A GENERALIZATION OF WEYL'S IDENTITY FOR D_n

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Abstract. The expansion in Schur functions of the product

$$\prod_{i<j}(1-x_ix_j)$$

is well-known. It is more or less equivalent to Weyl's identity for root-systems of type D_n. In this paper we obtain the expansion in Schur functions of the product

$$\prod_{i<j}\left(1-\sum_{\ell>0}a_\ell x_ix_j\,\frac{x_i^\ell-x_j^\ell}{x_i-x_j}\right),$$

thus generalizing Weyl's identity. We obtain this result by systematic calculation in fermionic Fock space. But familiarity with the latter is not a prerequisite for reading this paper. We develop from scratch the modest amount of theory that we need in elementary and purely algebraic fashion, taking pains to integrate the theory with classical symmetric function theory. The tools developed in this paper ought to have many further applications, e. g., to random matrix theory and to computational abelian function theory.

1. Introduction

1.1. Schur function expansions.

1.1.1. *Weyl's identity for D_n.* The expansion in Schur functions of the product $\prod_{i<j}(1-x_ix_j)$ is well-known. It is more or less equivalent to Weyl's identity for root-systems of type D_n. The apparatus of [Macdonald] taken for granted, Weyl's identity can be put in the form

$$\prod_{i<j}(1-x_ix_j)=\sum_{\lambda=(\alpha_1\ldots\alpha_r|\beta_1\ldots\beta_r)}(-1)^{\beta_1+\cdots+\beta_r}\det_{i,j=1}^{r}\delta_{\alpha_i,\beta_j-1}\cdot s_\lambda,$$

where $(\alpha_1\ldots\alpha_r|\beta_1\ldots\beta_r)$ is the Frobenius notation for the partition λ, and s_λ denotes the S-function (Schur function) of the x_i indexed by λ. For further discussion of this identity and also of its analogues for root-systems other than D_n, see [Macdonald, I, 5, Ex. 9].

1.1.2. *The generalization.* Define coefficients $M_{\alpha\beta}$ by the two-variable power series identity

$$\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} M_{\alpha\beta}x^\alpha y^\beta = -\frac{y}{x}\frac{\partial}{\partial y}\log\left(1-\sum_{\ell>0} a_\ell xy\,\frac{x^\ell-y^\ell}{x-y}\right).$$

We shall prove (Theorem 7.1 below) that

$$\prod_{i<j}\left(1-\sum_{\ell>0} a_\ell x_i x_j\,\frac{x_i^\ell-x_j^\ell}{x_i-x_j}\right)$$

$$=\sum_{\lambda=(\alpha_1\ldots\alpha_r|\beta_1\ldots\beta_r)}(-1)^{\beta_1+\cdots+\beta_r}\det_{i,j=1}^{r} M_{\alpha_i\beta_j}\cdot s_\lambda.$$

To the best of our knowledge this Schur function expansion is new.

1.2. **Technical tools.** We now discuss our method.

1.2.1. *Fermionic Fock space.* We take $\mathbb{C}$ as the scalar field for linear algebra. Let $\mathbb{C}((1/t))$ be the field consisting of Laurent series $\sum_i a_i t^i$ in t with complex coefficients a_i vanishing for $i \gg 0$. Given $f = \sum_i a_i t^i \in \mathbb{C}((1/t))$, put $f_{<0} = \sum_{i<0} a_i t^i$ and $\mathrm{Res}(f) = a_{-1}$. Given linear operators X and Y on a common vector space, put $\{X, Y\} = XY + YX$. Suppose now that we are given

- a vector space $\mathcal{H}$ equipped with a nonzero vector $|\bullet\rangle$, and
- linear maps $\left.\begin{array}{c} f\mapsto f^\sharp \\ g\mapsto g^\flat\end{array}\right\} : \mathbb{C}((1/t)) \to \left(\begin{array}{c}\text{space of linear}\\ \text{operators on } \mathcal{H}\end{array}\right)$

such that

- $\{f^\sharp, g^\sharp\} = 0$, $\{f^\flat, g^\flat\} = 0$, $\{f^\sharp, g^\flat\} = \mathrm{Res}(fg)$, $(f_{<0})^\sharp|\bullet\rangle = 0$, and $(g_{<0})^\flat|\bullet\rangle = 0$ for all $f, g \in \mathbb{C}((1/t))$, and
- for all subspaces $V \subset \mathcal{H}$ such that $|\bullet\rangle \in V$, if for all $f, g \in \mathbb{C}((1/t))$ and $v \in V$ we have $f^\sharp v, g^\flat v \in V$, then $V = \mathcal{H}$.

In this situation we say that the quadruple

$$\left(\mathcal{H},\quad |\bullet\rangle,\quad f\mapsto f^\sharp,\quad g\mapsto g^\flat\right)$$

is a model of *fermionic Fock space.* It is physics folklore that up to the evident notion of isomorphism there exists only one model of fermionic Fock space. Ultimately the source for the notion of fermionic Fock space is Dirac's theory of electrons and positrons. In a purely algebraic (physics-free) way we prove existence and uniqueness in §2 and we develop basic rules of calculation in §3 and §4.

1.2.2. *A natural linearly independent set in $\mathcal{H}$ indexed by partitions.* As comes out in the course of the discussion of existence and uniqueness (see Lemma 2.7.3 below), a basis for $\mathcal{H}$ is formed by the family of vectors of the form

$$(t^{\alpha_1})^\sharp \cdots (t^{\alpha_r})^\sharp (t^{\beta_1})^\flat \cdots (t^{\beta_s})^\flat |\bullet\rangle$$

where $\{\alpha_i\}_{i=1}^r$ and $\{\beta_j\}_{j=1}^s$ are any strictly decreasing sequences of nonnegative integers. Notice that we need not have $r = s$ here. Given now a partition $\lambda = (\alpha_1 \cdots \alpha_r | \beta_1 \cdots \beta_r)$ put

$$|\lambda\rangle = (-1)^{\beta_1+\cdots+\beta_r} (t^{\alpha_1})^\sharp \cdots (t^{\alpha_r})^\sharp (t^{\beta_r})^\flat \cdots (t^{\beta_1})^\flat |\bullet\rangle \in \mathcal{H},$$

and let $\mathcal{H}_0 \subset \mathcal{H}$ be the span of the independent family $\{|\lambda\rangle\}$. Quantities of interest from the point of view of classical symmetric function theory can often be interpreted as the entries of matrices representing naturally occurring operators on $\mathcal{H}_0$ with respect to the basis $\{|\lambda\rangle\}$. In §5, §6 and §7 we work out such interpretations for all the quantities of interest in this paper, thus to a large extent reducing the proof of our generalization of Weyl's identity to the manipulation of commutation relations. We learned this powerful method from a variety of sources, above all [Segal-Wilson 1985, §8 and §10]. Other important sources for the circle of ideas explored here are [Tate 1968], [AdCK 1987], [DJKM 1981], [Pressley-Segal], and [Raina 1990].

