{P} \in \mathrm{Part}_n \mid \mathcal{P} \cap \{\{1\},\ldots,\{n\}\} = \emptyset\}$.
(χ is meant to be an allusion to Khinchin.) ("no singletons")

For example, $\{\{1,2\},\{3,4,5\},\{6,7\}\} \in \mathrm{Part}_7^\chi$.

For a nonexample, $\{\{1,2\},\{3\},\{4,5,6,7\}\} \in \mathrm{Part}_7 \setminus \mathrm{Part}_7^\chi$.

$\mathrm{Part}_{2k}^{\chi\chi} = \{\mathcal{P} \in \mathrm{Part}_{2k}^\chi \mid \mathcal{P} \cap \{\{1,2\},\ldots,\{2k-1,2k\}\} = \emptyset\}$.
("no singletons nor any special doubletons of form $\{2\alpha-1, 2\alpha\}$")
(not needed now—preview of quadratic case)

Combinatorial proof of MZ

For simplicity we assume that $X_1, \dots, X_N$ are real.

Notation: $\langle n \rangle = \{1, \dots, n\}$.

First steps:

$$\|X_1 + \cdots + X_N\|_{2k}^{2k} = \mathbf{E}(X_1 + \cdots + X_N)^{2k} = \sum_{\mathbf{i}:\langle 2k \rangle \to \langle N \rangle} \mathbf{E} \prod_{\alpha=1}^{2k} X_{\mathbf{i}(\alpha)}$$

$$= \sum_{\substack{\mathbf{i}:\langle 2k \rangle \to \langle N \rangle \\ \text{s.t. } \mathbf{i} \text{ takes no value} \\ \text{exactly once}}} \mathbf{E} \prod_{\alpha=1}^{2k} X_{\mathbf{i}(\alpha)} \le \sum_{\substack{\mathbf{i}:\langle 2k \rangle \to \langle N \rangle \\ \text{s.t. } \mathbf{i} \text{ takes no value} \\ \text{exactly once}}} \prod_{\alpha=1}^{2k} \left\| X_{\mathbf{i}(\alpha)} \right\|_{2k}$$

Combinatorial proof of M.-Z. (continued)

$$\|X_1 + \cdots + X_N\|_{2k}^{2k} \overset{\text{carry down}}{\leq} \sum_{\substack{\mathbf{i}:\langle 2k\rangle \to \langle N\rangle \\ \text{s.t. } \mathbf{i} \text{ takes no value} \\ \text{exactly once}}} \prod_{\alpha=1}^{2k} \left\|X_{\mathbf{i}(\alpha)}\right\|_{2k}$$

$$\overset{\text{use definition of } \mathrm{Part}_{2k}^{\chi}}{\leq} \sum_{\mathcal{P} \in \mathrm{Part}_{2k}^{\chi}} \sum_{\substack{\mathbf{i}:\langle 2k\rangle \to \langle N\rangle \text{ s.t.} \\ \mathbf{i} \text{ is constant} \\ \text{on blocks of } \mathcal{P}}} \prod_{\alpha=1}^{2k} \left\|X_{\mathbf{i}(\alpha)}\right\|_{2k}$$

Combinatorial proof of M.-Z. (concluded)

Simple facts to keep in mind:

$\left(\sum_{i=1}^n |z_i|^{2k}\right)^{\frac{1}{2k}} \leq \left(\sum_{i=1}^n |z_i|^2\right)^{1/2}$

Notation: $|S|$ = cardinality of S

$|\mathrm{Part}_n^\chi| \leq |\{\mathbf{i} : \langle n\rangle \to \langle\lfloor \frac{n}{2}\rfloor\rangle\}| \leq \left(\frac{n}{2}\right)^n.$

$$\|X_1 + \cdots + X_N\|_{2k}^{2k} \overset{\text{carry down}}{\leq} \sum_{\mathcal{P}\in\mathrm{Part}_{2k}^\chi} \sum_{\substack{\mathbf{i}:\langle 2k\rangle\to\langle N\rangle \text{ s.t.} \\ \mathbf{i} \text{ is constant} \\ \text{on blocks of } \mathcal{P}}} \prod_{\alpha=1}^{2k} \left\|X_{\mathbf{i}(\alpha)}\right\|_{2k}$$

$$= \sum_{\mathcal{P}\in\mathrm{Part}_{2k}^\chi} \prod_{A\in\mathcal{P}} \sum_{i=1}^N \|X_i\|_{2k}^{|A|} \leq k^{2k} \left(\sum_{i=1}^N \|X_i\|_{2k}^2\right)^k.$$

□

Summary

The key idea above boils down to the estimate

$$\left| \sum_{\substack{\mathbf{i}:\langle n\rangle \to \langle N\rangle \\ \text{s.t. } \mathbf{i} \text{ takes no value} \\ \text{exactly once}}} \prod_{\alpha=1}^{n} z_{\mathbf{i}(\alpha)}^{(\alpha)} \right| \le \left(\frac{n}{2}\right)^n \prod_{\alpha=1}^{n} \left(\sum_{i=1}^{N} \left| z_i^{(\alpha)} \right|^2 \right)^{1/2},$$

which generalizes the Cauchy-Schwarz inequality.

