The self-adjoint linearization trick and algebraicity problems

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Outline

Part I:
The self-adjoint linearization trick

The linearization trick

Haagerup, U. and Thorbjørnsen, S. *A new application of random matrices:* $\mathrm{Ext}(C^*(\mathbb{F}_2))$ *is not a group.* Ann. of Math. **162**(2005)711–775. MR2183281

Haagerup, U., Schultz, H. and Thorbjørnsen, S. A random matrix approach to the lack of projections in $C^*(\mathbb{F}_2)$. Adv. Math. **204**(2006) 1–83. MR2233126

Anderson, G. W., Guionnet, A. and Zeitouni, O. *An Introduction to Random Matrices.* Cambridge Studies in Advanced Mathematics 118. Cambridge Univ. Press, Cambridge 2010. MR2760897 (See Chap. 5, Sec. 5)

The self-adjointness-preserving variant of the L. T.

Anderson, G. W., *Convergence of the largest singular value of a polynomial in independent Wigner matrices,* Ann. of Probab. **41**(2013), 2103–2181. MR3098069 `arXiv:1103.4825`

Anderson, G., *Support properties of spectra of polynomials in Wigner matrices.* (Lecture notes, IMA, June 2012) `z.umn.edu/selfadjlintrick`

Belinschi, S., Mai, T. and Speicher, R., *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem.* `arXiv:1303.3196`

This just in...

Helton, J., McCullough, S., Vinnikov, V. *Noncommutative convexity arises from linear matrix inequalities.* Journal of Functional Analysis **240**(2006), 105–191.

If somebody tells you where to look (see Lemma 4.1), you can actually see the self-adjoint linearization trick in this paper.

The trick does not originate there. The cited lemma just enumerates certain key facts from "the classical theory of descriptor realizations for NC rational functions." The theory "in quotes" is used in engineering and some theoretical background for it can be found, e.g., in this book:

J. Berstel, C. Reutenauer, Rational Series and Their Languages, Texts Theoret. Comput. Sci. EATCS Ser., Springer, Berlin, 1984.

I'm just starting to learn all this new material starting last week based on correspondence with Bill Helton, with an eye toward simplifying the parts typically needed in FP.

Statement of the trick

Now we turn to a lightning course in the self-adjoint linearization trick designed for easy use in free probability. Let $\mathrm{GL}_n(\mathcal{A})$ denote the group of invertible n-by-n matrices with entries in $\mathcal{A}$.

Lemma 1

For each $f \in \mathrm{Mat}_p(\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle)$ there exists some $n > p$ and some $L \in \mathrm{GL}_n(\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle)$ with all entries linear forms in $1, \mathbf{X}_1, \ldots, \mathbf{X}_q$ such that f is the p-by-p block in the upper left of L^{-1}.

This presentation of the trick is strongly influenced by the "engineering" I am just learning about.

Matrix inversion formula

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and d are invertible,

then so is the *Schur complement* $a - bd^{-1}c$ and

$$\begin{aligned}
&\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \\
= \;& \left(\begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix} \right)^{-1} \\
= \;& \begin{bmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{bmatrix} \begin{bmatrix} (a - bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix} \begin{bmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{bmatrix} \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & d^{-1} \end{bmatrix} + \begin{bmatrix} 1 \\ -d^{-1}c \end{bmatrix} (a - bd^{-1}c)^{-1} \begin{bmatrix} 1 & -bd^{-1} \end{bmatrix}.
\end{aligned}$$

Everything follows from this formula.

Typical application of the lemma

Let $\mathcal{A}$ be an algebra. Let

$$f = \begin{bmatrix} \mathbf{I}_p & 0 \end{bmatrix} L^{-1} \begin{bmatrix} \mathbf{I}_p \\ 0 \end{bmatrix}$$

be as in the lemma statement. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_q) \in \mathcal{A}^q$ and $\Lambda \in \mathrm{Mat}_p(\mathcal{A})$ be such that $\Lambda - f(\mathbf{x}) \in \mathrm{Mat}_p(\mathcal{A})$ is invertible. Then

$$\begin{aligned}
&\begin{bmatrix} \Lambda & \begin{bmatrix} \mathbf{I}_p & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbf{I}_p \\ 0 \end{bmatrix} & L(\mathbf{x}) \end{bmatrix}^{-1} \\
= \quad & \begin{bmatrix} 0 & 0 \\ 0 & L(\mathbf{x})^{-1} \end{bmatrix} \\
& + \begin{bmatrix} \mathbf{I}_p \\ -L(\mathbf{x})^{-1} \begin{bmatrix} \mathbf{I}_p \\ 0 \end{bmatrix} \end{bmatrix} (\Lambda - f(\mathbf{x}))^{-1} \begin{bmatrix} \mathbf{I}_p & -\begin{bmatrix} \mathbf{I}_p & 0 \end{bmatrix} L(\mathbf{x})^{-1} \end{bmatrix}.
\end{aligned}$$

Preservation of self-adjointness

Suppose that with respect to the involution satisfying $\mathbf{X}_i^* = \mathbf{X}_i$ for $i = 1, \ldots, q$ the given f is self-adjoint: $f = f^*$. Then with

$$L \text{ and } f = [\ \mathbf{I}_p \quad 0\ ] L^{-1} \begin{bmatrix} \mathbf{I}_p \\ 0 \end{bmatrix},$$

as in the lemma statement, and

$$\tilde{L} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_p & 0 & \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_{n-p} & 0 & 0 \\ \mathbf{I}_p & 0 & -\mathbf{I}_p & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n-p} \end{bmatrix} \begin{bmatrix} 0 & L \\ L^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & 0 & \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_{n-p} & 0 & 0 \\ \mathbf{I}_p & 0 & -\mathbf{I}_p & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n-p} \end{bmatrix}$$

one has

$$f = \begin{bmatrix} \mathbf{I}_p & 0 \end{bmatrix} \tilde{L}^{-1} \begin{bmatrix} \mathbf{I}_p \\ 0 \end{bmatrix}.$$

Ironically enough, the self-adjointness-preserving aspect will be irrelevant for algebraicity...