1.2.3. *The operators T_h.* We briefly discuss a class of naturally occurring operators playing an especially important role in the proof of our generalization of Weyl's identity. Given linear operators X and Y on a common vector space, put $[X, Y] = XY - YX$. For each $h \in \mathbb{C}((1/t))$ there exists a unique linear operator T_h on $\mathcal{H}$ such that

$$[T_h, f^\sharp] = (hf)^\sharp, \quad [T_h, g^\flat] = (-hg)^\flat,$$

for all $f, g \in \mathbb{C}((1/t))$ and

$$T_h|\bullet\rangle = \sum_{\alpha \geq 0} \sum_{\beta \geq 0} \mathrm{Res}(t^{-2-\alpha-\beta} h)(t^\alpha)^\sharp (t^\beta)^\flat |\bullet\rangle.$$

Existence is established by a general construction (Theorem 3.4 below) combined with some explicit calculations undertaken in §5. Uniqueness is guaranteed by a result of Schur Lemma type (Theorem 3.2 below). On account of the uniqueness of T_h we necessarily have commutation relations

$$[T_{h_1}, T_{h_2}] = \mathrm{Res}(h_1 h_2').$$

Thus the operators T_h "almost" commute; all the "errors" are scalar multiples of the identity. We remark that these commutation relations are an important point of contact between the fermionic Fock space formalism and the theory of residues given in [Tate 1968]. From the point of view taken in this paper, our generalization of Weyl's identity springs from the existence and uniqueness of the operators T_h. Now put $T_i = T_{t^i}$. Note that members of the family $\{T_i\}_{i>0}$ commute among themselves. For any symmetric function f let $f(T)$ denote the result of expanding f as a polynomial in the power sum symmetric functions p_i and subsequently setting $p_i = T_i$ for all $i > 0$. We then have (see Theorem 6.4 below)

$$s_\lambda(T)|\bullet\rangle = |\lambda\rangle$$

for all partitions λ. To a physicist (see [Pressley-Segal, §10.7]) this is the *boson-fermion correspondence* in the zero-charge sector. The boson-fermion correspondence is the main "bridge" between the fermionic Fock space formalism and classical symmetric function theory.

1.2.4. *Remark.* To a combinatorist the boson-fermion correspondence is "just" a lightly disguised version of the Murnaghan-Nakayama rule. The operators T_i for $i > 0$ in effect "slap on rim-hooks" of weight i with a certain alteration of sign. Analogously the operators T_{-i} for $i > 0$ strip off rim-hooks. The rim-hook interpretation becomes (almost) visible in Lemmas 6.5.1 and 6.5.2 below, and in their proofs. But we do not follow up on the idea because we prefer to stay on our operator-theoretic path. Still, perhaps there is something to be gained by following the path we did not.

1.2.5. *The Wick-Lieb identity.* There exists a unique linear functional $\langle\bullet|$ on $\mathcal{H}$ such that

$$\langle\bullet|(t^{\alpha_1})^\sharp\cdots(t^{\alpha_r})^\sharp(t^{\beta_1})^\flat\cdots(t^{\beta_s})^\flat|\bullet\rangle$$
$$= \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are empty sequences,} \\ 0 & \text{otherwise,} \end{cases}$$

for all finite strictly decreasing sequences $\alpha = \{\alpha_i\}_{i=1}^r$ and $\beta = \{\beta_j\}_{j=1}^s$ of nonnegative integers. Let Pf denote the operation associating to an upper triangular array of numbers (a square matrix with just the superdiagonal entries filled in) its *pfaffian.* We have

$$\begin{aligned}\langle\bullet|(f_1^\sharp + g_1^\flat)\cdots(f_n^\sharp + g_n^\flat)|\bullet\rangle &= \mathrm{Pf}_{i,j=1}^n \operatorname{Res}\left((f_i)_{<0}\, g_j + (g_i)_{<0}\, f_j\right) \\ &= \mathrm{Pf}_{i,j=1}^n \langle\bullet|(f_i^\sharp + g_i^\flat)(f_j^\sharp + g_j^\flat)|\bullet\rangle\end{aligned}$$

for all finite sequences $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ in $\mathbb{C}((1/t))$. (When n is odd we set the pfaffians equal to 0.) This we call the *Wick-Lieb identity.* We got it from the papers [Wick 1950] and [Lieb 1968]. The Wick-Lieb identity is to fermionic Fock space calculus as the Chain Rule is to freshman calculus—absolutely indispensable.

1.3. **Concluding remarks.** As the reader might at this point suspect, our generalization of Weyl's identity is really just an excuse to get the fermionic Fock space formalism "out in the open". We now briefly discuss two possible further applications.

1.3.1. *Random matrix theory.* Many interesting random matrix probabilities are represented by *Hankel determinants*, i. e., determinants of the form $\det_{i,j=0}^{n-1} \mu_{i+j}$ where the μ_i are the moments of some measure μ on the real line. Typically one considers μ varying in a family and tries to derive nonlinear relations among the various partial derivatives of the corresponding Hankel determinants. It is a remarkable fact that such relations exist in cases connected with the classical families of orthogonal polynomials and moreover that in such relations n appears merely as a parameter. Now Hankel determinants can be represented naturally as matrix coefficients in the fermionic Fock space context, and become in that way (so we contend) relatively easy to manipulate. In particular, appropriate generalizations of the Wick-Lieb identity (e. g., the one given in §4.4.2 below) can be used to detect nonlinear relations among derivatives. Thus it is possible to approach in a somewhat simpler way various fundamental results, e. g., those proved in [AVM 2001], and perhaps to carry calculations of this nature further.