The special case

$$\left| \left\{ \substack{\mathbf{i}:\langle n\rangle \to \langle N\rangle \\ \text{s.t. } \mathbf{i} \text{ takes no value} \\ \text{exactly once}} \right\} \right| \le \left(\frac{n}{2}\right)^n N^{n/2}$$

is especially important in the sequel.

Part II:

Application:
an asymptotic freeness result from EE

An asymptotic freeness result derived with M.-Z. ideas

Recently (in effect) the M.-Z. circle of ideas was used in the following paper to get an asymptotic freeness result:

A. M. Tulino, G. Caire, S. Shamai, S. Verdú, *Capacity of channels with frequency-selective and time-selective fading.* IEEE Trans. Inform. Theory **56** (2010), no. 3, 1187–1215. MR2723670

Theorem (Special case of Lemma 1, p. 1194 of TCSV)

Let X and Y be bounded real random variables. Let X_N and Y_N be independent N-by-N diagonal random matrices with i.i.d. copies of X and Y, respectively, down the diagonal. Let F_N be the discrete Fourier transform matrix with entries $F_N(i,j) = \zeta_N^{ij}/\sqrt{N}$ ($\zeta_N = \exp(2\pi\sqrt{-1}/N)$). Then X_N and $F_N Y_N F_N^*$ are asymptotically free as $N \to \infty$.

Sketch of proof of TCSV

Let

$$Z = (Z^{(1)}, \ldots, Z^{(2k)})$$

be a real random vector with bounded entries all of mean zero, e.g.,

$$Z^{(1)} = X^{m_1} - \mathbf{E}X^{m_1}, \;\; Z^{(2)} = Y^{m_2} - \mathbf{E}Y^{m_2}, \ldots$$

and let

$$C = \max_i \left\| Z^{(i)} \right\|_\infty .$$

Sketch of proof of TCSV (continued)

Let

$$Z_N^{(1)}, \ldots, Z_N^{(2k)}$$

be N-by-N diagonal random matrices with expected traces equal to zero such that the random vectors

$$(Z_N^{(1)}(i,i), \ldots, Z_N^{(2k)}(i,i)) \text{ for } i = 1, \ldots, N$$

are independent copies of Z, e.g.,

$$X_N^{m_1} - (\mathbf{E}X^{m_1})I_N, Y_N^{m_2} - (\mathbf{E}Y^{m_1})I_N, \ldots$$

Sketch of proof of TCSV (continued)

Notation: $\mathrm{tr} X = \sum X(i,i)$ (un-normalized trace)

After making the standard reductions (familiar to this crowd) it will be more than enough to prove that

$$\left| \mathbf{E} \mathrm{tr} Z_N^{(1)} F_N Z_N^{(2)} F_N^* \cdots Z_N^{(2k-1)} F_N Z_N^{(2k)} F_N^* \right| \leq (kC)^{2k}.$$

This one can prove by counting and bounding the nonzero terms.

Sketch of proof of TCSV (continued)

Let us open the brackets to get a big sum:

$$\sum_{\mathbf{i}:\langle 2k\rangle\to\langle N\rangle}$$

$$\mathbf{E} Z_N^{(1)}(\mathbf{i}(1),\mathbf{i}(1))\cdots Z_N^{(2k)}(\mathbf{i}(2k),\mathbf{i}(2k))$$

$$\times F_N(\mathbf{i}(1),\mathbf{i}(2))F_N^*(\mathbf{i}(2),\mathbf{i}(3))\cdots F_N(\mathbf{i}(2k-1),\mathbf{i}(2k))F_N^*(\mathbf{i}(2k),\mathbf{i}(1))$$

Here we have N^{2k} terms each of which is bounded in absolute value by C^{2k}/N^k. This gives a very bad upper bound $C^{2k}N^k$.

Sketch of proof (concluded)

But many terms in the big sum are obliged to vanish.

More precisely, if $\mathbf{i} : \langle 2k \rangle \to \langle N \rangle$ is a function which takes some particular value exactly once, then

$$\mathbf{E} \prod_{\alpha=1}^{2k} Z_N^{(\alpha)}(\mathbf{i}(\alpha), \mathbf{i}(\alpha)) = 0$$

because the diagonal entries of the $Z_N^{(\alpha)}$ are of mean zero, and independent if they fall in different rows.

Thus there are at most

$$|\{\mathbf{i} : \langle 2k \rangle \to \langle N \rangle \mid \mathbf{i} \text{ takes no values exactly once}\}| \leq k^{2k} N^k$$

nonzero terms in the big sum each bounded in absolute value by C^{2k}/N^k, whence the result. □

Remarks

(i) The proof from TCSV which we paraphrased above is not at all delicate. One could replace the sequence $\{F_N\}_{N=1}^\infty$ by any sequence $\{U_N\}_{N=1}^\infty$ of deterministic unitaries for which $\max_{i,j=1}^N |U_N(i,j)|$ does not grow too fast as a function of N. In particular, F_N could be replaced by $H_N/\sqrt{N}$ where H_N is any N-by-N complex Hadamard matrix.

(ii) In TCSV one assumes that X_N and Y_N are independent but in the proof sketched above, less need be assumed, namely, it is enough merely that the random vectors $(X_N(i,i), Y_N(i,i))$ for $i = 1, \ldots, N$ be independent copies of (X, Y). It is not necessary to assume that X and Y are independent.

