Toward local limit laws for polynomials in independent Wigner matrices

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Outline

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Part I:
Introduction and goal of talk

Overall motivation

Given the great progress of the last several years on universality for Wigner matrices, it is natural to attempt to generalize toward polynomials in independent Wigner matrices, aiming at first for local limit laws. We report on the steps we took in this direction and on the questions that came up.

Local semicircle law background (just a tiny sample)

Erdös, L., Knowles, A., Yau, H.-T. and Yin, J., *The local semicircle law for a general class of random matrices.* `arXiv:1212.0164`

Erdös, L, Yau, H.-T. and Yin, J., *Rigidity of Eigenvalues of Generalized Wigner Matrices* `arXiv:1007.4652`

Erdös, L, Yau, H.-T. and Yin, J., *Bulk universality for generalized Wigner matrices*, Probability Theory and Related Fields **154**(2012), 341–407. MR2981427 `arXiv:1001.3453v8` ← *greatest influence*

Tao, T. and Vu, V., *Random matrices: Universality of local eigenvalue statistics.* Acta Math. **206**(2011), 127–204. MR2784665

Local limit laws and delocalization

Local limit laws are a key ingredient in universality results. These are limit laws which are strong enough to permit one to prove delocalization of eigenvectors of a Wigner matrix.

Roughly speaking, an eigenvector is delocalized if one has a "good" bound for the ℓ^∞-norm of an eigenvector in terms of its ℓ^2-norm. It turns out that if one can bound all diagonal entries of the resolvent close to the real axis, then one can get delocalization.

The algebra behind delocalization

Let us recall a simple point of algebra that begins to explain the last somewhat cryptic remark.

Let X be an N-by-N hermitian matrix. Let $\{\lambda_i\}_{i=1}^N$ be the family of eigenvalues of X and let $\{v_i\}_{i=1}^N$ be an orthonormal system of eigenvectors for X. The resolvent

$$R = (X - z\mathbf{I}_N)^{-1} = \sum_{j=1}^{N} \frac{v_j v_j^*}{\lambda_j - z}$$

satisfies

$$\Im R(i,i) = \sum_{j=1}^{N} \frac{(\Im z)|v_j(i)|^2}{(\Re z - \lambda_j)^2 + (\Im z)^2}.$$

The algebra behind delocalization (continued)

Now suppose that

$$z = \lambda_{i_0} + \mathrm{i}\sigma$$

with $\sigma > 0$. We then have

$$\sigma \Im R(i,i) = \sum_{j=1}^{N} \frac{\sigma^2 |v_j(i)|^2}{(\lambda_{i_0} - \lambda_j)^2 + \sigma^2} \geq |v_{i_0}(i)|^2.$$

Suppose now X is a Wigner matrix with entries of variance $\frac{1}{N}$. Then the eigenvalues of X with high probability are contained $[-2-\epsilon, 2+\epsilon]$ where ϵ is fixed (independent of N). If one can gain control of R, getting, say, a bound

$$\max_{i=1}^{N} |R(i,i)| = O(1) \text{ for } \Im z = O(1/N) \text{ and } |\Re z| \leq 2 - \epsilon,$$

then one has

$$\max_{i_0 : |\lambda_{i_0}| \leq 2-\epsilon} \max_{i} |v_{i_0}(i)| = O(1/\sqrt{N}).$$

The algebra behind delocalization (concluded)

Actually the scenario contemplated above is perfect delocalization, better than which one cannot do, because the best bound for the ℓ_∞ norm of a unit vector in $\mathbb{C}^N$ is $\frac{1}{\sqrt{N}}$.

Of course perfect delocalization never happens. In practice factors of $(\log N)^c$ must be accepted everywhere.

Overview of results of methods

We have worked through a "test problem," namely that of getting a local limit law for $XY + YX$ where X and Y are independent Wigner matrices. The actual local limit law we get is nowhere near as refined as what one nowadays has for individual Wigner matrices, but it is good enough to get delocalization of eigenvectors for eigenvalues in the bulk of the spectrum.

In order to do this, we find some way to transfer basic estimates learned from Erdös, Yau, Yin, et al. to $XY + YX$; plausibly this step can be generalized to arbitrary self-adjoint polynomials in Wigner matrices via SALT. We can also identify a certain problem in free probability as an important obstruction to obtaining a general local limit law for polynomials in independent Wigner matrices.

Goal of talk

After filling in background concerning what we proved about $XY + YX$, we will indicate briefly where the self-adjoint linearization trick fits into the picture, and then at greater length we will motivate and explain the free probability problem we have identified as a major obstruction to getting a general local limit law for polynomials in independent Wigner matrices.

Part II:
The model and the main result

References

We will be talking about a result from

Anderson, G. *A local limit law for the empirical spectral distribution of the anticommutator of independent Wigner matrices*, `arXiv:1308.4668`, to appear in AIHP

Key background references are as follows:

Füredi, Z. and Komlós, J., *The eigenvalues of random symmetric matrices.* Combinatorica **1**(1981) 233–241. MR0637828

Nica, A. and Speicher, R., *Commutators of free random variables.* Duke Math. J. **92**(1998), 553–592. MR1620518

Vu, V. H., *Spectral norm of random matrices.* Combinatorica **27** (2007), 721–736. MR2384414

Notation

$\|Z\|_p = (\mathbf{E}|Z|^p)^{1/p}$

$x \vee y = \max(x, y)$

$x \wedge y = \min(x, y)$

Given $A \in \mathrm{Mat}_{k\times\ell}(\mathbb{C})$ with singular values $\{\mu_i\}$, put $[\![A]\!] = \max \mu_i$ and $[\![A]\!]_p = \left(\sum_i \mu_i^p\right)^{1/p}$.