1.3.2. *Computational abelian function theory.* For the study of rational points on a curve one needs access to the group law of the corresponding Jacobian variety in its native algebraic form rather than in its linearization via the Abel map so that one can talk about fields of definition and reductions modulo p. It is possible to gain such access by manipulation of classical complex analytic identities relating determinants on products of copies of the curve to theta functions on the Jacobian, but it is a tricky business. Also it is possible to analyze those theta identities within a fermionic Fock space framework, as in [Raina 1990]. So our simple algebraic approach to fermionic Fock space could in principle be cultivated in characteristic p and over number fields as a substitute for classical abelian function theory over the complex numbers, thus becoming a new tool for the study of diophantine equations. The first step in carrying out the program is easy: just replace $\mathbb{C}((1/t))$ in the definition of fermionic Fock space by the adele ring of a curve; with no essential change the general theory developed in §2, §3, and §4 below may be transported to the adelic setting.

2. Fermionic Fock space: existence and uniqueness

2.1. **Notation.**

2.1.1. The cardinality of a set S is denoted $\#S$.

2.1.2. The scalar field for linear algebra in this paper is the field $\mathbb{C}$ of complex numbers. Given linear operators X and Y on a common vector space, we write

$$XY = X \circ Y, \quad [X,Y] = XY - YX, \quad \{X,Y\} = XY + YX.$$

A *basis* $\{v_i\}_{i\in I}$ for a vector space V is understood always to be a *Hamel basis*, i. e., every $v \in V$ is supposed to be a *finite* linear combination of the v_i in a unique way.

2.1.3. Let t be a variable. Let $\mathbb{C}((1/t))$ be the ring of power series of the form

$$\sum_{i\in\mathbb{Z}} a_i t^i \quad (a_i \in \mathbb{C}, \;\; a_i = 0 \text{ for } i \gg 0).$$

Note that since division by nonzero elements makes sense, the ring $\mathbb{C}((1/t))$ is actually a field. We write

$$\begin{array}{rcl} \left(\sum_i a_i t^i\right)_{\geq 0} &=& \sum_{i\geq 0} a_i t^i, \\ \left(\sum_i a_i t^i\right)_{<0} &=& \sum_{i<0} a_i t^i, \\ \mathrm{Res}\left(\sum_i a_i t^i\right) &=& a_{-1}, \\ \deg\left(\sum_i a_i t^i\right) &=& \max(\{i \mid a_i \neq 0\} \cup \{-\infty\}), \\ \left(\sum_i a_i t^i\right)' &=& \sum_i i a_i t^{i-1}. \end{array}$$

Also we write

$$\left(\sum_i a_i t^i\right) \circ v = \sum_i a_i v^i$$

for any $v \in \mathbb{C}((1/t))$ such that $\deg v > 0$; the sum on the right makes sense because $\deg v > 0 \Rightarrow \lim_{i\to-\infty} \deg v^i = -\infty$.

2.2. **Definition.**

Suppose we are given

- a vector space $\mathcal{H}$ equipped with a nonzero vector $|\bullet\rangle$, and
- linear maps $\left.\begin{array}{c} f \mapsto f^\sharp \\ g \mapsto g^\flat \end{array}\right\} : \mathbb{C}((1/t)) \to \left(\begin{array}{c}\text{space of linear} \\ \text{operators on } \mathcal{H}\end{array}\right)$

such that

- $\{f^\sharp, g^\sharp\} = 0$, $\{f^\flat, g^\flat\} = 0$, $\{f^\sharp, g^\flat\} = \mathrm{Res}(fg)$, $(f_{<0})^\sharp|\bullet\rangle = 0$, and $(g_{<0})^\flat|\bullet\rangle = 0$ for all $f, g \in \mathbb{C}((1/t))$, and

- for all subspaces $V \subset \mathcal{H}$ such that $|\bullet\rangle \in V$, if for all $f, g \in \mathbb{C}((1/t))$ and $v \in V$ we have $f^\sharp v, g^\flat v \in V$, then $V = \mathcal{H}$.

In this situation we say that the quadruple

$$\big(\mathcal{H}, \quad |\bullet\rangle, \quad f \mapsto f^\sharp, \quad g \mapsto g^\flat\big)$$

is a model of *fermionic Fock space.*

2.3. **Isomorphism of models.** Suppose that we have two models

$$\Big(\mathcal{H}_i, \quad |\bullet\rangle_i, \quad f \mapsto f_i^\sharp, \quad g \mapsto g_i^\flat\Big) \quad (i = 1, 2)$$

of fermionic Fock space. By definition an *isomorphism* from the first model to the second is an isomorphism

$$\phi : \mathcal{H}_1 \xrightarrow{\sim} \mathcal{H}_2$$

of vector spaces such that

$$\phi f_1^\sharp = f_2^\sharp \phi, \qquad \phi g_1^\flat = g_2^\flat \phi, \qquad \phi|\bullet\rangle_1 = |\bullet\rangle_2$$

for all $f, g \in \mathbb{C}((1/t))$.

Theorem 2.4. *There exists a model of fermionic Fock space, and moreover from any model to another there exists a unique isomorphism.*

The theorem is physics folklore. The uniqueness part of the theorem is especially important: it is going to serve us as efficient means for constructing useful examples of linear operators on fermionic Fock space. For lack of a convenient reference handling exactly our set of hypotheses, we sketch a proof of this theorem below. The sketch takes up the rest of §2.

2.5. **The standard model.** Under this heading we prove the existence part of Theorem 2.4 by a simple explicit construction. After giving the construction we comment on the sources for it.