(iii) In one important respect TCSV is more general: the assumption of independence along the diagonal is weakened to stationarity of diagonal entries for one of X_N or Y_N. This is a very natural type of generalization which we do not consider in this talk.

Part III:
Yin's Lemma

Partitions (reminder)

Part_n = family of partitions of the set $\langle n \rangle$.

For example, $\{\{1,2\},\{3,4,7\},\{6\},\{5,8,9,10\}\} \in \mathrm{Part}_{10}$.

$\mathrm{Part}_n^\chi = \{\mathcal{P} \in \mathrm{Part}_n \mid \mathcal{P} \cap \{\{1\},\dots,\{n\}\} = \emptyset\}$.
(χ is meant to be an allusion to Khinchin.) ("no singletons")

$\mathrm{Part}_{2k}^{\chi\chi} = \{\mathcal{P} \in \mathrm{Part}_{2k}^\chi \mid \mathcal{P} \cap \{\{1,2\},\dots,\{2k-1,2k\}\} = \emptyset\}$.
("no singletons nor any special doubletons of form $\{2\alpha - 1, 2\alpha\}$")

For example, $\{\{2,3\},\{4,5\},\{6,7\},\{1,8\}\} \in \mathrm{Part}_8^{\chi\chi}$.
For nonexample, $\{\{1,2\},\{4,5\},\{6,7\},\{3,8\}\} \in \mathrm{Part}_8^\chi \setminus \mathrm{Part}_8^{\chi\chi}$.

Yin's Lemma (statement)

Reminder: $[\![A]\!]_2 = \sqrt{\mathrm{tr} AA^*}$.

Lemma (Yin's Lemma)

Let $A^{(1)}, \ldots, A^{(k)}$ be N-by-N matrices. For $\mathcal{P} \in \mathrm{Part}_{2k}^{\chi\chi}$ one has

$$\left| \sum_{\substack{\mathbf{i} : \langle 2k \rangle \to \langle N \rangle \text{ s.t. } \mathbf{i} \text{ is} \\ \text{constant on blocks of } \mathcal{P}}} \prod_{\alpha=1}^{k} A^{(\alpha)}(\mathbf{i}(2\alpha - 1), \mathbf{i}(2\alpha)) \right| \leq \prod_{\alpha=1}^{k} \left[\!\left[A^{(\alpha)} \right]\!\right]_2 .$$

The preceding is a specialization (using quite different notation) of Lemma 3.4 from

Yin, Y. Q. *Limiting spectral distribution for a class of random matrices.* J. Multivariate Anal. **20**(1986), 50–68. MR0862241

Yin's paper gets a lot of citations for its main result concerning $X_N T_N X_N^*$ with minimal moment assumptions (not for Lemma 3.4).

Example

For example, for

$$k = 5 \text{ and } \mathcal{P} = \{\{1,3,5\},\{2,7\},\{4,8,9\},\{6,10\}\} \in \mathrm{Part}_{10}^{\chi\chi},$$

the lemma says that

$$\left| \sum_{i,j,\ell,m=1}^{N} A^{(1)}(i,j) A^{(2)}(i,\ell) A^{(3)}(i,m) A^{(4)}(j,\ell) A^{(5)}(\ell,m) \right|$$

$$\leq [\![A^{(1)}]\!]_2 [\![A^{(2)}]\!]_2 [\![A^{(3)}]\!]_2 [\![A^{(4)}]\!]_2 [\![A^{(5)}]\!]_2.$$

Enhanced Yin's Lemma

For application to study of asymptotic liberation we will need a cheap enhancement.

Lemma (Enhanced Yin's Lemma)

Let $A^{(1)}, \ldots, A^{(k)}$ be N-by-N matrices with vanishing traces. For $\mathcal{P} \in \mathrm{Part}_{2k}^{\chi}$ one has

$$\left| \sum_{\substack{\mathbf{i} : \langle 2k \rangle \to \langle N \rangle \text{ s.t. } \mathbf{i} \text{ is} \\ \text{constant on blocks of } \mathcal{P}}} \prod_{\alpha=1}^{k} A^{(\alpha)}(\mathbf{i}(2\alpha - 1), \mathbf{i}(2\alpha)) \right| \leq \prod_{\alpha=1}^{k} \left[\!\left[A^{(\alpha)} \right]\!\right]_2.$$

Proof This is a very cheap enhancement indeed. In the cases $\mathcal{P} \in \mathrm{Part}_{2k}^{\chi} \setminus \mathrm{Part}_{2k}^{\chi\chi}$ not already covered by Yin's Lemma as originally stated, the left side simply vanishes! □

Yin's Lemma (generalization permitting inductive proof)

The following is a mild generalization of Yin's Lemma 3.4.