Fix constants $\alpha_0 > 0$ and $\alpha_1 \geq 1$. Let $N \geq 2$ be a integer. Let $U, V \in \mathrm{Mat}_N$ be random hermitian matrices with the following properties:

$$\sup_{p \in [2,\infty)} p^{-\alpha_0} \left(\bigvee_{i,j=1}^{N} \|U(i,j)\|_p \vee \bigvee_{i,j=1}^{N} \|V(i,j)\|_p \right) \leq \sqrt{\frac{\alpha_1}{N}}.$$

The family $\{U(i,j), V(i,j)\}_{1 \leq i \leq j \leq N}$ is independent.

All entries of U and V have mean zero.

$\|U(i,j)\|_2 = \|V(i,j)\|_2 = \dfrac{1}{\sqrt{N}}$ for distinct $i, j = 1, \ldots, N$.

(Here $U(i,j)$ is the (i,j)-entry of U.) This is a class of Wigner matrices similar to that considered in Erdös-Yau-Yin.

Apparatus from free probability

Let **u** and **v** be freely independent semicircular noncommutative random variables. Let $\mu_{\{\mathbf{uv}\}}$ denote the law of $\{\mathbf{uv}\} = \mathbf{uv} + \mathbf{vu}$ and let

$$m_{\{\mathbf{uv}\}}(z) = \int \frac{\mu_{\{\mathbf{uv}\}}(dt)}{t - z} \quad \text{for } z \in \mathfrak{h} = \{z \in \mathbb{C} \mid \Im z > 0\}$$

denote the Stieltjes transform of that law. We write briefly $m = m_{\{\mathbf{uv}\}}(z)$, with dependence on z understood. It was shown by Nica-Speicher that m satisfies the equation

$$zm^3 - m^2 - zm - 1 = 0.$$

(Caution: our sign-convention for the Stieltjes transform is opposite to that of N.-S.) It follows that the support of $\mu_{\{\mathbf{uv}\}}$ is the interval $[-\zeta, \zeta]$ where

$$\zeta = \sqrt{\frac{11 + 5\sqrt{5}}{2}} \cong 3.33.$$

Side-remark: What is the anticommutator density?

In case you were wondering, here is the explicit description of the measure μ for which $m_{\{\mathbf{uv}\}}(z) = S_\mu(z) = \int \frac{\mu(dt)}{t-z}$, due to Nica and Speicher. It is absolutely continuous with respect to Lebesgue measure and its density is

$$\frac{d\mu(t)}{dt} = \frac{\sqrt{3}}{2\pi|t|}\left(\frac{3t^2+1}{9h(t)} - h(t)\right), \quad |t| \leq \sqrt{(11+5\sqrt{5})/2},$$

where

$$h(t) = \sqrt[3]{\frac{18t^2+1}{27} + \sqrt{\frac{t^2(1+11t^2-t^4)}{27}}}.$$

Fortunately our approach completely avoids having to use this frightening formula; we just use Stieltjes transforms.

This distribution has recently attracted some interest in connection with stochastic calculus, both classical and free. See for example Deya-Nourdin `arXiv:1107.3538`, MR2893412.

Statement of theorem

For $A \in \mathrm{Mat}_n(\mathbb{C})$, let $[\![A]\!]$ denote the largest singular value of A.
For $z \in \mathfrak{h}$ let $h = |z+\zeta| \wedge |z-\zeta| \wedge 1$.

Theorem (A., `arXiv:1308.4668`, to appear in AIHP)

Notation and assumptions are as above. There exists a random variable $\mathbf{K} \geq 1$ *with the following two properties:*

On the event $[[\![U]\!] \vee [\![V]\!] \leq 4]$ *one has*

$$\bigvee_{i=1}^{N} \left| (\{UV\} - z\mathbf{I}_N)^{-1}(i,i) - m_{\{\mathbf{uv}\}}(z) \right| \leq \frac{\mathbf{K}}{\sqrt{Nh\Im z}}$$

for $z \in \mathfrak{h}$ *such that* $|\Re z| \vee \Im z \leq 64$ *and* $\mathbf{K}^2/N \leq h^2 \Im z$.

For every $t > 0$ *one has* $\Pr(\mathbf{K} > t^{2\alpha_0+1}) \leq \beta_0 N^{\beta_1} \exp(-\beta_2 t)$, *for positive constants* β_0 *and* β_2 *depending only on* α_0 *and* α_1 *and a positive absolute constant* β_1.

Remarks

In particular, β_0, β_1 and β_2 are independent of N.

The theorem is certainly not so sharp as the sharpest available concerning the local semicircle law.

Concerning $\mathbf{K}$, the main point is that the tail of $\mathbf{K}$ decays rather rapidly. For intuitive understanding of statements involving $\mathbf{K}$, just put $\mathbf{K} = O((\log N)^c)$.

The probability $\Pr([\![U]\!] \vee [\![V]\!]\, 4)$ decays as $c_0 \exp(-c_1 N^{c_2})$ by the classical Füredi-Komlös estimate. (See also Vu for a corrected and sharpened version of these estimates.) So conditioning on $[\![U]\!] \vee [\![V]\!] \leq 4$ is not costly.