Proof of the lemma

If every entry of f belongs to the $\mathbb{C}$-linear span of $1, \mathbf{X}_1, \ldots, \mathbf{X}_q$, then, say,

$$L = \begin{bmatrix} 0 & \mathbf{I}_p \\ \mathbf{I}_p & -f \end{bmatrix}$$

is a linearization of f. Thus it will be enough to demonstrate that given

$$f_i = \begin{bmatrix} \mathbf{I}_p & 0 \end{bmatrix} L_i^{-1} \begin{bmatrix} \mathbf{I}_p \\ 0 \end{bmatrix}$$

with

$$L_i \in \mathrm{GL}_{n_i}(\mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle) \text{ for } i = 1, 2,$$

$$\text{entries linear forms in } 1, \mathbf{X}_1, \ldots, \mathbf{X}_q,$$

we can suitably linearize $f_1 + f_2$ and $f_1 f_2$. We may assume without loss of generality that $n = p + N = n_1 = n_2$ after tacking a block $\mathbf{I}_k$ onto L_1 or L_2 to make the matrix sizes equal.

Proof of the lemma (case of $f_1 + f_2$)

It is not hard to check that

$$\frac{1}{4}\begin{bmatrix} \mathbf{I}_p & 0 & \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_N & 0 & 0 \\ \mathbf{I}_p & 0 & -\mathbf{I}_p & 0 \\ 0 & 0 & 0 & \mathbf{I}_N \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & 0 & \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_N & 0 & 0 \\ \mathbf{I}_p & 0 & -\mathbf{I}_p & 0 \\ 0 & 0 & 0 & \mathbf{I}_N \end{bmatrix}$$

linearizes $f_1 + f_2$.

Proof of the lemma (case of $f_1 f_2$)

It is not hard to check that

$$\begin{bmatrix} 0 & L_2 \\ L_1 & -\begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

linearizes $f_1 f_2$. □

The secondary trick

If A is invertible then so is $\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$ and

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BA^{-1} \\ 0 & A^{-1} \end{bmatrix}.$$

This is not much of a trick! But for study of the Schwinger-Dyson equation it played an important role in A's Annals of Probab. paper and is again important in the proof of Theorem 2 in connection with the generalized Schwinger-Dyson equation.

Last remark on SALT

Based on what I have learned from the engineering literature so far, there is in fact a CANONICAL linearization procedure but that refinement is not needed for the present purpose. For future developments it will be mandatory.

Part II:
Preservation of algebraicity: statement of the main result

Main result

Theorem 2 (A., `arXiv:1406.6664`)

Let $(\mathcal{A}, \phi)$ *be a noncommutative probability space. Let*

$$x_1, \ldots, x_q \in \mathcal{A}$$

be freely independent noncommutative random variables. Let

$$X \in \mathrm{Mat}_p(\mathbb{C}\langle x_1, \ldots, x_q \rangle) \subset \mathrm{Mat}_p(\mathcal{A})$$

be a matrix. If the laws of $x_1, \ldots, x_q$ *are algebraic, then so is the law of* X.

The theorem answers a question raised in the paper

Shlyakhtenko, D., Skoufranis, P. *Freely Independent Random Variables with Non-Atomic Distributions.* `arXiv:1305.1920` (about which more later)

Notation and terminology: noncommutative random variables

The setting for the theorem is formal and algebraic. All questions about positivity are ignored.

The noncommutative probability space $(\mathcal{A}, \phi)$ is simply a unital algebra $\mathcal{A}$ with scalar field $\mathbb{C}$ along with a $\mathbb{C}$-linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1_{\mathcal{A}}) = 1$.

A *law* $\mu : \mathbb{C}\langle \mathbf{X} \rangle \to \mathbb{C}$ in this setup is just a linear functional such that $\mu(1_{\mathbb{C}\langle \mathbf{X} \rangle}) = 1$. The sequence $\{\mu(\mathbf{X}^i)\}_{i=1}^{\infty}$ of complex numbers can be arbitrarily prescribed.

The *law* of $x \in \mathcal{A}$ is by definition the linear functional $(f(\mathbf{X}) \mapsto \phi(f(x))) : \mathbb{C}\langle \mathbf{X} \rangle \to \mathbb{C}$.

Notation and terminology: free matrix-polynomial combinations

We need not repeat the definition of freeness here.

$\mathbb{C}\langle x_1, \ldots, x_q \rangle \subset \mathcal{A}$ denotes the $\mathbb{C}$-linear span of all monomials in $x_1, \ldots, x_q \in \mathcal{A}$, including the empty monomial $1_{\mathcal{A}}$. This is a subalgebra of $\mathcal{A}$.

$\mathrm{Mat}_p(\mathcal{A})$ denotes the algebra of p-by-p matrices with entries in $\mathcal{A}$. Elements of $\mathrm{Mat}_p(\mathcal{A})$ are regarded as noncommutative random variables with respect to the state $A \mapsto \frac{1}{p} \sum_{i=1}^{p} \phi(A(i,i))$.

In the sequel we refer to elements of $\mathrm{Mat}_p(\mathbb{C}\langle x_1, \ldots, x_q \rangle)$ as *free matrix-polynomial combinations*.

Notation and terminology: algebraicity of laws

Given a law $\mu : \mathbb{C}\langle \mathbf{X} \rangle \to \mathbb{C}$ the (formal) *Stieltjes transform* is defined as

$$S_\mu(z) = \sum_{i=0}^{\infty} \mu(\mathbf{X}^i) z^{-1-i} \in \mathbb{C}((1/z)).$$

Here $\mathbb{C}((1/z))$ is the field consisting of (formal) Laurent series of the form

$$\sum_{i=-\infty}^{\infty} c_i z^i$$

where $c_i \in \mathbb{C}$ and $c_i = 0$ for $i \gg 0$.