2.5.1. Let $\mathcal{H}$ be a vector space equipped with a basis $\{|S\rangle\}$ indexed by subsets $S \subset \mathbb{Z}$ such that $n \in S$ for $n \ll 0$ and $n \notin S$ for $n \gg 0$. Put

$$|\bullet\rangle = |\{s \in \mathbb{Z} | s < 0\}\rangle \in \mathcal{H}.$$

Given any

$$f = \sum_i a_i t^i \in \mathbb{C}((1/t)), \quad g = \sum_j b_j t^j \in \mathbb{C}((1/t)),$$

we define linear operators $f^\sharp$ and $g^\flat$ on $\mathcal{H}$ by the rules

$$f^\sharp|S\rangle = \sum_{i\in\mathbb{Z}\setminus S} a_i(-1)^{\#S_i}\,|S\cup\{i\}\rangle,\quad g^\flat|S\rangle = \sum_{j\in S} b_{-1-j}(-1)^{\#S_j}\,|S\setminus\{j\}\rangle,$$

where

$$S_i = \{s\in S|s>i\}.$$

Note that the sets S_i are finite. Note that the sums make sense because only finitely many nonzero terms appear in them. We claim that the quadruple

$$\big(\mathcal{H},\quad |\bullet\rangle,\quad f\mapsto f^\sharp,\quad g\mapsto g^\flat\big)$$

thus constructed is a model of fermionic Fock space.

2.5.2. Fix arbitrarily a subset $S\subset\mathbb{Z}$ such that $n\in S$ for $n\ll 0$ and $n\notin S$ for $n\gg 0$. Fix $f,g\in\mathbb{C}((1/t))$ arbitrarily. Write $f=\sum_i a_it^i$ and $g=\sum_j b_jt^j$. We have

$$\begin{aligned} f^\sharp g^\flat|S\rangle &= \sum_{j\in S}\sum_{i\in\mathbb{Z}\setminus(S\setminus\{j\})} a_ib_{-1-j}(-1)^{\#(S\setminus\{j\})_i+\#S_j}|S\setminus\{j\}\cup\{i\}\rangle,\\ g^\flat f^\sharp|S\rangle &= \sum_{i\in\mathbb{Z}\setminus S}\sum_{j\in S\cup\{i\}} b_{-1-j}a_i(-1)^{\#(S\cup\{i\})_j+\#S_i}|S\cup\{i\}\setminus\{j\}\rangle. \end{aligned}$$

Note also that

$$(-1)^{\#(S\setminus\{j\})_i+\#S_j}(-1)^{\#(S\cup\{i\})_j+\#S_i} = \begin{cases} -1 & \text{if } i\neq j,\\ 1 & \text{if } i=j.\end{cases}$$

Therefore

$$\{f^\sharp,g^\flat\}|S\rangle = \mathrm{Res}(fg)|S\rangle.$$

Since S was chosen arbitrarily it follows in turn that $\{f^\sharp,g^\flat\}=\mathrm{Res}(fg)$. The relations $\{f^\sharp,g^\sharp\}=0$ and $\{f^\flat,g^\flat\}=0$ can be proved by similar calculations safely left to the reader in their entirety. The relations $(f_{<0})^\sharp|\bullet\rangle=0$ and $(g_{<0})^\flat|\bullet\rangle=0$ are immediate consequences of the definitions.

2.5.3. Again fix arbitrarily a set $S\subset\mathbb{Z}$ such that $n\in S$ for $n\ll 0$ and $n\notin S$ for $n\gg 0$. Fix arbitrarily a subspace $V\subset\mathcal{H}$ such that $|\bullet\rangle\in V$ and $f^\sharp v, g^\flat v\in V$ for all $f,g\in\mathbb{C}((1/t))$ and $v\in V$. Put

$$I=\{0,1,2,\dots\}\cap S,\quad J=\{0,1,2,\dots\}\setminus\{-1-s\mid s\in S\},$$

thus defining finite sets of nonnegative integers, and write

$$I=\{\alpha_1>\cdots>\alpha_p\},\quad J=\{\beta_1>\cdots>\beta_q\}.$$

Then we clearly have

$$S=\{\alpha_1,\dots,\alpha_p\}\cup(\{-1,-2,-3,\dots,\}\setminus\{-1-\beta_1,\dots,-1-\beta_q\}),$$

and the relation

$$|S\rangle = \pm(t^{\alpha_1})^\sharp \cdots (t^{\alpha_p})^\sharp (t^{\beta_1})^\flat \cdots (t^{\beta_q})^\flat |\bullet\rangle$$

can be verified by a straightforward calculation safely left to the reader. It follows that $|S\rangle \in V$. Since S was chosen arbitrarily, it follows in turn that $V = \mathcal{H}$.

2.5.4. Clearly the observations made in §2.5.2 and §2.5.3 suffice to prove the claim made in §2.5.1. Thus the existence part of Theorem 2.4 is proved. We call the model explicitly constructed above the *standard model* of fermionic Fock space, and we call the family of vectors $\{|S\rangle\}$ the *standard basis*. Ultimately the standard model is derived from Dirac's theory of electrons and positrons; in the Dirac picture each set S would be interpreted as a pattern of occupation of a system of integer energy levels by indistinguishable particles obeying Fermi-Dirac statistics. For the construction of the standard model presented here we have also drawn inspiration from [Tate 1968], [DJKM 1981], [Pressley-Segal], [AdCK 1987], and [Segal-Wilson 1985, §10].

2.5.5. Because it foreshadows our approach to the proof of the uniqueness part of Theorem 2.4, it is worth remarking that the construction $S \mapsto (I, J)$ considered in §2.5.3 puts the sets $S \subset \mathbb{Z}$ such that $n \in S$ for $n \ll 0$ and $n \notin S$ for $n \gg 0$ in bijective correspondence with ordered pairs (I, J) where I and J are finite sets of nonnegative integers.

2.6. **The anticommutator trick and related operator identities.**

2.6.1. We have

$$X^2 = \{X, X\}/2.$$

In particular, a linear operator anticommuting with itself squares to 0.

2.6.2. Let $X_1 \ldots, X_n$ be linear operators on a common vector space, where $n \geq 2$. We have

$$\begin{aligned} & X_1 X_2 \cdots X_n \\ = \; & (-1)^{n-1} X_2 \cdots X_n X_1 + \sum_{j=2}^{n} (-1)^j X_2 \ldots X_{j-1} \{X_1, X_j\} X_{j+1} \ldots X_n. \end{aligned}$$

We call this relation the *anticommutator trick*. If $\{X_1, X_j\} = 0$ for $j = 2, \ldots, n$, we have simply

$$X_1 X_2 \cdots X_n = (-1)^{n-1} X_2 \cdots X_n X_1.$$

The latter we call the *trivial case* of the anticommutator trick.