Delocalization

Corollary

Evaluate $\{UV\}$ *and* $\mathbf{K}$ *at a sample point of the event* $[[\![U]\!] \vee [\![V]\!] \leq 4]$. *We still write* $\{UV\}$ *and* $\mathbf{K}$ *for these evaluations, respectively. Let* λ *be any eigenvalue of* $\{UV\}$ *and let* v *be a corresponding unit-length (right) eigenvector. Let* $\rho = \mathbf{K}^2/N$ *and for simplicity assume that* $\rho < 1$. *Let* $\sigma \in [\rho, \rho^{1/3}]$ *be defined by the equation*

$$\rho = h^2 \Im z|_{z=\lambda+\mathrm{i}\sigma}.$$

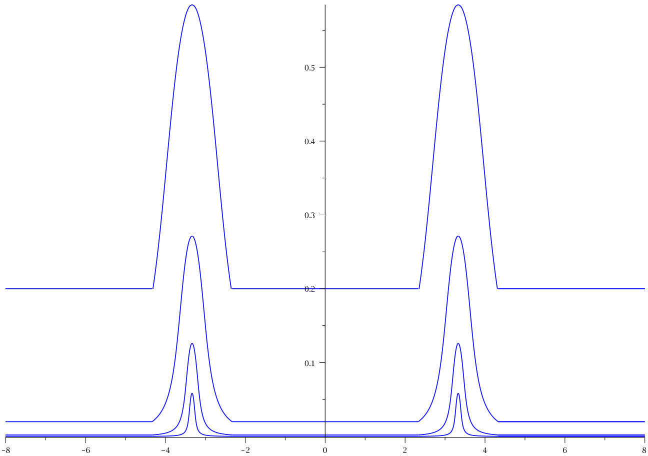
Then we have

$$\bigvee_{i=1}^{N} |v(i)| \leq \sqrt{2\sigma}.$$

Figure 1 shows σ as a function of λ for $\rho = 0.2, 0.02, 0.002, 0.0002$.

Visual aid

Figure: Closest permissible approach σ to the real axis as a function of λ for $\rho = 0.2, 0.02, 0.002, 0.0002$



Remarks

Very roughly speaking the bound on $\bigvee_{i=1}^{N} |v(i)|$ for λ in the bulk of the spectrum works out to be $O((\log N)^c/\sqrt{N})$. But the bound weakens toward spectrum edge, and becomes non-optimal.

In the course of proving the theorem we actually give an explicit (if gruesome) construction of $\mathbf{K}$. The construction seems susceptible to generalization using the SALT.

Proof of the corollary

Let $32 \geq \lambda_1 \geq \cdots \geq \lambda_N \geq -32$ be the eigenvalues of $\{UV\}$ and let $v_1, \ldots, v_N$ be corresponding unit-length eigenvectors. We may assume that $\lambda = \lambda_{i_0}$ and $v = v_{i_0}$ for a suitable index i_0. Let $z_0 = \lambda + \mathrm{i}\sigma$ and $h_0 = h|_{z=z_0}$, noting that

$$|\lambda| \vee \sigma = |\Re z_0| \vee \Im z_0 \leq 64 \quad \text{and} \quad \frac{\mathbf{K}}{\sqrt{N h_0 \Im z_0}} = \sqrt{h_0} \leq 1$$

by our assumption that $[\![U]\!] \vee [\![V]\!] \leq 4$ and simplifying assumption that $\rho < 1$. Thus we have

$$\begin{aligned} 2 \;&\geq\; 1 + \frac{\mathbf{K}}{\sqrt{N h_0 \Im z_0}} \;\geq\; \Im(\{UV\} - z_0 \mathbf{I}_N)^{-1}(i,i) \\ &=\; \sum_{j=1}^{N} \frac{\sigma |v_j(i)|^2}{(\lambda_j - \lambda_{i_0})^2 + \sigma^2} \;\geq\; \frac{|v(i)|^2}{\sigma} \end{aligned}$$

by Theorem 1 and the uniform bound $|m| < 1$ which we will explain later in the talk. □

Part III:

Generalized resolvents and related apparatus

Goal

Keep in mind that we aim to show that

$$\bigvee_{i=1}^{N} \left| (\{UV\} - z\mathbf{I}_N)^{-1}(i,i) - m_{\{\mathbf{uv}\}}(z) \right|$$

is small.

Here and below $z \in \mathfrak{h}$ is an arbitrary point in the upper half-plane.

Basic (random) matrices Λ, X, and W

To that end, let

$$
\begin{aligned}
\Lambda &= \begin{bmatrix} z & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_3, \\
X &= \begin{bmatrix} 0 & \frac{U-V}{\sqrt{2}} & \frac{-U-V}{\sqrt{2}} \\ \frac{U-V}{\sqrt{2}} & 0 & 0 \\ \frac{-U-V}{\sqrt{2}} & 0 & 0 \end{bmatrix} \in \mathrm{Mat}_{3N} \text{ and} \\
W &= \begin{bmatrix} \mathbf{I}_N & 0 & 0 \\ \frac{-U+V}{\sqrt{2}} & \mathbf{I}_N & 0 \\ \frac{-U-V}{\sqrt{2}} & 0 & \mathbf{I}_N \end{bmatrix} \in \mathrm{Mat}_{3N}.
\end{aligned}
$$

Generalized resolvent R

Continuing to define auxiliary objects, let

$$R = (X - \Lambda \otimes \mathbf{I}_N)^{-1} = W \begin{bmatrix} (\{UV\} - z\mathbf{I}_N)^{-1} & 0 & 0 \\ 0 & \mathbf{I}_N & 0 \\ 0 & 0 & -\mathbf{I}_N \end{bmatrix} W^*,$$

which we call the *generalized resolvent* for anticommutators.