Lemma (Generalized Yin's Lemma)

Let S be a finite set equipped with two partitions $\mathcal{P}$ and $\mathcal{Q}$. Assume that every block of $\mathcal{P}$ meets at least two blocks of $\mathcal{Q}$. For each $B \in \mathcal{Q}$ let $f_B : \langle N \rangle^B \to \mathbb{C}$ be a function. Then one has

$$\left| \sum_{\substack{\text{functions } \mathbf{i}:S \to \langle N \rangle \\ \text{s.t. } \mathbf{i} \text{ is constant on} \\ \text{each block } A \in \mathcal{P}}} \prod_{B \in \mathcal{Q}} f_B(\mathbf{i}|_B) \right|^2 \leq \prod_{B \in \mathcal{Q}} \sum_{\substack{\text{functions } \mathbf{j}:B \to \langle N \rangle \\ \text{s.t. } \mathbf{j} \text{ is constant on } A \cap B \\ \text{for each block } A \in \mathcal{P}}} |f_B(\mathbf{j})|^2.$$

To recover Yin's Lemma in the form stated above, take $\mathcal{P} \in \mathrm{Part}_{2k}^{\chi\chi}$, $\mathcal{Q} = \{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}$ and $A^{(\alpha)}(\mathbf{i}(2\alpha-1),\mathbf{i}(2\alpha)) = f_{\{2\alpha-1,2\alpha\}}(\mathbf{i}|_{\{2\alpha-1,2\alpha\}})$.

Generic example

$$S = \{1,2,3,4,5,6,7,8,9\} = \langle 9 \rangle,$$

$$\mathcal{P} = \{A_1, A_2, A_3, A_4\}, \quad \mathcal{Q} = \{B_1, B_2, B_3, B_4, B_5\},$$

where

$$(A_1, A_2, A_3, A_4) = (\{1,2,7\}, \{3,5\}, \{4,8\}, \{6,9\}),$$

$$(B_1, B_2, B_3, B_4, B_5) = (\{1\}, \{2,3,4\}, \{5,6\}, \{7,8\}, \{9\}).$$

	A_1	A_2	A_3	A_4
B_1	$*$			
B_2	$*$	$*$	$*$	
B_3		$*$		$*$
B_4	$*$		$*$	
B_5				$*$

(Verification of hypothesis–each column has at least two $*$'s)

Generic example (concluded)

$$(i, i, j, k, j, \ell, i, k, \ell)$$

is the general example of a 9-tuple constant on blocks of $\mathcal{P}$.

$$(a, b, b, b, c, c, d, d, e)$$

is the general example of a 9-tuple constant on blocks of $\mathcal{Q}$.
The specialization of Yin's Lemma in the present case is then

$$\left| \sum_{i,j,k,\ell=1}^{N} a(i)b(i,j,k)c(j,\ell)d(i,k)e(\ell) \right|^2$$

$$\leq \sum_{i=1}^{N} |a(i)|^2 \cdot \sum_{i,j,k=1}^{N} |b(i,j,k)|^2 \cdot \sum_{i,j=1}^{N} |c(i,j)|^2 \cdot \sum_{i,j=1}^{N} |d(i,j)|^2 \cdot \sum_{i=1}^{N} |e(i)|^2.$$

Gist of the proof of generalized Yin's Lemma

The idea of the proof is to proceed by induction on $|\mathcal{P}|$, with the inductive step accomplished by "summing out an index." We illustrate this procedure on the generic example.

Here we will "sum out" i. Let

$$\begin{aligned}
\hat{a}() &= \left(\sum_{i=1}^{N} |a(i)|^2\right)^{1/2} \quad \text{(a constant)}, \\
\hat{b}(j,k) &= \left(\sum_{i=1}^{N} |b(i,j,k)|^2\right)^{1/2}, \\
\hat{c}(j,\ell) &= |c(j,\ell)|, \\
\widehat{d}(k) &= \left(\sum_{i=1}^{N} |d(i,k)|^2\right)^{1/2}, \quad \hat{e}(\ell) = |e(\ell)|.
\end{aligned}$$

Gist of proof (continued)

Recall that $\left(\sum_{i=1}^n |z_i|^{2k}\right)^{\frac{1}{2k}} \leq \left(\sum_{i=1}^n |z_i|^2\right)^{1/2}$.
We also use the Hölder inequality.

$$\left|\sum_{i,j,k,\ell=1}^{N} a(i)b(i,j,k)c(j,\ell)d(i,k)e(\ell)\right|^2$$

$$\leq \left(\hat{a}() \sum_{j,k,\ell=1}^{N} \hat{b}(j,k)\hat{c}(j,\ell)\widehat{d}(k)\hat{e}(\ell)\right)^2$$

$$\leq \hat{a}()^2 \cdot \sum_{i,j=1}^{N} \hat{b}(i,j)^2 \cdot \sum_{i,j=1}^{N} \hat{c}(i,j)^2 \cdot \sum_{i=1}^{N} \widehat{d}(i)^2 \cdot \sum_{i=1}^{N} \hat{e}(i)^2$$

$$= \sum_{i=1}^{N} |a(i)|^2 \cdot \sum_{i,j,k=1}^{N} |b(i,j,k)|^2 \cdot \sum_{i,j=1}^{N} |c(i,j)|^2 \cdot \sum_{i,j=1}^{N} |d(i,j)|^2 \cdot \sum_{i=1}^{N} |e(i)|^2.$$

Gist of proof (concluded)

To finish off this explanation, let us make explicit which case of Yin's Lemma we are using to achieve the induction.