A law $\mu : \mathbb{C}\langle \mathbf{X} \rangle \to \mathbb{C}$ is called *algebraic* if its (formal) Stieltjes transform $S_\mu(z) \in \mathbb{C}((1/z))$ is algebraic over $\mathbb{C}(z) \subset \mathbb{C}((1/z))$.

Hereafter we will drop the adjective "formal" since all our work is in the formal setting.

Summary catchphrase

"Algebraicity of laws is preserved by free matrix-polynomial combination of noncommutative random variables."

Part III:
Background and examples from the literature

Background: Green functions

Let G be a group. Let $\Theta \in \mathbb{C}[G]$ be a group-ring element, i.e., $\Theta = \sum_{g \in G} \Theta(g)g$ where $\Theta(\cdot)$ is finitely supported. If Θ has nonnegative coefficients summing to 1 then Θ defines a finitely-supported random walk. Consider the formal sum

$$(z - \Theta)^{-1} = \sum_{n=0}^{\infty} \Theta^n z^{-n-1} \in (\mathbb{C}[G])((1/z)).$$

The expansion

$$(z - \Theta)^{-1} = \sum_{g \in G} \Phi_g(z) g \;\; (\Phi_g(z) \in \mathbb{C}((1/z)))$$

defines the *Green function*

$$\{\Phi_g(z)\}_{g \in G}$$

associated with Θ.

Background: Green functions (continued and concluded)

More generally one can define the Green function of a random walk; this is the resolvent of the Markov matrix giving the transition probabilities for a step.

Green functions for groups and for Markov matrices have long been studied. Many algebraic tricks from that setting transfer to free probability.

Note that $\Phi_1(z)$ is the Stieltjes transform of the law of Θ with respect to the usual trace $\sum_g \theta_g g \mapsto \theta_1$ on the group ring.

Aomoto's algebraicity result

Reference:
Aomoto, K. *Spectral theory on a free group and algebraic curves.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31**(1984), no. 2, 297–318. MR0763424

For a fairly general class of random walks on a finitely generated free group, Aomoto proved algebraicity of the Green function. In particular, Green functions for group ring elements were proved algebraic under a mild nondegeneracy hypothesis.

Aomoto's results—connection to free probability

Aomoto's results in the case of $\Phi_1(z)$ imply when translated to free probability language that a polynomial in free unitary noncommutative random variables (under a certain nondegeneracy hypothesis) has an algebraic law.

The latter result can be recovered from Theorem 2 above by observing that a unitary noncommutative random variable factors as the product of two free Bernoulli random variables.

Note that the nondegeneracy constraint is not needed, and one gets the result automatically for matrix-polynomials, too.

Woess's result

Reference:
Woess, W. *Context-free languages and random walks on groups.* Discrete Math. **67**(1987), no. 1, 81–87. MR0908187

In this paper the algebraicity of the Green function of a finitely supported random walk on a group with a finitely generated free subgroup of finite index is proved, without any nondegeneracy hypothesis. The result is proved quite concisely, taking for granted machinery from formal language theory. The bibliography to the paper is a good syllabus in related materials, including a classic by Chomsky and Schützenberger.

The result as it pertains to the value $\Phi_1(z)$ at the identity of the Green function can also be recovered from Theorem 2, since this is the Stieltjes transform of the law of a matrix-polynomial combination of free unitary variables.

Remark

It should be possible to recover the full results of Aomoto and Woess from the methods of proof of Theorem 2 since the method of proof already involves a Green function analogue. But details remain to be worked out.

Conversely, it would be interesting to see what could be said about Green-functions from the perspective of Shlyakhtenko-Skoufranis, Belinschi-Mai-Speicher, etc.

Zeta-functions

References:

Kontsevich, M. *Noncommutative identities* `arXiv:1109.2469`
Kassel, C. and Reutenauer, C., *Algebraicity of the zeta function associated to a matrix over a free group algebra* `arXiv:1303.3481`

Here are quick summaries in free-probability language. (The latter is not the language of either paper.)

The first paper shows (among other things) that if μ is the law of an integer polynomial in free unitary variables, then

$$\exp\left(\sum_{i=1}^{\infty}\mu(\mathbf{X}^i)\frac{t^i}{i}\right)\in\mathbb{Z}[[t]]\ \ (\text{“zeta-function” of }\mu)$$

and moreover this power series is algebraic.

The second paper shows the same thing for μ the law of a matrix-polynomial with integer coefficients in free unitary variables.

Zeta-functions (continued and concluded)

The logarithmic derivative of the zeta-function is more or less the Stieltjes transform:

$$t^2 \frac{d}{dt} \log \exp \left(\sum_{i=1}^{\infty} \mu(\mathbf{X}^i) \frac{t^i}{i} \right) \Bigg|_{t=1/z} = \sum_{i=1}^{\infty} \mu(\mathbf{X}^i) z^{-i-1} = S_\mu(z).$$

The value added here by the cited papers is somehow to get integrality and algebraicity from that of the logarithmic derivative. In Kassel-Reutenauer, tools from arithmetic geometry are exploited to make this inference.

As far as I know the free-probabilistic generalizations that naturally spring to mind here have not yet been investigated.

Reference:

Voiculescu, D. V., Dykema, K. J. and Nica, A., *Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. CRM Monograph Series* **1**. American Mathematical Society, Providence, RI, 1992. MR1217253

R-transforms and algebraicity (continued)

For a law $\mu : \mathbb{C}\langle \mathbf{X} \rangle \to \mathbb{C}$ the *R-transform*

$$R_\mu(t) = \sum_{i=1}^{\infty} \kappa_i(\mu) t^{i-1} \in \mathbb{C}[[t]]$$

is the generating function of the free cumulants $\kappa_i(\mu)$. For convenience consider the *modified* R-transform

$$\tilde{R}_\mu(z) = z + \sum_{i=1}^{\infty} \kappa_i(\mu) z^{1-i} \in z + \mathbb{C}[[1/z]].$$

Note that $z + \mathbb{C}[[1/z]]$ is a group under composition. The functional equation

$$\tilde{R}_\mu \circ \frac{1}{S_\mu} = \frac{1}{S_\mu} \circ \tilde{R}_\mu = z$$

can be taken to define the free cumulants.