The construction of R is an example of SALT in action.

The "block trace" G

For $i = 1, \ldots, N$, let $e_i \in \mathrm{Mat}_{1\times N}$ denote the i^{th} row of $\mathbf{I}_N$ and let $\mathbf{e}_i = \mathbf{I}_3 \otimes e_i \in \mathrm{Mat}_{3\times 3N}$. ($\otimes$ is Kronecker product.) Then let

$$G_i \; = \; \mathbf{e}_i R \mathbf{e}_i^* \in \mathrm{Mat}_3, \;\; G \; = \; \frac{1}{N}\sum_{i=1}^{N} G_i \in \mathrm{Mat}_3 \;\; (\text{"block trace"}).$$

Crucially:

$$G(1,1) = \frac{1}{N}\mathrm{tr}(\{UV\} - z\mathbf{I}_3)^{-1}.$$

So the Stieltjes transform we want is contained in G. With that justification, hereafter the focus is on controlling G.

A solution of the Schwinger-Dyson equation

With $m = m_{\{\mathbf{uv}\}}(z)$, let

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & -\frac{1}{m-1} & 0 \\ 0 & 0 & -\frac{1}{m+1} \end{bmatrix} \in \mathrm{Mat}_3 \quad (m = m_{\{\mathbf{uv}\}}(z))$$

Let $\Phi \in B(\mathrm{Mat}_3)$ be the (constant) linear map defined by

$$\Phi(A) = (e_{12} + e_{21})A(e_{12} + e_{21}) + (e_{13} + e_{31})A(e_{13} + e_{31}) \tag{1}$$

where $\{e_{ij}\}_{i,j=1}^{3}$ is the standard basis for Mat_3 consisting of elementary matrices. Then

$$\mathbf{I}_3 + M(\Lambda + \Phi(M)) = 0,$$

which is an instance of the *Schwinger-Dyson equation*.

Background on Schwinger-Dyson equation

Anderson, G., Lecture notes, June 2012,
`z.umn.edu/selfadjlintrick`

Anderson, G., Guionnet, A., and Zeitouni, O., *An Introduction to Random Matrices.* MR2760897 (Chap. 5)

Helton, J., Rashidi Far, R. and Speicher, R. Int. Math. Res. Not., no. 22, Art. ID rnm086, 2007. MR2376207

Nica, A. and Speicher, R., *Lectures on the combinatorics of free probability.* MR2266879

...and references from previous lectures in this series.

Further objects

For $i = 1, \ldots, N$ let

$$
\begin{aligned}
R_i &= (\hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^* - \Lambda \otimes \mathbf{I}_{N-1})^{-1} \in \mathrm{Mat}_{3(N-1)}, \\
\widehat{G}_i &= \frac{1}{N} \sum_{j=1}^{N} \mathbf{e}_j \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i \mathbf{e}_j^* \in \mathrm{Mat}_3, \\
Q_i &= \mathbf{e}_i X \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i X \mathbf{e}_i^* - \mathbf{e}_i X \mathbf{e}_i^* - \Phi(\widehat{G}_i) \in \mathrm{Mat}_3, \\
\mathfrak{K}_i &= 1 \vee \frac{[\![Q_i]\!]}{\frac{1}{\sqrt{N}} \left(1 \vee \frac{[\![R_i]\!]_2}{\sqrt{N}} \right)} \in [1, \infty) \text{ and } \mathfrak{K} = \bigvee_{i=1}^{N} \mathfrak{K}_i.
\end{aligned}
$$

Ultimately we define the random variable $\mathbf{K}$ in Theorem 1 in terms of $\mathfrak{K}$.

Key relations

Notation: $x_\bullet = \max(x,1) = x \vee 1$.

By not very pleasant calculations we prove that

$$\begin{aligned}
\bigvee_{i=1}^{N} [\![G_i - M]\!] &\leq \frac{2^7([\![U]\!] \vee [\![V]\!] \vee 1)^2}{\Im z} \\
\left[\!\left[G - \widehat{G}_i\right]\!\right] &\leq 16[\![W]\!]^2 \frac{(\Im z)_\bullet}{N\Im z}[\![G_i]\!]_\bullet \left[\!\left[G_i^{-1}\right]\!\right], \text{ and} \\
\left[\!\left[G_i^{-1} + \Lambda + \Phi(\widehat{G}_i)\right]\!\right] &\leq 4\mathfrak{K}[\![W]\!]\sqrt{\frac{(\Im z)_\bullet}{N\Im z}}\left[\!\left[\widehat{G}_i\right]\!\right]_\bullet^{1/2}.
\end{aligned}$$

These are estimates in the spirit of Erdös-Yau-Yin.

The problem we are then faced with is that of making $\bigvee_{i=1}^{N} [\![G_i - M]\!]$ small using the interlocking set of inequalities presented above.

The key idea: stability

One wants to make $\bigvee_{i=1}^{N} [\![G_i - M]\!]$ small. But this quantity is not directly accessible to estimation. Since one has exactly

$$\mathbf{I}_3 + (\Lambda + \Phi(M))M = 0$$

and because this equation is *stable* it is possible to bound $\bigvee_{i=1}^{N} [\![G_i - M]\!]$ in terms of $[\![E]\!]$ where

$$\mathbf{I}_3 + (\Lambda + \Phi(G))G = E.$$

We focus on stability for the rest of the talk.