2.6.3. Again let $X_1, \ldots, X_n$ be linear operators on a common vector space, where $n \geq 2$. The anticommutator trick is very similar in form to the *Leibniz identity*

$$[X_1, X_2 \cdots X_n] = \sum_{j=2}^{n} X_2 \ldots X_{j-1}[X_1, X_j]X_{j+1} \ldots X_n$$

indispensable to the representation theory of Lie algebras.

2.6.4. In this line we should also mention the *Jacobi identity*

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]]$$

satisfied by any three linear operators X, Y and Z on a common vector space. We have written the Jacobi identity in the asymmetrical form best suited to the applications we have in mind for it.

2.6.5. Of the various operator identities discussed here, only the anticommutator trick is needed for the proof of Theorem 2.4. But it is natural at this juncture to mention the Leibniz and Jacobi identities because, in conjunction with the anticommutator trick, they frequently come up in applications of the theorem.

2.7. **A trio of technical lemmas.** Here is the common setting for the following three lemmas. Fix a model

$$\big(\mathcal{H}, \quad |\bullet\rangle, \quad f \mapsto f^\sharp, \quad g \mapsto g^\flat\big)$$

of fermionic Fock space arbitrarily. Given finite sets

$$I = \{\alpha_1 > \cdots > \alpha_p\}, \quad J = \{\beta_1 > \cdots > \beta_q\}$$

of nonnegative integers, put

$$|I, J\rangle = (t^{\alpha_1})^\sharp \cdots (t^{\alpha_p})^\sharp (t^{\beta_1})^\flat \cdots (t^{\beta_q})^\flat |\bullet\rangle \in \mathcal{H}.$$

We remark that in the case of the standard model, the family $\{|I, J\rangle\}$ thus constructed agrees up to signs with the standard basis, cf. the remarks of §2.5.3 and §2.5.5.

Lemma 2.7.1. *We have*

$$(t^i)^\sharp|I,J\rangle = \begin{cases} (-1)^{\#\{\alpha\in I|\alpha>i\}}|I\cup\{i\},J\rangle & \text{if } i\notin I,\\ 0 & \text{if } i\in I,\end{cases}$$

$$(t^j)^\flat|I,J\rangle = \begin{cases} (-1)^{\#I+\#\{\beta\in J|\beta>j\}}|I,J\cup\{j\}\rangle & \text{if } j\notin J,\\ 0 & \text{if } j\in J,\end{cases}$$

$$(f_{<0})^\sharp|I,J\rangle = \sum_{j\in J}(-1)^{\#I+\#\{\beta\in J|\beta>j\}}\operatorname{Res}(ft^j)|I,J\setminus\{j\}\rangle,$$

$$(g_{<0})^\flat|I,J\rangle = \sum_{i\in I}(-1)^{\#\{\alpha\in I|\alpha>i\}}\operatorname{Res}(gt^i)|I\setminus\{i\},J\rangle$$

for all integers $i,j\geq 0$, finite sets I and J of nonnegative integers, and Laurent series $f,g\in\mathbb{C}((1/t))$.

Proof. All the formulas are proved by applying the anticommutator trick in a straightforward way. In fact only the trivial case of the anticommutator trick is needed to prove the first two identities. □

Lemma 2.7.2. *Let I_0 and J_0 be finite sets of nonnegative integers. Write*

$$I_0=\{\alpha_1>\cdots>\alpha_p\},\quad J_0=\{\beta_1>\cdots>\beta_q\},$$

and put

$$X=(t^{-1-\beta_q})^\sharp\cdots(t^{-1-\beta_1})^\sharp(t^{-1-\alpha_p})^\flat\cdots(t^{-1-\alpha_1})^\flat.$$

Then we have

$$\#I+\#J\leq\#I_0+\#J_0\Rightarrow X|I,J\rangle=\delta_{II_0}\delta_{JJ_0}|\bullet\rangle$$

for all finite sets I and J of nonnegative integers.

Proof. By repeated application of the third and fourth identities stated in Lemma 2.7.1 we have $X|I_0,J_0\rangle=|\bullet\rangle$. If $(I,J)\neq(I_0,J_0)$, then by hypothesis either we have $I_0\setminus I\neq\emptyset$ or we have $J_0\setminus J\neq\emptyset$, and hence by repeated application of the third and fourth identities stated in Lemma 2.7.1 we have $X|I,J\rangle=0$. □

Lemma 2.7.3. *The family $\{|I,J\rangle\}$ is a basis for $\mathcal{H}$.*

Proof. Let V be the span of the family $\{|I,J\rangle\}$. By Lemma 2.7.1 we have $f^\sharp v,g^\flat v\in V$ for all $f,g\in\mathbb{C}((1/t))$ and $v\in V$. Moreover, we have $|\bullet\rangle=|\emptyset,\emptyset\rangle\in V$. Therefore we have $V=\mathcal{H}$ by the definition of fermionic Fock space. In other words, the family $\{|I,J\rangle\}$ spans $\mathcal{H}$. We turn now to the proof of independence. Suppose that there exists a nontrivial relation

$$\sum c(I,J)|I,J\rangle = 0$$

of linear dependence. Choose finite sets I_0 and J_0 of nonnegative integers such that $c(I_0, J_0) \neq 0$, with $\#I_0 + \#J_0$ as large as possible. By Lemma 2.7.2 there exists a linear operator X on $\mathcal{H}$ such that

$$0 = \sum c(I,J)X|I,J\rangle = c(I_0,J_0)|\bullet\rangle \neq 0,$$

which is a contradiction. Therefore the family $\{|I,J\rangle\}$ is linearly independent, and hence a basis of $\mathcal{H}$. □