$$\left(\hat{a}() \sum_{j,k,\ell=1}^{N} \hat{b}(j,k)\hat{c}(j,\ell)\widehat{d}(k)\hat{e}(\ell) \right)^2$$

$$\leq \hat{a}()^2 \cdot \sum_{i,j=1}^{N} \hat{b}(i,j)^2 \cdot \sum_{i,j=1}^{N} \hat{c}(i,j)^2 \cdot \sum_{i=1}^{N} \widehat{d}(i)^2 \cdot \sum_{i=1}^{N} \hat{e}(i)^2$$

The diagram $\begin{matrix} j & k & j & \ell & k & \ell \\ b & b & c & c & d & e \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$ shows we are using the instance

$$\widehat{\mathcal{P}} = \{\{1,3\},\{2,5\},\{4,6\}\}, \quad \widehat{\mathcal{Q}} = \{\{1,2\},\{3,4\},\{5\},\{6\}\}$$

of generalized Yin's Lemma, where clearly $|\widehat{\mathcal{P}}| < |\mathcal{P}|$. □

Part IV:
The quadratic M.-Z. inequality

The quadratic M.-Z. inequality

There are many results of quadratic M.-Z. type, some quite sophisticated. They are in common use in RMT.

For example, the Hanson-Wright inequality has lately been much "in the news" in connection with universality.

Hanson-Wright inequality

Notation: For a matrix A with singular values μ_i, let
$[\![A]\!]_2 = \left(\sum \mu_i^2\right)^{1/2}$ (Hilbert-Schmidt norm) and
$[\![A]\!]_\infty = [\![A]\!] = \max \mu_i$ (operator norm). Also: $\wedge = \min$

Theorem (Hanson-Wright inequality)

Fix a finite positive constant K. Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent components X_i which satisfy $\sup_{p \geq 1} \frac{\|X_i\|_p}{\sqrt{p}} \leq K$ and $\mathbf{E}X_i = 0$. Let $A \neq 0$ be an n-by-n matrix with real entries. Then, for every $t \geq 0$,

$$\Pr\left(|X^{\mathrm{T}}AX - \mathbf{E}X^{\mathrm{T}}AX| > t\right) \leq 2\exp\left(-c\left(\frac{t^2}{K^4[\![A]\!]_2^2} \wedge \frac{t}{K^2[\![A]\!]}\right)\right)$$

for an absolute constant c.

See `arXiv:1306.2872` by Rudelson-Vershynin for a proof using the standard modern toolkit of large deviations, and background.

Whittle's result

In this talk we have less sophisticated interests. We just need the quadratic analogue of the M.-Z. simplified upper bound. The following paper supplies such a result. I do not know if it is the oldest such paper.

Whittle, P., *Bounds for the moments of linear and quadratic forms in independent variables.* (Russian summary) Teor. Verojatnost. i Primenen. **5**(1960) 331–335. transl. Theory Probab. Appl. **5**(1960) 303–305. MR0133849

Whittle's result

Notation: Let $\Theta(s) = \frac{2^{s/2}}{\sqrt{\pi}}\Gamma\left(\frac{s+1}{2}\right)$ for $s \geq 0$.

Theorem (Whittle (1960))

Fix $p \in [2, \infty)$. Let $X_1, \ldots, X_n$ be independent real random variables in L^{2p} and of mean zero. Let $A \in \mathrm{Mat}_n$ be a matrix with real entries. Then we have

$$\begin{aligned} &\left\| \sum_{i,j=1}^{N} A(i,j)(X_i X_j - \mathbf{E} X_i X_j) \right\|_p \\ \leq \quad & 2^3 \Theta(p)^{\frac{1}{p}} \Theta(2p)^{\frac{1}{2p}} \left(\sum_{i,j=1}^{N} A(i,j)^2 \|X_i\|_{2p}^2 \|X_j\|_{2p}^2 \right)^{1/2}. \end{aligned}$$

Since $\sup_{s \geq 2} \frac{\Theta(s)^{\frac{1}{s}}}{\sqrt{s}} \leq 1$, the Whittle bound simplifies nicely.

Remarks

(i) Whittle gave a version of the M.-Z. simplified upper bound with explicit constant of a similar form:

$$\left.\begin{array}{r} p \geq 2 \text{ and} \\ X_1, \ldots, X_N \in L^p \\ \text{of mean zero} \end{array}\right\} \Rightarrow \left\| \sum_{i=1}^{N} X_i \right\|_p \leq 2\Theta(p)^{\frac{1}{p}} \left(\sum_{i=1}^{N} \|X_i\|_p^2 \right)^{1/2}$$

(ii) Whittle derived these results using sharp upper constants for the Khinchin inequality; the latter are also derived in this very interesting (short!) paper.

Combinatorial proof of Whittle's result for $p = 2k$

To motivate Yin's Lemma given above, we sketch a combinatorial proof of Whittle's result in the case $p = 2k$ (with a bad constant).

Without loss of generality we may assume that

$$\|X_i\|_{4k} = \frac{1}{\sqrt{2}} \text{ for } i = 1, \ldots, N,$$

in which case it is enough to show that

$$\left\| \sum_{i,j=1}^{N} A(i,j)(X_i X_j - \mathbf{E} X_i X_j) \right\|_{2k} \leq c [\![A]\!]_2$$

where c is a constant depending only on k.

For simplicity, i.e., out of laziness, we also assume that $X_1, \ldots, X_N$ are i.i.d., which does entail loss of generality. The loss can in principle be recouped with more "elbow grease."