R-transforms and algebraicity (concluded)

Lemma 3

$\tilde{R}_\mu(z)$ is algebraic over $\mathbb{C}(z)$ if and only if $S_\mu(z)$ is so.

Proof: For $0 \neq F(x, y) \in \mathbb{C}[x, y]$ such that $F(z, S_\mu(z)) = 0$ we necessarily have $F(z, S_\mu(z)) \circ R_\mu(z) = F(R_\mu(z), 1/z) = 0$. Thus algebraicity of $S_\mu(z)$ implies that of $R_\mu(z)$. The argument is evidently reversible. □

Now the R-transform is additive for free convolution of noncommutative random variables, and the sum of Laurent series algebraic over $\mathbb{C}(z)$ is again algebraic.

Thus it is implicit in the theory of the R-transform that algebraicity is stable under free convolution.

Theory of $\sqrt{-1}(xy - yx)$

References:

Nica, A., Speicher, R., *Commutators of free random variables.* Duke Math. J. **92**(1998), no. 3, 553–592. MR1620518

Without going into the details, suffice it to say that the explicit method developed in this paper for computing the law of an (anti)commutator of free random variables is more than enough to show that if x and y are (i) self-adjoint, (ii) freely independent and (iii) have algebraic laws, then the self-adjoint random variable $\sqrt{-1}(xy - yx)$ has again an algebraic law.

Algebraicity result of Shlyakhtenko and Skoufranis

Reference:
Shlyakhtenko, D., Skoufranis, P. *Freely Independent Random Variables with Non-Atomic Distributions.* `arXiv:1305.1920`

The main result of this paper gives precise information about the laws of matrix-polynomial combinations of free random variables with non-atomic distributions. As a complement to the main result the authors prove that matrix-polynomial combinations of free semicircular variables have algebraic laws.

The general question about preservation of algebraicity raised in this paper is what Theorem 2 answers.

Remark

One can ask for more, namely, to have very precise control of algebraic equations and in particular branch points. This aspect is not addressed by Theorem 2 at all. It is very much an open problem to clarify it.

Part IV:
Algebraicity criteria

An algebraicity criterion from the random walk literature

Now we examine tools for proving algebraicity in general situations where explicit two-variable equations are not obviously on offer.

Reference:
Lalley, S. *Random walks on regular languages and algebraic systems of generating functions.* Algebraic methods in statistics and probability (Notre Dame, IN, 2000), 201–230, Contemp. Math., 287, Amer. Math. Soc., Providence, RI, 2001. MR1873677

Proposition 5.1 of the cited paper gives a clear, general algebraicity criterion and then applies it in an attractive way. I do not know how to track this general idea down to its first occurrence. To some extent it is part of formal language theory.

Gist of the criterion

The gist of Prop. 5.1 in Lalley's paper is that whenever the hypotheses of the Implicit Function Theorem are satisfied by a system of polynomial functions at a point so as to implicitly define near that point a collection of one-variable functions, the functions so defined must all be algebraic.

Side-remark

There are some extremely interesting ideas in Lalley's paper about the relationship of algebraicity and positivity, leading to local limit laws for random walk on certain groups, ideas we do not touch in connection with Theorem 2. It might be worthwhile to attempt to develop these ideas in the direction of free probability.

Algebraicity criterion used to prove Theorem 1

We recast Lalley's Proposition 5.1 in the following simpler form:

Proposition 1

Let K/K_0 be an extension of fields. Let $x = (x_1, \ldots, x_n)$ be an n-tuple of variables and let $K_0[x]$ denote the (commutative) polynomial ring generated over K_0 by these variables. Let $f = (f_1, \ldots, f_n) = f(x) \in K_0[x]^n$ be an n-tuple of polynomials. Let $J(x) = \det \frac{\partial f}{\partial x} \in K_0[x]$ be the Jacobian determinant of f. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in K^n$ be an n-tuple such that $f(\alpha) = 0$ but $J(\alpha) \neq 0$. Then every entry of the vector α is algebraic over K_0.

We have uppermost in mind the extension $\mathbb{C}((1/z))/\mathbb{C}(z)$.

Remarks

The algebraicity criterion we have stated has a more elementary proof than Lalley's criterion. Instead of quoting a result on dimension theory as does Lalley, we use Hilbert's Nullstellensatz and a theorem of Krull asserting that $\bigcap I^n = 0$ for any nonunit ideal I in a noetherian domain A.

I would like to acknowledge helpful correspondence with my colleague Christine Berkesch Zamaere concerning the underlying commutative algebra here.

Remark

Reference:
Anderson, G., Zeitouni, O.: *A law of large numbers for finite-range dependent random matrices.* Comm. Pure Appl. Math. **61**(2008), 1118–1154. MR2417889

In this paper there is an algebraicity criterion given along similar lines to that of Lalley, albeit with a rather more elaborate statement and proof. Unfortunately, the authors did not know at that time of Lalley's paper.

Remark

Reference:
Nagnibeda, T., Woess, W. *Random walks on trees with finitely many cone types.* J. Theoret. Probab. **15**(2002), no. 2, 383–422. MR1898814

This paper independently and contemporaneously proves results roughly similar to those in the cited paper by Lalley.

Part V: The generalized Schwinger-Dyson equation

$\mathbb{C}((1/z))$ as a complete valued field

We start looking at technical tools for proving Theorem 2.