Part IV:
Geometry of $m_{\{\mathbf{uv}\}}(z)$

Rationale for this interlude

We look at the geometry of $m = m_{\{\mathbf{uv}\}}(z)$ because plausibly and in fact we will have to understand it in order to analyze stability of the instance of the Schwinger-Dyson equation satisfied by M. We cheerfully skip the proofs because they are pure high school algebra do-able with a computer.

The rational map $m \mapsto \frac{1}{m-1} + \frac{1}{m+1} - \frac{1}{m}$

There are various apts ways to rewrite the equation

$$zm^3 - m^2 - zm - 1 = 0,$$

e.g.,

$$z = \frac{m^2 + 1}{m^3 - m} = \frac{1}{m-1} + \frac{1}{m+1} - \frac{1}{m}.$$

By and large, to understand the geometry of $m_{\{\mathbf{uv}\}}(z)$ means to understand the geometry of the rational map

$$m \mapsto \frac{1}{m-1} + \frac{1}{m+1} - \frac{1}{m}.$$

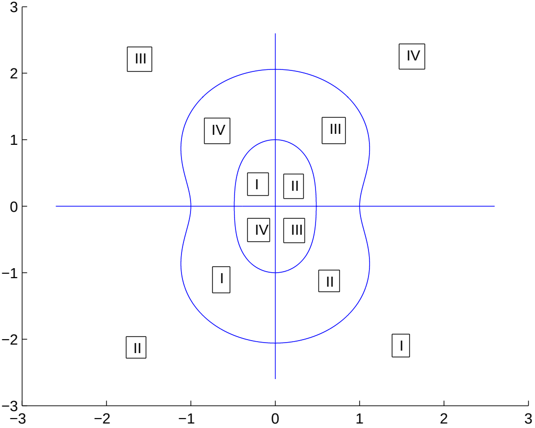
Preparation for quadrant lifting diagram

Let $m = u + \mathrm{i}v$ with u and v real. Then for $m^3 - m \neq 0$ we have formulas

$$\begin{aligned} \Re\frac{m^2+1}{m^3-m} &= \frac{u((u^2+v^2)^2-4v^2-1)}{|m^3-m|^2}, \\ \Im\frac{m^2+1}{m^3-m} &= -\frac{v((u^2+v^2)^2+4u^2-1)}{|m^3-m|^2} \end{aligned}$$

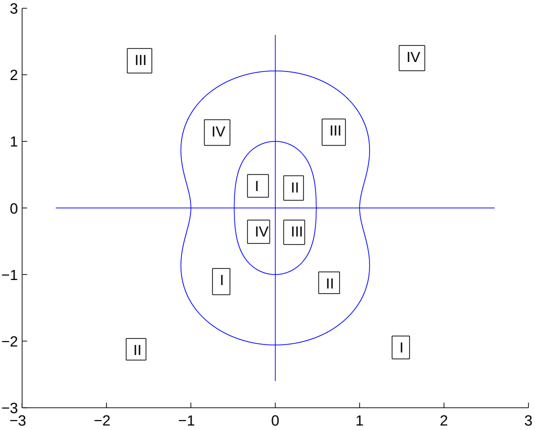
making it possible to determine for each point m which quadrant of the complex plane $\frac{m^2+1}{m^3-m}$ falls in. The following diagram summarizes the answer.

Quadrant lifting diagram



The image of $\mathfrak{h}$ under the map $z \mapsto m_{\{\mathbf{uv}\}}(z)$ is the upper-half-small-oval.

Quadrant lifting diagram



$\pm\sqrt{-1}$: endpoints of major axis of little oval.
± 1: endpoints of minor axis of big oval.

The special number ω

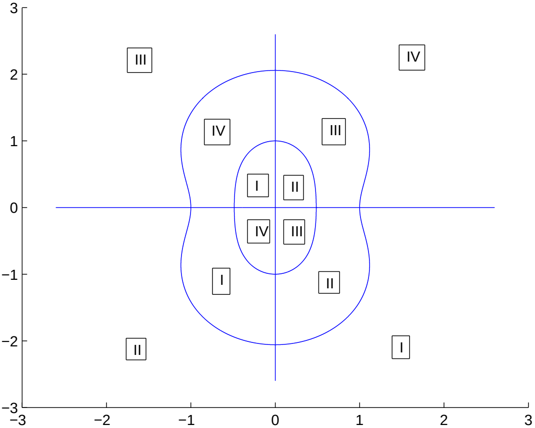
Let

$$\omega = \sqrt{\sqrt{5} - 2} \stackrel{\sim}{=} 0.4858682712,$$

which is the unique positive root of the polynomial

$$m^4 + 4m^2 - 1 = (m - \omega)(m + \omega)(m - \mathrm{i}/\omega)(m + \mathrm{i}/\omega).$$

Quadrant lifting diagram



$\pm\omega$: endpoints of minor axis of little oval.
$\pm\mathrm{i}/\omega$: endpoints of major axis of big oval.

The special number ζ

Repeating the definition from earlier in the talk, let

$$\zeta = \sqrt{\frac{11 + 5\sqrt{5}}{2}} \simeq 3.330190676.$$

This is the unique positive root of the polynomial

$$z^4 - 11z^2 - 1 = (z - \zeta)(z + \zeta)(z - \mathrm{i}/\zeta)(z + \mathrm{i}/\zeta).$$

Why the special numbers are special

It can be shown that the system of equations

$$\begin{aligned} zm^3 - m^2 - zm - 1 &= 0 \\ \frac{\partial}{\partial m}\left(zm^3 - m^2 - zm - 1\right) &= 0 \end{aligned} \qquad (2)$$

has exactly four complex solutions, namely

$$(z, m) = (-\zeta, \omega), (\zeta, -\omega), (-\mathrm{i}/\zeta, \mathrm{i}/\omega), (\mathrm{i}/\zeta, -\mathrm{i}/\omega). \qquad (3)$$

These four points in $\mathbb{C}^2$ are where the Implicit Function Theorem *fails* to yield locally a solution $m = m(z)$ of the equation

$$zm^3 - m^2 - zm - 1 = 0$$

depending analytically on z.