Combinatorial proof of Whittle (continued)

For $\mathbf{i} : \langle 4k \rangle \to \langle N \rangle$ let

$$\varphi(\mathbf{i}) = \mathbf{E} \prod_{\alpha=1}^{2k} \left(X_{\mathbf{i}(2\alpha-1)} X_{\mathbf{i}(2\alpha)} - \mathbf{E} X_{\mathbf{i}(2\alpha-1)} X_{\mathbf{i}(2\alpha)} \right).$$

Then

$$|\varphi(\mathbf{i})| \leq 1, \ \varphi(\mathbf{i}) \text{ depends only on } \pi(\mathbf{i}) \in \mathrm{Part}_{4k},$$

and

$$\left\| \sum_{i,j=1}^{N} A(i,j)(X_i X_j - \mathbf{E} X_i X_j) \right\|_{2k}^{2k}$$

$$= \sum_{\mathbf{i}:\langle 4k \rangle \to \langle N \rangle} \varphi(\mathbf{i}) \prod_{\alpha=1}^{2k} A(\mathbf{i}(2\alpha-1), \mathbf{i}(2\alpha)).$$

Combinatorial proof of Whittle (continued)

Suppose that $\mathbf{i}$ takes some value exactly once, say

$$\mathbf{i}(4k) \notin \{\mathbf{i}(\beta) \mid \beta \in \langle 4k-1 \rangle\}.$$

Then

$$\begin{aligned} & \varphi(\mathbf{i}) \\ = \; & \mathbf{E}\left[\left(\prod_{\alpha=1}^{2k-1}\left(X_{\mathbf{i}(2\alpha-1)}X_{\mathbf{i}(2\alpha)} - \mathbf{E}X_{\mathbf{i}(2\alpha-1)}X_{\mathbf{i}(2\alpha)}\right)\right) X_{\mathbf{i}(4k-1)}X_{\mathbf{i}(4k)}\right] \\ = \; & \mathbf{E}\left[\left(\prod_{\alpha=1}^{2k-1}\left(X_{\mathbf{i}(2\alpha-1)}X_{\mathbf{i}(2\alpha)} - \mathbf{E}X_{\mathbf{i}(2\alpha-1)}X_{\mathbf{i}(2\alpha)}\right)\right) X_{\mathbf{i}(4k-1)}\right] \mathbf{E}X_{\mathbf{i}(4k)} \\ = \; & 0. \end{aligned}$$

Upshot: $\varphi(\mathbf{i}) \neq 0 \Rightarrow \pi(\mathbf{i}) \in \mathrm{Part}_{4k}^{\chi}$ where (recall)

$$\mathrm{Part}_n^{\chi} = \{\mathcal{P} \in \mathrm{Part}_n \mid \mathcal{P} \text{ has no singleton members}\}.$$

Combinatorial proof of Whittle (continued)

Now suppose $\pi(\mathbf{i}) \in \mathrm{Part}_{4k}$ has a block of the form $\{2\alpha-1, 2\alpha\}$. By symmetry we might as well assume that

$$\mathbf{i}(4k-1) = \mathbf{i}(4k) \notin \{\mathbf{i}(\beta) \mid \beta \in \langle 4k-2 \rangle\}.$$

Then

$$\begin{aligned}
\varphi(\mathbf{i}) &= \mathbf{E}\left[\left(\prod_{\alpha=1}^{2k-1}\left(X_{\mathbf{i}(2\alpha-1)}X_{\mathbf{i}(2\alpha)} - \mathbf{E}X_{\mathbf{i}(2\alpha-1)}X_{\mathbf{i}(2\alpha)}\right)\right)\right] \\
&\quad \times \mathbf{E}\left(X_{\mathbf{i}(4k-1)}X_{\mathbf{i}(4k)} - \mathbf{E}X_{\mathbf{i}(4k-1)}X_{\mathbf{i}(4k)}\right) \\
&= 0.
\end{aligned}$$

Upshot: $\varphi(\mathbf{i}) \neq 0 \Rightarrow \pi(\mathbf{i}) \in \mathrm{Part}_{4k}^{\chi\chi}$, where (recall)

$$\mathrm{Part}_{2\ell}^{\chi\chi} = \{\mathcal{P} \in \mathrm{Part}_{2\ell}^{\chi} \mid \mathcal{P} \cap \{\{1,2\},\dots,\{2\ell-1,2\ell\}\} = \emptyset\}.$$

("No singletons and no special doubletons of form $\{2\alpha-1, 2\alpha\}$.")

Combinatorial proof of Whittle

Thus we have

$$\begin{aligned}
&\left\| \sum_{i,j=1}^{N} A(i,j)(X_i X_j - \mathbf{E} X_i X_j) \right\|_{2k}^{2k} \\
\leq & \sum_{\mathcal{P} \in \mathrm{Part}_{4k}^{\chi\chi}} \left| \sum_{\substack{\mathbf{i}:\langle 4k\rangle \to \langle N\rangle \\ \text{s.t. } \mathcal{P}=\pi(\mathbf{i})}} \prod_{\alpha=1}^{2k} A^{(\alpha)}(\mathbf{i}(2\alpha-1), \mathbf{i}(2\alpha)) \right| \\
\leq & \; c \max_{\mathcal{P} \in \mathrm{Part}_{4k}^{\chi\chi}} \left| \sum_{\substack{\mathbf{i}:\langle 4k\rangle \to \langle N\rangle \\ \text{s.t. } \mathbf{i} \text{ is constant} \\ \text{on blocks of } \mathcal{P}}} \prod_{\alpha=1}^{2k} A^{(\alpha)}(\mathbf{i}(2\alpha-1), \mathbf{i}(2\alpha)) \right|
\end{aligned}$$

where c is a constant depending only on k. The last step is accomplished by Möbius inversion. Done by Yin's Lemma. □

Remarks

Up to bad constants, for study of asymptotic liberation, Yin's Lemma and Whittle's bound are more or less equivalent.