2.8. **Completion of the proof of Theorem 2.4.** Existence has already been proved. We have only to prove uniqueness. Suppose we have two models

$$\left(\mathcal{H}_i, \quad |\bullet\rangle_i, \quad f \mapsto f_i^\sharp, \quad g \mapsto g_i^\flat\right) \qquad (i = 1,2)$$

of fermionic Fock space. For all finite sets I and J of nonnegative integers and indices $i = 1,2$, let the vector $|I,J\rangle_i \in \mathcal{H}_i$ be defined by the rule given in §2.7. Then for $i = 1,2$ the family $\{|I,J\rangle_i\}$ is a basis for $\mathcal{H}_i$ by Lemma 2.7.3. Further, with the help of Lemma 2.7.1, it can be verified that there exists exactly one linear isomorphism $\mathcal{H}_1 \xrightarrow{\sim} \mathcal{H}_2$ underlying an isomorphism from the first model of fermionic Fock space to the second, namely that sending $|I,J\rangle_1$ to $|I,J\rangle_2$ for all I and J. We can safely leave the details of that verification to the reader. Thus uniqueness is proved. □

3. Fundamental structural facts

3.1. **The setting for the rest of the paper.** We now fix a model

$$\left(\mathcal{H}, \quad |\bullet\rangle, \quad f \mapsto f^\sharp, \quad g \mapsto g^\flat\right)$$

of fermionic Fock space arbitrarily. As in §2.7, we put

$$|I,J\rangle = (t^{\alpha_1})^\sharp \cdots (t^{\alpha_p})^\sharp (t^{\beta_1})^\flat \cdots (t^{\beta_q})^\flat \in \mathcal{H}$$

for all finite sets

$$I = \{\alpha_1 > \cdots > \alpha_p\}, \quad J = \{\beta_1 > \cdots > \beta_q\}$$

of nonnegative integers. Recall that by Lemma 2.7.3 the family $\{|I,J\rangle\}$ thus constructed is a basis for $\mathcal{H}$. Recall also that

$$|\bullet\rangle = |\emptyset, \emptyset\rangle.$$

As becomes clear presently, it is extraordinarily convenient to define $\langle\bullet|$ to be the unique linear functional on $\mathcal{H}$ such that

$$\langle\bullet|I,J\rangle = \begin{cases} 1 & \text{if } I=\emptyset \text{ and } J=\emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all finite sets I and J of nonnegative integers.

Theorem 3.2 (The Schur Lemma)**.** *The following statements hold:*

(1) *Let a vector $\psi \in \mathcal{H}$ be given. We have*

$$(f_{<0})^\sharp\psi = 0, \quad (g_{<0})^\flat\psi = 0$$

for all $f, g \in \mathbb{C}((1/t))$ if and only if ψ is a scalar multiple of $|\bullet\rangle$.

(2) *Let a linear functional ψ^* on $\mathcal{H}$ be given. We have*

$$\psi^*(f_{\geq 0})^\sharp = 0, \quad \psi^*(g_{\geq 0})^\flat = 0$$

for all $f, g \in \mathbb{C}((1/t))$ if and only if ψ^* is a scalar multiple of $\langle\bullet|$.

(3) *Let a linear operator X on $\mathcal{H}$ be given. We have $[f^\sharp, X] = 0$ and $[g^\flat, X] = 0$ for all $f, g \in \mathbb{C}((1/t))$ if and only if X is a scalar multiple of the identity operator on $\mathcal{H}$.*

(4) *Let a nonzero subspace $V \subset \mathcal{H}$ be given. If for all $f, g \in \mathbb{C}((1/t))$ and $v \in V$ we have $f^\sharp v, g^\flat v \in V$, then $V = \mathcal{H}$.*

Proof. 1($\Rightarrow$). Suppose for the purpose of deriving a contradiction that ψ is not a scalar multiple of $|\bullet\rangle$. Let $\psi = \sum c(I,J)|I,J\rangle$ be the expansion of ψ in terms of the basis $\{|I,J\rangle\}$. Choose finite sets I_0 and J_0 of nonnegative integers such that $c(I_0,J_0) \neq 0$ and $n = \#I_0 + \#J_0$ is as large as possible. We must have $(I_0,J_0) \neq (\emptyset,\emptyset)$ and hence $n > 0$ lest ψ be a scalar multiple of $|\bullet\rangle = |\emptyset,\emptyset\rangle$. By Lemma 2.7.2 there exists a linear operator X on $\mathcal{H}$ factoring as a product of n operators of the form $(f_{<0})^\sharp$ or $(g_{<0})^\flat$ with $f, g \in \mathbb{C}((1/t))$ such that $X\psi = c(I_0,J_0)|\bullet\rangle \neq 0$. But by hypothesis we also have $X\psi = 0$ (since $n > 0$), which is a contradiction.

1($\Leftarrow$). This holds by definition of fermionic Fock space.

2($\Rightarrow$). By hypothesis ψ^* annihilates the basis $\{|I,J\rangle\}_{(I,J)\neq(\emptyset,\emptyset)}$ for the hyperplane in $\mathcal{H}$ annihilated by the linear functional $\langle\bullet|$.

2($\Leftarrow$). With the help of the first two parts of Lemma 2.7.1 it can easily be verified that the linear functional $\langle\bullet|$ annihilates the spaces $(f_{\geq 0})^\sharp\mathcal{H}$ and $(g_{\geq 0})^\flat\mathcal{H}$ for all $f, g \in \mathbb{C}((1/t))$.

3($\Rightarrow$). By hypothesis we have $(f_{<0})^\sharp X|\bullet\rangle = X(f_{<0})^\sharp|\bullet\rangle = 0$ and $(g_{<0})^\flat X|\bullet\rangle = X(g_{<0})^\flat|\bullet\rangle = 0$ for all $f, g \in \mathbb{C}((1/t))$. It follows by the first part of the theorem already proved that we have $X|\bullet\rangle = c|\bullet\rangle$ for some scalar c. By a second application of the hypothesis we have $X|I,J\rangle = c|I,J\rangle$ for all finite sets I and J nonnegative integers. Since the family $\{|I,J\rangle\}$ is a basis for $\mathcal{H}$, we have $X = c$.

3($\Leftarrow$). Trivial.