Let

$$\operatorname{val}\left(\sum_{i=-\infty}^{\infty} c_i z^i\right) = \sup\{i \in \mathbb{Z} \mid c_i \neq 0\}.$$

Then:

$$\begin{aligned} \operatorname{val} f = -\infty &\Leftrightarrow f = 0, \\ \operatorname{val}(f_1 f_2) &= \operatorname{val} f_1 + \operatorname{val} f_2, \\ \operatorname{val}(f_1 + f_2) &\leq \max(\operatorname{val} f_1, \operatorname{val} f_2). \end{aligned}$$

Thus val is (the logarithm of) a nonarchimedean valuation with respect to which $\mathbb{C}((1/z))$ is complete. Thus it becomes possible to use metric space ideas to reason about $\mathbb{C}((1/z))$.

$\mathrm{Mat}_n(\mathbb{C}((1/z)))$ as Banach algebra

In particular, we can make $\mathrm{Mat}_n(\mathbb{C}((1/z)))$ into a Banach algebra by defining

$$\mathrm{val}\, A = \max_{i,j=1}^{n} \mathrm{val} A(i,j).$$

We take this structure for granted going forward.

The generalized Schwinger-Dyson equation (beginning of statement)

Valuation theory in hand, we next review key functional equations.

Suppose we are given

$$\begin{cases} \bullet \text{ a positive integer } n, \\ \bullet \text{ matrices } a^{(0)}, g \in \mathrm{Mat}_n(\mathbb{C}((1/z))), \\ \bullet \text{ matrices } a^{(\theta)} \in \mathrm{Mat}_n(\mathbb{C}) \text{ for } \theta = 1, \ldots, q, \text{ and} \\ \bullet \text{ a family } \{\{\kappa_j^{(\theta)}\}_{j=2}^{\infty}\}_{\theta=1}^{q} \text{ of complex numbers.} \end{cases} \tag{1}$$

The generalized Schwinger-Dyson equation (conclusion of statement)

Suppose the data (1) satisfy the following conditions:

$$\lim_{j\to\infty} \mathrm{val}\,(a^{(\theta)}g)^j = -\infty \ \text{ for } \theta = 1, \ldots, q. \tag{2}$$

$$I_n + a^{(0)}g + \sum_{\theta=1}^{q}\sum_{j=2}^{\infty} \kappa_j^{(\theta)}(a^{(\theta)}g)^j = 0. \tag{3}$$

The linear map (4)

$$\left(\gamma \mapsto a^{(0)}\gamma + \sum_{\theta=1}^{q}\sum_{j=2}^{\infty}\sum_{\nu=0}^{j-1} \kappa_j^{(\theta)}(a^{(\theta)}g)^{\nu}(a^{(\theta)}\gamma)(a^{(\theta)}g)^{j-1-\nu}\right)$$

$: \mathrm{Mat}_n(\mathbb{C}((1/z))) \to \mathrm{Mat}_n(\mathbb{C}((1/z)))$ is invertible.

Then we say that the data (1) constitute a solution of the *generalized Schwinger-Dyson equation*.

Remarks

In view of the Banach algebra structure over $\mathbb{C}((1/z))$ with which we have equipped $\mathrm{Mat}_n(\mathbb{C}((1/z)))$ at least the three relations (2), (3), and (4) jointly make sense.

Note that the function considered in (4) arises by formally differentiating the function considered in (3). Our method does not require us to make sense of this observation rigorously but the intuition guides our calculations.

Condition (4) is a sort of nondegeneracy condition which turns out (somewhat indirectly) to match up with the hypotheses for Proposition 1.

Remark

The generalized Schwinger-Dyson equation (3) (up to signs and various trivial alterations) is of course familiar from operator-valued free probability theory.

In the semicircular case condition (3) takes the form

$$I_n + \left(a^{(0)} + \sum_{\theta=1}^{q} \kappa_2^{(\theta)} a^{(\theta)} g a^{(\theta)} \right) g = 0$$

familiar, say, from study of polynomials in independent GUE matrices.

Part VI:
(Thm. 2)$=$(Prop. 2)$+$ (Prop. 3)

Large-scale structure of the proof of Theorem 2

We prove Theorem 2 by splitting it into two statements both of which concern aspects of the generalized Schwinger-Dyson equation. We discuss each of these two statements in turn below.

First half of the split

Proposition 2

In the setup of Theorem 2, for indices $\theta = 1, \ldots, q$ *and* $j = 2, 3, 4, \ldots$ *let* $\kappa_j^{(\theta)} = \kappa(\mu_{x_\theta})$*. Then the family* $\{\{\kappa_j^{(\theta)}\}_{j=2}^{\infty}$ *for some integer* $n \geq p$ *can be completed to a family*

$$\left(n, a^{(0)}, g, \left\{a^{(\theta)}\right\}_{\theta=1}^{q}, \left\{\left\{\kappa_j^{(\theta)}\right\}_{j=2}^{\infty}\right\}_{\theta=1}^{q}\right)$$

of the form (1) *satisfying* (2), (3) *and* (4) *along with the further conditions*

$$a_0 \in \mathrm{Mat}_n(\mathbb{C}[z]) \text{ and } S_{\mu_X} = -\frac{1}{p}\sum_{i=1}^{p} g(i,i).$$

This roughly speaking is a formal algebraic version of the main results of Belinschi-Mai-Speicher.

Second half of split

Proposition 3

Let data of the form (1) *satisfy* (2), (3), *and* (4). *Assume furthermore that*

$$a_0 \in \mathrm{Mat}_n(\mathbb{C}(z)) \text{ and } z + \sum_{j=1}^{\infty} \kappa_{j+1}^{(\theta)} z^{-j} \in \mathbb{C}((1/z)) \text{ is algebraic for } \theta = 1, \ldots, g.$$

Then every entry of the matrix g is algebraic.

In view of Lemma 3 above, to the effect that S_μ is algebraic if and only if $\widetilde{R}_\mu$ is algebraic, it is clear that Propositions 2 and 3 prove Theorem 2.

Part VII:
Notes on the proof of Proposition 2

Large-scale plan

We will again divide and conquer. We will explain how

(Prop. 2)= (Prop. 5)+(Prop. 6)

but we will not drill down too far into the proofs of the latter propositions.