These quadratic variants of Marcinkiewicz-Zygmund are the stock-in-trade for people working on large covariance matrices.

Part V:
Application: Fake Haar unitaries

Setup for result on fake Haar unitaries

We now recall a result from

Anderson, G., Farrell, B., *Asymptotically liberating sequences of random unitary matrices,* Advances in Mathematics **255**(2014), 381–413.

Theorem (A.-Farrell, Corollary 3.5)

Let X_N and Y_N be N-by-N hermitian matrices with operator norms bounded uniformly in N. Assume that the empirical distributions of eigenvalues of X_N and Y_N converge weakly. Let H_N be an N-by-N complex Hadamard matrix. Let W_N be a uniformly distributed N-by-N signed permutation matrix. Let $U_N = \frac{W_N H_N W_N^*}{\sqrt{N}}$. Then the pair $(X_N, U_N Y_N U_N^*)$ is asymptotically free.

With U_N replaced by a Haar-distributed random unitary, the preceding statement is a classical result of Voiculescu. In this sense U_N is a "fake" Haar unitary.

Proof of the A.-F. result

After making reductions of a form familiar to a free probabilist, the proof boils down to the following issue. (See the paper of A.-F. for discussion of such reductions.) Let $A^{(1)}, \ldots, A^{(2k)}$ be N-by-N matrices with complex entries such that $\mathrm{tr} A^{(\alpha)} = 0$ for $\alpha = 1, \ldots, 2k$. It is (more than enough) to show that

$$\left| \mathbf{E} \mathrm{tr} A^{(1)} U_N A^{(2)} U_N^* \cdots A^{(2k-1)} U_N A^{(2k)} U_N^* \right|$$

$$\leq c \left[\left[A^{(1)} \right]\right] \cdots \left[\left[A^{(2k)} \right]\right]$$

where the constant c depends only on k (not N). (The focus of the paper of A.-F. is on this type of estimate and various natural generalizations.)

Proof of the A.-F. result (continued)

For $\mathbf{i} : \langle 4k \rangle \to \langle N \rangle$ let

$$\varphi(\mathbf{i}) = \mathbf{E}\Big[U_N(\mathbf{i}(1), \mathbf{i}(2)) U_N^*(\mathbf{i}(3), \mathbf{i}(4))$$

$$\times \cdots \times U_N(\mathbf{i}(4k-3), \mathbf{i}(4k-2)) U_N^*(\mathbf{i}(4k-1), \mathbf{i}(4k)) \Big].$$

Proof of the A.-F. result (continued)

Then

$$|\varphi(\mathbf{i})| \leq \frac{1}{\sqrt{N}}, \quad \varphi(\sigma \circ \mathbf{i}) = \varphi(\mathbf{i}) \text{ for } \sigma \in S_{4k},$$

and

$$\mathbf{E}\,\mathrm{tr} A^{(1)} U_N A^{(2)} U_N^* \cdots A^{(2k-1)} U_N A^{(2k)} U_N^*$$

$$= \sum_{\mathbf{i}:\langle 4k\rangle \to \langle N\rangle} \varphi(\mathbf{i}) \prod_{\alpha=1}^{2k} A^{(\alpha)}(\mathbf{i}(2\alpha-1), \mathbf{i}(2\alpha)).$$

It follows that $\varphi(\mathbf{i})$ depends only on $\pi(\mathbf{i}) \in \mathrm{Part}_{4k}$ and furthermore, because of invariance of the law of U_N under conjugation by *signed* permutation matrices, in fact $\varphi(\mathbf{i}) \neq 0$ implies that $\pi(\mathbf{i}) \in \mathrm{Part}_{4k}^{\chi}$.

Proof of the A.-F. result (concluded)

The endgame plays out like that of the combinatorial proof of Whittle's result under simplifying i.i.d. hypotheses:

$$\left|\mathbf{E}\,\mathrm{tr}\, A^{(1)} U_N A^{(2)} U_N^* \cdots A^{(2k-1)} U_N A^{(2k)} U_N^*\right|$$

$$\leq \quad \frac{1}{N^k} \sum_{\mathcal{P}\in \mathrm{Part}_{4k}^{\chi}} \left| \sum_{\substack{\mathbf{i}:\langle 4k\rangle \to \langle N\rangle \\ \text{s.t. } \mathcal{P}=\pi(\mathbf{i})}} \prod_{\alpha=1}^{2k} A^{(\alpha)}(\mathbf{i}(2\alpha-1), \mathbf{i}(2\alpha)) \right|$$