4. By hypothesis there exists $0 \neq \psi \in V$. As in the proof of the first part of theorem, write $\psi = \sum_I \sum_J c(I,J)|I,J\rangle$ and choose I_0 and J_0 such that $c(I_0, J_0) \neq 0$ and $n = \#I_0 + \#J_0$ is as large as possible. By Lemma 2.7.2 there exists an operator X factoring as the product of n operators of the form $(f_{<0})^\sharp$ or $(g_{<0})^\flat$ with $f, g \in \mathbb{C}((1/t))$ such that $X\psi = c(I_0, J_0)|\bullet\rangle$. It follows that $|\bullet\rangle \in V$, and hence that $V = \mathcal{H}$ by the definition of fermionic Fock space. □

Theorem 3.3 (The lifting construction)**.** *Let A and B be invertible (one-to-one and onto) linear operators on $\mathbb{C}((1/t))$ such that*

$$\mathrm{Res}((Af)(Bg)) = \mathrm{Res}(fg)$$

for all $f, g \in \mathbb{C}((1/t))$. Let there also be given $0 \neq \psi_0 \in \mathcal{H}$ such that

$$(A(f_{<0}))^\sharp \psi_0 = 0, \quad (B(g_{<0}))^\flat \psi_0 = 0$$

for all $f, g \in \mathbb{C}((1/t))$. Then there exists a unique invertible linear operator γ on $\mathcal{H}$ such that

$$\gamma f^\sharp = (Af)^\sharp \gamma, \quad \gamma g^\flat = (Bg)^\flat \gamma, \quad \gamma|\bullet\rangle = \psi_0$$

for all $f, g \in \mathbb{C}((1/t))$.

Proof. We first prove uniqueness. Suppose γ_1 and γ_2 both have the desired properties. Then the operator $\phi = \gamma_2^{-1}\gamma_1$ commutes with $f^\sharp$ and $g^\flat$ for all $f, g \in \mathbb{C}((1/t))$, hence $\phi = c$ for some scalar c by the third part of the Schur Lemma, and finally $c = 1$ because $\phi|\bullet\rangle = |\bullet\rangle$. Thus uniqueness is proved.

We turn now to the proof of existence. By the fourth part of the Schur Lemma the smallest subspace of $\mathcal{H}$ stable under the action of $f^\sharp$ and $g^\flat$ for all $f, g \in \mathbb{C}((1/t))$ and to which ψ_0 belongs is $\mathcal{H}$. It follows under our hypotheses that the quadruple

$$\left(\mathcal{H}, \quad \psi_0, \quad f \mapsto (Af)^\sharp, \quad g \mapsto (Bf)^\flat\right)$$

is a model of fermionic Fock space. In turn, Theorem 2.4 provides us with an isomorphism γ to the model above from our fixed model of fermionic Fock space. Clearly this isomorphism γ has all the desired properties. Thus existence is proved. □

Theorem 3.4 (The Lie-lifting construction)**.** *Let A and B be linear operators on $\mathbb{C}((1/t))$ such that*

$$\mathrm{Res}((Af)g) + \mathrm{Res}(f(Bg)) = 0$$

for all $f, g \in \mathbb{C}((1/t))$. Let there also be given $\psi_0 \in \mathcal{H}$ such that

$$(f_{<0})^\sharp\psi_0 + (A(f_{<0}))^\sharp|\bullet\rangle = 0, \quad (g_{<0})^\flat\psi_0 + (B(g_{<0}))^\flat|\bullet\rangle = 0$$

for all $f, g \in \mathbb{C}((1/t))$. Then there exists a unique linear operator X on $\mathcal{H}$ such that

$$[X, f^\sharp] = (Af)^\sharp, \quad [X, g^\flat] = (Bg)^\flat, \quad X|\bullet\rangle = \psi_0$$

for all $f, g \in \mathbb{C}((1/t))$.

Proof. We first prove uniqueness. Suppose that X_1 and X_2 both have the desired properties. Then $\phi = X_1 - X_2$ commutes with $f^\sharp$ and $g^\flat$ for all $f, g \in \mathbb{C}((1/t))$, hence $\phi = c$ for some scalar c by the third part of the Schur Lemma, and finally $c = 0$ because $\phi|\bullet\rangle = 0$. Thus uniqueness is proved.

We turn now to the proof of existence. Let $\mathcal{H}^{(2)}$ be the space of column vectors of length 2 with entries in $\mathcal{H}$. Let $\tilde{\mathcal{H}}$ be the intersection of all subspaces $V \subset \mathcal{H}^{(2)}$ with the following two properties: firstly, V is stable under the action of the operators

$$\begin{bmatrix} f^\sharp & (Af)^\sharp \\ 0 & f^\sharp \end{bmatrix}, \quad \begin{bmatrix} g^\flat & (Bg)^\flat \\ 0 & g^\flat \end{bmatrix}$$

for all $f, g \in \mathbb{C}((1/t))$; and secondly, the vector

$$\begin{bmatrix} \psi_0 \\ |\bullet\rangle \end{bmatrix}$$

belongs to V. Then the quadruple

$$\left(\tilde{\mathcal{H}}, \quad \begin{bmatrix} \psi_0 \\ |\bullet\rangle \end{bmatrix}, \quad f \mapsto \begin{bmatrix} f^\sharp & (Af)^\sharp \\ 0 & f^\sharp \end{bmatrix}, \quad g \mapsto \begin{bmatrix} g^\flat & (Bg)^\flat \\ 0 & g^\flat \end{bmatrix}\right)$$

is a model of fermionic Fock space, as can be verified by a straightforward calculation with two by two matrices that we omit. In turn, Theorem 2.4 provides us with a unique isomorphism to the model above from our fixed model of fermionic Fock space. Let's write that isomorphism in the form

$$\psi \mapsto \begin{bmatrix} X\psi \\ Y\psi \end{bmatrix}$$

where X and Y are linear operators on $\mathcal{H}$. By construction we have

$$\begin{bmatrix} Xf^\sharp\psi \\ Yf^\sharp\psi \end{bmatrix} = \begin{bmatrix} f^\sharp & (Af)^\sharp \\ 0 & f^\sharp \end{bmatrix} \begin{bmatrix} X\psi \\ Y\psi \end{bmatrix},$$

$$\begin{bmatrix} Xg^\flat\psi \\ Yg^\flat\psi \end{bmatrix} = \begin{bmatrix} g^\flat & (Bg)^\flat \\ 0 & g^\flat \end{bmatrix} \begin{bmatrix} X\psi \\ Y\psi \end{bmatrix},$$

$$\begin{bmatrix} X|\bullet\rangle \\ Y|\bullet\rangle \end{bmatrix} = \begin{bmatrix} \psi_0 \\ |\bullet\rangle \end{bmatrix}$$

for all $f, g \in \mathbb{C}((1/t))$ and $\psi \in \mathcal{H}$. In particular, Y commutes with $f^\sharp$ and $g^\flat$ for all $f, g \in \mathbb{C}((1/t))$, and moreover $Y|\bullet\rangle = |\bullet\rangle$, so we have $Y = 1$ by the third part of the Schur Lemma. It is then clear that X has all the desired properties. Thus existence is proved. □

4. The Wick-Lieb identity

4.1. **A very brief review of pfaffians.** Let A and B denote n by n matrices with entries in the complex numbers.