The big matrix algebra $\mathfrak{M}$

Let $\mathbb{N}$ denote the set of **nonnegative** integers.
Let $\mathfrak{M}$ denote the vector space over $\mathbb{C}$ consisting of $\mathbb{N}$-by-$\mathbb{N}$ matrices M such that for each $j \in \mathbb{N}$ there exist only finitely many $i \in \mathbb{N}$ such that $M(i,j) \neq 0$.
Every upper-triangular matrix belong to $\mathfrak{M}$.
We write $\mathbf{1} = 1_{\mathfrak{M}}$ to abbreviate notation.
We equip $\mathfrak{M}$ with the state $\phi(M) = M(0,0)$, thus defining a noncommutative probability space $(\mathfrak{M}, \phi)$.

Elementary matrices

Let $\mathbf{e}[i,j] \in \mathfrak{M}$ denote the elementary matrix with entries given by the rule

$$\mathbf{e}[i,j](k,\ell) = \delta_{ik}\delta_{j\ell} \text{ for } i,j,k,\ell \in \mathbb{N}.$$

For $M \in \mathfrak{M}$ supported in a set $S \subset \mathbb{N} \times \mathbb{N}$ we abuse notation by writing

$$M = \sum_{(i,j)\in S} M(i,j)\mathbf{e}[i,j]$$

as a convenient shorthand to indicate placement of entries.

Example

Consider the infinite matrix

$$C = \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 & \dots & & & & \\ 1 & \kappa_1 & \kappa_2 & \kappa_3 & \dots & & & \\ & 1 & \kappa_1 & \kappa_2 & \kappa_3 & \dots & & \\ & & 1 & \kappa_1 & \kappa_2 & \kappa_3 & \dots & \\ & & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \in \mathfrak{M} \quad (\kappa_j \in \mathbb{C}).$$

Lemma 4 (Voiculescu)

The j^{th} free cumulant of the law of C is κ_j.

This is an important observation on which the theory of the R-transform is based.

Example (continued and concluded)

In terms of the elementary matrices $\mathbf{e}[i,j] \in \mathfrak{M}$ we have

$$C = \sum_{k \in \mathbb{N}} \left(\mathbf{e}[1+k,k] + \sum_{j \in \mathbb{N}} \kappa_{j+1} \mathbf{e}[k,j+k] \right).$$

The matrix C displays the (upper) *Hessenberg* pattern: $i > j+1 \Rightarrow C(i,j) = 0$ for $i,j \in \mathbb{N}$. The matrix C also displays the *Toeplitz* pattern: $C(i+1,j+1) = C(i,j)$ for $i,j \in \mathbb{N}$.

Stars and diamonds

In grade school one learns to represent nonnegative integers to the base q using place-notation and digits selected from the set $\{0, \ldots, q-1\}$. It is not hard to see that using instead digits selected from the set $\{1, \ldots, q\}$ one still gets a unique representation for every member of $\mathbb{N}$, it being understood that 0 is represented by the empty digit string $\emptyset$. Here is "improper" counting in base 3:

$$\emptyset, 1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, \ldots$$

Let $\star$ be the unital binary operation on $\mathbb{N}$ corresponding to concatenation of digit strings with respect to such "improper" base q expansions and let $\diamond$ denote exponentiation.

"Arborized" Boltzmann-Fock space

Proposition 4

Let $\{\{\kappa_j^{(\theta)}\}_{j=1}^{\infty}\}_{\theta=1}^{q}$ *be any family of complex numbers. Then the family*

$$\sum_{k\in\mathbb{N}} \mathbf{e}[\theta\star k,k]+\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}\kappa_{j+1}^{(\theta)}\mathbf{e}[k,\theta^{\diamond j}\star k]\in\mathfrak{M}\ \ \text{for } \theta=1,\ldots,q$$

of noncommutative random variables is freely independent and moreover the j^{th} *free cumulant of the* θ^{th} *noncommutative random variable equals* $\kappa_j^{(\theta)}$.

This is just an offbeat way of describing the standard model for free random variables in terms of raising and lowering operators on Boltzmann-Fock space used by Voiculescu to prove additivity of the R-transform for addition of free noncommutative random variables.

The "Laurentization" $\mathfrak{M}((1/z))$

Let $\mathfrak{M}((1/z))$ denote the set of $\mathbb{N}$-by-$\mathbb{N}$ matrices M with entries in $\mathbb{C}((1/z))$ satisfying one and hence both of the following equivalent conditions:

- There exists a *Laurent expansion* $M = \sum_{n\in\mathbb{Z}} M_n z^n$ with coefficients
  $M_n \in \mathfrak{M}$ such that $M_n = 0$ for $n \gg 0$.
- One has $\lim_{i\to\infty} \mathrm{val}\, M(i,j) = -\infty$ for each $j \in \mathbb{N}$ (without any requirement of uniformity in j) and furthermore one has $\sup_{i,j\in\mathbb{N}} \mathrm{val}\, M(i,j) < \infty$.

Let $\mathrm{val}\, M = \sup_{i,j\in\mathbb{N}} \mathrm{val}\, M(i,j)$ for $M \in \mathfrak{M}((1/z))$.
With respect to the valuation function val thus extended to $\mathfrak{M}((1/z))$ the latter becomes a unital Banach algebra over $\mathbb{C}((1/z))$.
We write $\mathbf{1} = 1_{\mathfrak{M}} = 1_{\mathfrak{M}((1/z))}$.