$$\overset{\text{Möbius}}{\leq} \quad \frac{c}{N^k} \max_{\mathcal{P}\in \mathrm{Part}_{4k}^{\chi}} \left| \sum_{\substack{\mathbf{i}:\langle 4k\rangle \to \langle N\rangle \\ \text{s.t. } \mathbf{i} \text{ is constant} \\ \text{on blocks of } \mathcal{P}}} \prod_{\alpha=1}^{2k} A^{(\alpha)}(\mathbf{i}(2\alpha-1), \mathbf{i}(2\alpha)) \right|$$

$$\overset{\text{Yin}}{\leq} \quad \frac{c}{N^k} \prod_{\alpha=1}^{2k} \left[\!\left[A^{(\alpha)}\right]\!\right]_2 \leq c \prod_{\alpha=1}^{2k} \left[\!\left[A^{(\alpha)}\right]\!\right]$$

where c is a constant depending only on k. □

Remarks

(a) The proof only uses the invariance of the law of U_N under signed permutation matrices and control of the L^p norms of entries of U_N which one naturally has. Same argument thus applies in the classical Voiculescu setup.

(b) There is nothing particularly special about the group of signed permutation matrices. It is an open problem to understand what finite groups of matrices could play a similar role.

(c) It is an interesting problem to devise fake Haar unitaries with as little randomness as possible. Male's theory of traffics `arXiv:1111.4662` can possibly be used to get more examples.

Part VI:
Remarks on asymptotic liberation

Asymptotic liberation

In A.-Farrell Advances in Math. **255**(2014) the point is to introduce and explore some consequences of the following sort of estimate which in practice can be easier to handle than asymptotic freeness.

Definition of asymptotic liberation

Let I be an index set. For each positive integer N and index $i \in I$ suppose one is given a random unitary matrix $U_i^{(N)} \in \mathrm{Mat}_N$ defined on a probability space depending only on N. We say that the sequence of families

$$\left\{\left\{U_i^{(N)}\right\}_{i\in I}\right\}_{N=1}^{\infty}$$

is *asymptotically liberating* if for $i_1, \ldots, i_\ell \in I$ satisfying

$$\ell \geq 2, \;\; i_1 \neq i_2, \;\; \ldots, \;\; i_{\ell-1} \neq i_\ell, \;\; i_\ell \neq i_1,$$

there exists a constant $c(i_1, \ldots, i_\ell)$ such that

$$\left|\mathbf{E}\,\mathrm{tr}\left(U_{i_1}^{(N)} A_1 U_{i_1}^{(N)*} \cdots U_{i_\ell}^{(N)} A_\ell U_{i_\ell}^{(N)*}\right)\right| \leq c(i_1, \ldots, i_\ell)[\![A_1]\!] \cdots [\![A_\ell]\!]$$

for all positive integers N and constant matrices $A_1, \ldots, A_\ell \in \mathrm{Mat}_N$ each of trace zero.

Remarks

If $\{U_N^{(i)}\}_{i\in I}$ is a family of independent N-by-N Haar-distributed unitaries then it is asymptotically liberating.

Perhaps the slightly stronger estimate

$$\left|\mathbf{E}\,\mathrm{tr}\left(U_{i_1}^{(N)}A_1U_{i_1}^{(N)*}\cdots U_{i_\ell}^{(N)}A_\ell U_{i_\ell}^{(N)*}\right)\right| \le c(i_1,\ldots,i_\ell)\frac{[\![A_1]\!]_2}{\sqrt{N}}\cdots\frac{[\![A_\ell]\!]_2}{\sqrt{N}}$$

should be made the definition, since in all cases where we can make an interesting estimate, this is what we actually prove.

But the other hand, in the general run of applications the definition as stated is what one actually uses. And the definition itself could be weakened, say, by permitting a factor of N^ϵ where $0<\epsilon<1$.

Asymptotic freeness from asymptotic liberation

Let $\{U_N^{(i)}\}_{i\in I}$ be asymptotically liberating. Let $\{A_N^{(i)}\}_{i\in I}$ be, say, a family of N-by-N hermitian matrices of which it is assumed that $\sup_{i,N}\left[\left[A_N^{(i)}\right]\right]<\infty$ and for each $i\in I$ the E.S.D. of $A_N^{(i)}$ tends weakly as $N\to\infty$ to a limit. Then $\{U_N^{(i)}A_N^{(i)}U_N^{(i)*}\}_{i\in I}$ is asymptotically free. In other words, to be asymptotically liberating is to have the capability of making other matrices free. Nothing is assumed, however, about the asymptotic freeness of the $U_N^{(i)}$ themselves.

Procedure for generating open problems

Heuristic: Think of any theorem involving Haar-distributed unitaries where the limits of the ESD's involved are calculable by free probability. Replace the Haar-distributed unitaries by less random unitaries for which the limits of the ESD's are again calculable by free probability. Attempt to prove a new theorem.

Sample result about Haar unitaries: The limit of the empirical distribution of singular values of a block of a Haar-distributed unitary of fixed aspect ratio can be calculated by using free probability. Actually these singular values form a determinantal process for which universality in the bulk and at the edge has been established.

New problem: For the singular values of a randomly chosen block of a Hadamard matrix of fixed aspect ratio, one thus reasonably conjectures a local limit law and more ambitiously universality in the bulk and at the edge.

Thank you!