Estimates for m

What we need of the geometry can then be reduced to the following estimates.

Proposition

If $z \in \mathfrak{h}$ and $m = m_{\{\mathbf{uv}\}}(z)$, then

$$
\begin{aligned}
|m| &\le 1 \wedge \frac{1}{\Im z}, \\
|\Re m| &\le \omega < \frac{1}{2} \quad \text{and} \\
|m^2 - \omega^2| &\ge \frac{\sqrt{|z+\zeta| \wedge |z-\zeta| \wedge 1}}{c},
\end{aligned}
$$

where $c \ge 1$ is an absolute constant.

Remark on high school algebra

To prove the last estimate we hacked away with a computer until we found the following exact identity. Let

$$\rho = \frac{\omega^3 + 5\omega}{2} \simeq 1.272019648.$$

Then we have

$$\frac{1}{|m^2 - \omega^2|} = \frac{|m^2 - \rho^2|^{1/2}}{|m^2 - 1|} \frac{1}{|m|} \frac{\zeta}{|z^2 - \zeta^2|^{1/2}}.$$

Further remarks

Note the crucial role of ζ and ω in pinpointing behavior of m. The preceding estimates make the point that the singularities of the plane algebraic curve

$$zm^3 - m^2 - zm - 1 = 0$$

have a dominant influence on the analysis.

For a general polynomial in free semicircular variables one would like to have a good geometric understanding of the plane algebraic curve where the Stieltjes transform and R-transform live. In particular, we would like to be rid of the massive amounts of high school algebra we used for analyzing $UV + VU$.

Part V:

Stability of the Schwinger-Dyson equation

A general form of the Schwinger-Dyson equation

Let $\mathcal{S}$ be a finite-dimensional unital Banach algebra. A triple

$$(\Lambda, M, \Phi) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$$

is said to satisfy the *Schwinger-Dyson equation* if

$$1_{\mathcal{S}} + (\Lambda + \Phi(M))M = 0,$$

in which case M is necessarily invertible. We emphasize that in our (somewhat eccentric) usage, a solution of the Schwinger-Dyson equation is not a function; rather, it is just a point in the space $\mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$.

Nondegeneracy

Now let $(\Lambda, M, \Phi) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S})$ be any solution of the Schwinger-Dyson equation. If the linear map

$$\left(x \mapsto M^{-1}x - \Phi(x)M\right) \in B(\mathcal{S})$$

is invertible we say that (Λ, M, Φ) is *nondegenerate* in which case we let

$$\kappa = \kappa_{\Lambda,M,\Phi}$$

denote the inverse and we also say that the quadruple

$$(\Lambda, M, \Phi, \kappa) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S}) \times B(\mathcal{S})$$

is a *nondegenerate* solution of the Schwinger-Dyson equation.

A general stability estimate

Notation: $x_\bullet = 1 \vee x = \max(1, x)$.

Proposition (Proposition 4.4 of `arXiv:1308.4668`)

Let $\mathcal{S}$ be a finite-dimensional unital Banach algebra. Let $(\Lambda_0, M_0, \Phi_0, \kappa_0)$ be a nondegenerate solution of the Schwinger-Dyson equation defined over $\mathcal{S}$. Fix $G_0 \in \mathcal{S}$ and let $E_0 = 1_{\mathcal{S}} + (\Lambda_0 + \Phi_0(G_0))G_0 \in \mathcal{S}$. We then have

$$\begin{aligned} & [\![G_0 - M_0]\!] \leq \frac{1}{8[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet} \\ \Rightarrow \quad & [\![G_0 - M_0]\!] \leq 20[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet [\![M_0]\!]_\bullet^2 [\![E_0]\!]. \end{aligned}$$

The message here is that the quantity $[\![\kappa_0]\!]_\bullet$ is what one must control in order to have stability in a useful quantitative sense.

Example: semicircular case

Proposition

Fix $z \in \mathfrak{h}$ and let $m = \frac{1}{2\pi}\int_{-2}^{2}\frac{\sqrt{4-t^2}\,dt}{t-z}$. (i) One has $\Im m > 0$, $z = -m - m^{-1}$, and $|m| \leq 1 \wedge \frac{1}{\Im z}$. (ii) Let $\kappa = (m^{-1} - m)^{-1})$. The quadruple $(z, m, 1, \kappa)$ is a nondegenerate solution of the Schwinger-Dyson equation defined over $\mathbb{C}$. (iii) One has

$$|\kappa|_{\bullet} \leq \frac{c}{\sqrt{1 \wedge |z-2| \wedge |z+2|}}$$

where c is an absolute constant.

Note that the last estimate holds all the way down to the real axis, staying away from spectrum edge.