Lemma 5

Fix $M \in \mathfrak{M}$ arbitrarily and let μ denote the law of M. Then the matrix

$$z\mathbf{1} - M \in \mathfrak{M}((1/z))$$

is invertible and

$$(z\mathbf{1} - M)^{-1}(0,0) = S_\mu(z).$$

Proof

$$(z\mathbf{1} - M)^{-1} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{M^k}{z^k} \in \mathfrak{M}((1/z)).$$

□

Kronecker products

Recall that for matrices of finite size the *Kronecker product*

$$A^{(1)} \otimes A^{(2)} \in \mathrm{Mat}_{k_1 k_2 \times \ell_1 \ell_2}(\mathbb{C}) \quad \left(A^{(\alpha)} \in \mathrm{Mat}_{k_\alpha \times \ell_\alpha}(\mathbb{C}) \text{ for } \alpha = 1, 2\right)$$

is defined by the rule

$$A^{(1)} \otimes A^{(2)} = \begin{bmatrix} A^{(1)}(1,1)A^{(2)} & \dots & A^{(1)}(1,\ell_1)A^{(2)} \\ \vdots & & \vdots \\ A^{(1)}(k_1,1)A^{(2)} & \dots & A^{(1)}(k_1,\ell_1)A^{(2)} \end{bmatrix}.$$

In the mixed infinite/finite case we define the *Kronecker product*

$$x \otimes a \in \mathfrak{M}((1/z)) \quad (x \in \mathfrak{M}((1/z)) \text{ and } a \in \mathrm{Mat}_n(\mathbb{C}((1/z))))$$

by the rule

$$x \otimes a = \begin{bmatrix} x(0,0)a & x(0,1)a & \dots \\ x(1,0)a & x(1,1)a & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

more or less the same as in the finite-size case.

Application of linearization

Proposition 5

Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. Let $x_1, \ldots, x_q \in \mathcal{A}$ be freely independent noncommutative random variables. Fix a matrix $X \in \mathrm{Mat}_p(\mathbb{C}\langle x_1, \ldots, x_q \rangle) \subset \mathrm{Mat}_p(\mathcal{A})$. Let $\kappa_j^{(\theta)} = \kappa_j(\mu_{x_\theta})$ for $j = 1, 2, \ldots$ and $\theta = 1, \ldots, q$. Then for some integer $N > 0$ there exist matrices $L_0, L_1, \ldots, L_q \in \mathrm{Mat}_{p+N}(\mathbb{C})$ all of which vanish identically in the upper left p-by-p block such that

$$L = \mathbf{1} \otimes \left(L_0 + \begin{bmatrix} zI_p & 0 \\ 0 & 0 \end{bmatrix} \right) + \sum_{\theta=1}^{q} \sum_{k \in \mathbb{N}} \mathbf{e}[\theta \star k, k] \otimes L_\theta$$

$$+ \sum_{\theta=1}^{q} \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \kappa_{j+1}^{(\theta)} \mathbf{e}[k, \theta^{\diamond j} \star k] \otimes L_\theta \in \mathfrak{M}((1/z)) \text{ is invertible and}$$

$$S_{\mu_X}(z) = \frac{1}{p} \sum_{i=0}^{p-1} L^{-1}(i,i).$$

Remark

Adopting the absurd point of view that probabilities can be square matrices with arbitrary complex number entries, the matrix L describes a random walk on the q-ary tree such that from a given vertex $x \in \mathbb{N}$, one may (i) step one unit back toward the root (if not already at the root), (ii) stay in place, or (iii) move away from the root arbitrarily far along along a geodesic $\{\theta^{\diamond i} \star x \mid i \in \mathbb{N}\}$ for some $\theta \in \{1, \ldots, q\}$. Whether or not absurd, this interpretation does make random walk intuition available to analyze L.

The generalized Schwinger-Dyson equation (beginning of reminder)

Data:

$$\begin{cases} \bullet \text{ a positive integer } n, \\ \bullet \text{ matrices } a^{(0)}, g \in \mathrm{Mat}_n(\mathbb{C}((1/z))), \\ \bullet \text{ matrices } a^{(\theta)} \in \mathrm{Mat}_n(\mathbb{C}) \text{ for } \theta = 1, \ldots, q, \text{ and} \\ \bullet \text{ a family } \{\{\kappa_j^{(\theta)}\}_{j=2}^{\infty}\}_{\theta=1}^{q} \text{ of complex numbers.} \end{cases}$$

Application of block-manipulation

Proposition 6

Fix data of the form (1)*. Consider the matrix*

$$A = -\mathbf{1} \otimes a^{(0)} - \sum_{\theta=1}^{q} \sum_{k\in\mathbb{N}} \mathbf{e}[\theta \star k, k] \otimes a^{(\theta)} \tag{5}$$
$$- \sum_{\theta=1}^{q} \sum_{k\in\mathbb{N}} \sum_{j=1}^{\infty} \kappa_{j+1}^{(\theta)} \mathbf{e}[k, \theta^{\diamond j} \star k] \otimes a^{(\theta)} \in \mathfrak{M}((1/z)).$$

Assume that

$$G = A^{-1} \in \mathfrak{M}((1/z)) \text{ exists, and} \tag{6}$$
$$g(i,j) = G(i-1, j-1) \ \text{ for } i,j = 1, \ldots, n. \tag{7}$$

Then (2)*,* (3)*, and* (4) *hold, i.e., the data* (1) *constitute a solution of the generalized Schwinger-Dyson equation.*

The generalized Schwinger-Dyson equation (end of reminder)

Conditions:

$$\lim_{j\to\infty} \mathrm{val}\,(a^{(\theta)}g)^j = -\infty \ \text{ for } \theta = 1,\dots,q. \tag{8}$$

$$I_n + a^{(0)}g + \sum_{\theta=1}^{q}\sum_{j=2}^{\infty} \kappa_j^{(\theta)}(a^{(\theta)}g)^j = 0. \tag{9}$$

The linear map (10)

$$\left(\gamma \mapsto a^{(0)}\gamma + \sum_{\theta=1}^{q}\sum_{j=2}^{\infty}\sum_{\nu=0}^{j-1} \kappa_j^{(\theta)}(a^{(\theta)}g)^{\nu}(a^{(\theta)}\gamma)(a^{(\theta)}g)^{j-1-\nu}\right)$$

$: \mathrm{Mat}_n(\mathbb{C}((1/z))) \to \mathrm{Mat}_n(\mathbb{C}((1/z)))$ is invertible.

End of the sketch of proof

Propositions 5 and 6 are enough together to prove Proposition 2. Finally, Proposition 2 and Proposition 6 are enough to prove Theorem 2.

Future directions

We sacrificed all the information about positivity and branch points. Quite possibly a critical examination of our proof will reveal an algebro-geometric setup which can be analyzed to gain sharper control of the solutions of the generalize Schwinger-Dyson equation as matrices of meromorphic functions on compact Riemann surfaces.

Refinements of the linearization trick, a.k.a. realization, coming from formal language theory and engineering may prove helpful in gaining the sharper control.

The study of zeta-functions also seems promising. It is an interesting problem to prove (or disprove) that a free integral-matrix-polynomial combination of free random variables with integral-algebraic zeta-functions has again an integral-algebraic zeta-function.

Part VIII: Notes on the proof of Proposition 3

“Easy” semicircular case

In the semicircular case, i.e., $\kappa_j^{(\theta)} = 0$ for $j > 2$, or more generally $\kappa_j^{(\theta)} = 0$ for $j \gg 2$, we can simply plug directly into Proposition 1 and get algebraicity of all entries of g under the hypotheses of Proposition 3. (This is the case into which above-mentioned result of Shlyakhtenko and Skoufranis falls.) But in general, infinitely many of the $\kappa_j^{(\theta)}$ are not zero, which presents an obstruction that we overcome with help from algebraic geometry (Newton-Puiseux series and desingularization of plane algebraic curves) and commutative algebra (formal Weierstrass Preparation Theorem).

Motivating remark

Inspiration is derived at least in part from the discussion of efficient computation of the matrix exponential in the undergraduate sophomore calculus text by Williamson and Trotter (the latter also of tridiagonal fame). Let A be an n-by-n matrix with complex entries. It would take a while to reproduce the W.-T. approach at sophomore level. But fortunately at graduate level it goes quickly as follows. Perform Weierstrass division (possible globally in this case) to obtain an identity relating two-variable entire functions of complex variables t and X. One gets

$$\exp(tX) = \sum_{k=0}^{n-1} y_k(t)X^k + Q(X,t)\det(XI_n - A).$$

Then by plugging in $X = A$ on both sides and using the Cayley-Hamilton Theorem one has

$$\exp(tA) = \sum_{k=0}^{n-1} y_k(t)A^k.$$

The Jordan canonical form of A is not needed!

Setup for "widget theory"

Variables:

$$\{u_i\}_{i=1}^{n} \cup \{v_i\}_{i=1}^{2n} \cup \{t, x, y\}.$$

For $A \in \mathrm{Mat}_n(\mathcal{A})$,

$$\det(1 + tA) = 1 + \sum_{i=1}^{n} e_i(A) t^i \in \mathcal{A}[t],$$
$$e(A) = (e_1(A), \ldots, e_n(A)) \in \mathcal{A}^n \text{ and}$$
$$A^\flat = \left[\begin{array}{ccccccc} A(1,1) & \ldots & A(1,n) & \ldots & A(n,1) & \ldots & A(n,n) \end{array}\right]^{\mathrm{T}}$$
$$\in \mathrm{Mat}_{n^2 \times 1}.$$

Cayley-Hamilton Theorem:

$$A^n + \sum_{i=1}^{n} (-1)^i e_i(A) A^{n-i} = 0$$

Negative spectral valuation

For $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ one has $\lim_{k\to\infty} \mathrm{val}\, A^k$ if and only if $e(A) \in (1/z)\mathbb{C}[[1/z]]^n$. Under these equivalent conditions we say that A has *negative spectral valuation*. The fact is proved by using Newton-Puiseux series.

Definition of a widget

We call

$$P(u,v) \in \mathbb{C}[u,v]^{2n}$$

a widget if

$$P(0,0) = 0, \quad \det_{i,j=1}^{2n} \frac{\partial P_i}{\partial v_j}(0,0) \neq 0$$

and for every $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ of negative spectral valuation $\gamma \in (1/z)\mathbb{C}[[1/z]]^{2n}$ such that $P(e(A),\gamma) = 0$, and integer $N \geq 0$, we have

$$\left[\begin{array}{ccc} (A^N)^\flat & \dots & (A^{N+2n-1})^\flat \end{array}\right] \frac{\partial P}{\partial v}(e(A),\gamma)^{-1} \frac{\partial P}{\partial u}(e(A),\gamma) = 0.$$

In fact by the formal Implicit Function Theorem, the equation $P(e(A),\gamma) = 0$ has a unique solution.

Main result on widgets

Proposition 7

Fix an algebraic power series

$$f(t) = \sum_{i=0}^{\infty} c_i t^i \in \mathbb{C}[[t]] \ \ (c_i \in \mathbb{C}). \tag{11}$$

Then for each integer $N \gg 0$ *there exists a widget* $P(u, v) \in \mathbb{C}[u, v]^{2n}$ *such that for each* $A \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$ *of negative spectral valuation the unique* $\gamma \in (1/z)\mathbb{C}[[1/z]]^{2n}$ *satisfying* $P(e(A), \gamma) = 0$ *also satisfies*

$$\sum_{j=2N}^{\infty} c_j \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^j = \sum_{j=1}^{2n} \gamma_j \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}^{N+j-1} \tag{12}$$

for every $B \in \mathrm{Mat}_n(\mathbb{C}((1/z)))$.

Remarks

The proof of Proposition 7 is a long exercise applying the Weierstrass Preparation Theorem and Newton-Puiseux series.

The application of Proposition 7 to the proof of Proposition 3 is made by augmenting the family of entries of g by a family of auxiliary parameters $\gamma_j^{(\theta)}$ such that the enlarged family satisfies a system of polynomial equations to which Proposition 1 applies.

Ultimately this last bit of trickery with the Weierstrass Preparation Theorem may be the only novel part of the algebraicity proof when viewed against the backdrop of the engineering and formal language literature.