Example: anticommutator case

Proposition

(i)For each $z \in \mathfrak{h}$ *the triple* (Λ, M, Φ) *is a nondegenerate solution of the Schwinger-Dyson equation. (ii) Furthermore, we have bounds*

$$[\![\Lambda]\!] \leq 1 + |z|,\ [\![\Phi]\!] \leq 8,\ [\![M]\!] \leq 2 \text{ and } [\![\Lambda|_{z=0} + M]\!] \leq 2\left(1 \wedge \frac{1}{\Im z}\right).$$

(iii) Let $\kappa = \kappa_{\Lambda, M, \Phi} \in B(\mathrm{Mat}_3)$. *Finally,*

$$[\![\kappa]\!]_{\bullet} \leq \frac{c}{\sqrt{|z+\zeta| \wedge |z-\zeta| \wedge 1}} \tag{4}$$

where (recall) $\zeta = \sqrt{\frac{11+5\sqrt{5}}{2}} \cong 3.33$ *and* $c \geq 1$ *is an absolute constant.*

Again, note that the last estimate holds all the way down to the real axis, staying away from spectrum edge.

Note on the proof of Proposition 2

As in the statement of Proposition 2, let

$$(\Lambda_0, M_0, \Phi_0, \kappa_0) \in \mathcal{S} \times \mathcal{S} \times B(\mathcal{S}) \times B(\mathcal{S})$$

be a nondegenerate solution of the Schwinger-Dyson equation.

Lemma

Assumptions and notation are as above. Fix $(\Lambda_1, M_1) \in \mathcal{S} \times \mathcal{S}$ *and write* $(\Theta, H) = (\Lambda_1 - \Lambda_0, M_1 - M_0)$*. Then the* (Λ_1, M_1, Φ_0) *is a solution of the Schwinger-Dyson equation if and only if the* H *is a solution of the fixed-point equation*

$$H = \kappa_0 \left(\Theta M_0 + \Theta H + \Phi_0(H) H\right).$$

The latter fixed-point equation can then be analyzed in a routine way using the Banach Fixed Point Theorem, leading to the result. Our analysis is rather crude. Doubtless our approach could be refined.

Part VI:
The self-consistent equation

Reminder of anticommutator-related objects

Recall that $z \in \mathfrak{h}$ is an arbitrary point in the upper half-plane.

Recall that

$$\Lambda = \begin{bmatrix} z & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_3,$$

$$X = \begin{bmatrix} 0 & \frac{U-V}{\sqrt{2}} & \frac{-U-V}{\sqrt{2}} \\ \frac{U-V}{\sqrt{2}} & 0 & 0 \\ \frac{-U-V}{\sqrt{2}} & 0 & 0 \end{bmatrix} \in \mathrm{Mat}_{3N} \text{ and}$$

$$W = \begin{bmatrix} \mathbf{I}_N & 0 & 0 \\ \frac{-U+V}{\sqrt{2}} & \mathbf{I}_N & 0 \\ \frac{-U-V}{\sqrt{2}} & 0 & \mathbf{I}_N \end{bmatrix} \in \mathrm{Mat}_{3N}.$$

Reminder (continued)

Recall that

$$R = (X - \Lambda \otimes \mathbf{I}_N)^{-1} = W \begin{bmatrix} (\{UV\} - z\mathbf{I}_N)^{-1} & 0 & 0 \\ 0 & \mathbf{I}_N & 0 \\ 0 & 0 & -\mathbf{I}_N \end{bmatrix} W^*,$$

which is the *generalized resolvent* for anticommutators.

Reminder (continued)

Here is the relevant nondegenerate solution of the Schwinger-Dyson equation:

$$
\begin{aligned}
\Lambda &= \begin{bmatrix} z & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Mat}_3, \\
M &= \begin{bmatrix} m & 0 & 0 \\ 0 & -\frac{1}{m-1} & 0 \\ 0 & 0 & -\frac{1}{m+1} \end{bmatrix} \in \mathrm{Mat}_3 \ \ (m = m_{\{\mathbf{uv}\}}(z)), \\
\Phi &= \left(\begin{bmatrix} x_1 & x_4 & x_6 \\ x_5 & x_2 & x_8 \\ x_7 & x_9 & x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_2 + x_3 & x_5 & x_7 \\ x_4 & x_1 & 0 \\ x_6 & 0 & x_1 \end{bmatrix} \right) \in B(\mathrm{Mat}_3), \\
\kappa &= \text{(complicated and therefore omitted here)}
\end{aligned}
$$

(All but Φ depend on z although the notation does not show it.)

For $i = 1, \ldots, N$, let $e_i \in \mathrm{Mat}_{1\times N}$ denote the i^{th} row of $\mathbf{I}_N$ and let $\mathbf{e}_i = \mathbf{I}_3 \otimes e_i \in \mathrm{Mat}_{3\times 3N}$. Then let

$$
\begin{aligned}
R &= (X - \Lambda \otimes \mathbf{I}_N)^{-1} \in \mathrm{Mat}_{3N}, \\
G_i &= \mathbf{e}_i R \mathbf{e}_i^* \in \mathrm{Mat}_3, \quad G = \frac{1}{N}\sum_{i=1}^{N} G_i \in \mathrm{Mat}_3, \\
R_i &= (\hat{\mathbf{e}}_i X \hat{\mathbf{e}}_i^* - \Lambda \otimes \mathbf{I}_{N-1})^{-1} \in \mathrm{Mat}_{3(N-1)}, \\
\widehat{G}_i &= \frac{1}{N}\sum_{j=1}^{N} \mathbf{e}_j \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i \mathbf{e}_j^* \in \mathrm{Mat}_3, \\
Q_i &= \mathbf{e}_i X \hat{\mathbf{e}}_i^* R_i \hat{\mathbf{e}}_i X \mathbf{e}_i^* - \mathbf{e}_i X \mathbf{e}_i^* - \Phi(\widehat{G}_i) \in \mathrm{Mat}_3, \\
\mathfrak{K}_i &= 1 \vee \frac{[\![Q_i]\!]}{\frac{1}{\sqrt{N}}\left(1 \vee \frac{[\![R_i]\!]_2}{\sqrt{N}}\right)} \in [1, \infty) \text{ and } \mathfrak{K} = \bigvee_{i=1}^{N} \mathfrak{K}_i.
\end{aligned}
$$

Key identities

$$\begin{aligned}
\bigvee_{i=1}^{N} [\![G_i - M]\!] &\leq \frac{2^7([\![U]\!] \vee [\![V]\!] \vee 1)^2}{\Im z} \\
\left[\!\left[G - \widehat{G}_i\right]\!\right] &\leq 16[\![W]\!]^2 \frac{(\Im z)_\bullet}{N \Im z} [\![G_i]\!]_\bullet \left[\!\left[G_i^{-1}\right]\!\right], \quad \text{and} \\
\left[\!\left[G_i^{-1} + \Lambda + \Phi(\widehat{G}_i)\right]\!\right] &\leq 4\mathfrak{K}[\![W]\!] \sqrt{\frac{(\Im z)_\bullet}{N \Im z}} \left[\!\left[\widehat{G}_i\right]\!\right]_\bullet^{1/2}.
\end{aligned}$$

Setup for the self-consistent equation

Here is a technical result similar in intent to Lemma 4.3 of Erdös-Yau-Yin but superficially different in form, to give a flavor of estimates involved. The result is an elaboration and refinement of Proposition 2.

Forget now about the specifics of the anticommutator setup.

Fix a finite-dimensional unital Banach algebra $\mathcal{S}$. Fix a nondegenerate solution

$$(\Lambda_0, M_0, \Phi_0, \kappa_0)$$

of the Schwinger-Dyson equation defined over $\mathcal{S}$. Fix a family

$$\{G_i, \widehat{G}_i\}_{i=1}^N$$

of elements of $\mathcal{S}$ where all the G_i are invertible. (It is not necessary to assume that the $\widehat{G}_i$ are invertible.)

Setup for the self-consistent equation (continued)

Consider the statistic

$$\mathfrak{E} \;=\; \bigvee_{i=1}^{N} \frac{\left[\!\left[G_i^{-1} + \Lambda_0 + \Phi_0(\widehat{G}_i) \right]\!\right]}{\left[\!\left[\widehat{G}_i \right]\!\right]_{\bullet}^{1/2}} \vee \bigvee_{i=1}^{N} \sqrt{\frac{\left[\!\left[\widehat{G}_i - \frac{1}{N}\sum_{i=1}^{N} G_i \right]\!\right]}{[\![G_i]\!]_{\bullet} \left[\!\left[G_i^{-1} \right]\!\right]}},$$

which is a gauge of error in this situation. The idea to emphasize the statistic $\mathfrak{E}$ comes from Lemma 4.3 of Erdös-Yau-Yin.

Refinement of Proposition 2

Proposition (Proposition 6.2 of arXiv:1308.4668)

Notation and assumptions are as above. We have

$$\bigvee_{i=1}^{N} [\![G_i - M_0]\!] \leq \frac{1}{8[\![\kappa_0]\!]_\bullet [\![\Phi_0]\!]_\bullet}$$

$$\Rightarrow \quad \bigvee_{i=1}^{N} [\![G_i - M_0]\!] \leq 2^{14}(1 + [\![M_0]\!])^7([\![\Phi_0]\!]_\bullet \vee [\![\Lambda_0]\!]_\bullet)^4 [\![\kappa_0]\!]_\bullet \mathfrak{E}.$$

This refinement applied in the anticommutator situation makes it possible to show not only that $[\![G - M]\!]$ is small but also that $\bigvee_{i=1}^{N} [\![G_i - M]\!]$ is small, leading ultimately to a result strong enough to imply delocalization of eigenvectors in the bulk.

Part VII:
A free probability problem

Formulation of the question

Let $f \in \mathbb{C}\langle \mathbf{X}_1, \ldots, \mathbf{X}_q \rangle$ be a self-adjoint noncommutative polynomial. Let $\mathbf{x}_1, \ldots, \mathbf{x}_q$ be free semicircular noncommutative random variables. The C^*-algebra generated by these variables has no projectors other than 0 and 1. So the spectrum of $f(\mathbf{x}_1, \ldots, \mathbf{x}_q)$ is connected and since the law is anyhow algebraic, the spectrum is a bounded closed interval, say $[\alpha, \beta]$. For f there is in principle an essentially unique minimal linearization L, and to the latter we may associate a Schwinger-Dyson equation and a numerical function $[\![\kappa]\!]$ of $z \in \mathfrak{h}$. The problem is then to prove an estimate of the form, say, $[\![\kappa]\!] \leq \frac{c}{\sqrt{|z-\alpha| \wedge |z-\beta| \wedge 1}}$ if possible, and otherwise to classify the exceptional types of estimates that might come up. Then by the methods outlined above one would get a local limit law for $f(X_1^{(N)}, \ldots, X_q^{(N)})$ where $X_1^{(N)}, \ldots, X_q^{(N)}$ are independent N-by-N Wigner matrices.

Analogue of algebro-geometric soliton theory?

We hope for a refined understanding of the algebraic geometry of the linearization (realization) of an operator of the form $f(\mathbf{x}_1, \ldots, \mathbf{x}_q)$ modeled on soliton theory that would render problems of stability of the Schwinger-Dyson equation transparent.

Thank you!