4.1.1. If n is even, put

$$\operatorname{Pf} A = \operatorname{Pf}_{i,j=1}^n A_{ij} = \sum (-1)^\sigma \prod_{i=1}^{n/2} A_{\sigma(2i-1),\sigma(2i)}$$

where the sum is extended over permutations σ of $\{1, \ldots, n\}$ such that

- $\sigma(2i-1) < \sigma(2i)$ for $i = 1, \ldots, n/2$,
- $\sigma(2i-1) < \sigma(2i+1)$ for $i = 1, \ldots, n/2-1$,

and $(-1)^\sigma$ denotes the sign of σ; otherwise, if n is odd, put $\operatorname{Pf} A = \operatorname{Pf}_{i,j=1}^n A_{ij} = 0$. We call $\operatorname{Pf} A$ the *pfaffian* of A.

4.1.2. For example, we have

$$\operatorname{Pf} A = \begin{cases} A_{12} & \text{if } n = 2, \\ A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} & \text{if } n = 4. \end{cases}$$

4.1.3. Rewritten recursively, the definition of the pfaffian takes the form

$$\operatorname{Pf} A = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ \sum_{j=2}^n (-1)^j A_{1j} \operatorname{Pf} A^{1j} & \text{if } n \geq 2, \end{cases}$$

where A^{1j} denotes the $n-2$ by $n-2$ matrix obtained from A by striking rows 1 and j and columns 1 and j.

4.1.4. When n is even and A is antisymmetric, the definition of $\mathrm{Pf}\, A$ given here agrees with the standard one, cf. [Macdonald, III, 8, p. 254], and in that case, as is well-known, we have

$$\det A = (\mathrm{Pf}\, A)^2, \quad \mathrm{Pf}\, BAB^T = (\det B)(\mathrm{Pf}\, A).$$

But note that $\mathrm{Pf}\, A$ is defined here for any A, antisymmetric or not, and for any n, even or not. Note further that $\mathrm{Pf}\, A$ as defined here depends only on the entries of A strictly above the main diagonal.

4.2. **Formulation and proof of the identity.**

4.2.1. By [Wick 1950], as reinterpreted in [Lieb 1968] in terms of pfaffians, we have

$$\begin{aligned}\langle\bullet|(f_1^\sharp + g_1^\flat)\cdots(f_n^\sharp + g_n^\flat)|\bullet\rangle &= \mathrm{Pf}_{i,j=1}^n \operatorname{Res}\left((f_i)_{<0}\, g_j + (g_i)_{<0}\, f_j\right)\\ &= \mathrm{Pf}_{i,j=1}^n \operatorname{Res}\left(f_i(g_j)_{\geq 0} + g_i(f_j)_{\geq 0}\right)\\ &= \mathrm{Pf}_{i,j=1}^n \langle\bullet|(f_i^\sharp + g_i^\flat)(f_j^\sharp + g_j^\flat)|\bullet\rangle\end{aligned}$$

for all finite sequences $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ in $\mathbb{C}((1/t))$. This relation we shall call the *Wick-Lieb identity*.

4.2.2. For the reader's convenience we quickly sketch a proof of the identity. For brevity we write

$$\begin{aligned} x_i &= ((f_i)_{\geq 0})^\sharp + ((g_i)_{\geq 0})^\flat,\\ y_i &= ((f_i)_{<0})^\sharp + ((g_i)_{<0})^\flat,\\ z_i &= x_i + y_i = (f_i)^\sharp + (g_i)^\flat.\end{aligned}$$

We have

$$\langle\bullet|z_i|\bullet\rangle = \langle\bullet|(x_i + y_i)|\bullet\rangle = 0$$

by the first and second parts of the Schur Lemma. It follows in particular that the Wick-Lieb identity holds in the case $n = 1$. We have

$$\begin{aligned}\langle\bullet|z_i z_j|\bullet\rangle &\overset{\mathrm{I}}{=} \langle\bullet|y_i z_j|\bullet\rangle\\ &\overset{\mathrm{II}}{=} \langle\bullet|\{y_i, z_j\}|\bullet\rangle\\ &= \{y_i, z_j\}\\ &= \operatorname{Res}\left((f_i)_{<0}\, g_j + (g_i)_{<0}\, f_j\right)\\ &= \operatorname{Res}\left((f_i)_{<0}\, (g_j)_{\geq 0} + (g_i)_{<0}\, (f_j)_{\geq 0}\right)\\ &= \operatorname{Res}\left(f_i(g_j)_{\geq 0} + g_i(f_j)_{\geq 0}\right)\end{aligned}$$

at I by the second part of Schur Lemma, at II by the first part of the Schur Lemma, and elsewhere by the definitions. It follows in particular that the Wick-Lieb identity holds in the case $n = 2$. Finally, for $n > 2$, we have

$$\begin{aligned}
\langle \bullet | z_1 \cdots z_n | \bullet \rangle &\overset{\mathrm{I}}{=} \langle \bullet | y_1 z_2 \cdots z_n | \bullet \rangle \\
&\overset{\mathrm{II}}{=} \textstyle\sum_{j=2}^n (-1)^j \langle \bullet | z_2 \cdots \{y_1, z_j\} \cdots z_n | \bullet \rangle \\
&\overset{\mathrm{III}}{=} \textstyle\sum_{j=2}^n (-1)^j \langle \bullet | z_1 z_j | \bullet \rangle \langle \bullet | z_2 \cdots z_{j-1} z_{j+1} \cdots z_n | \bullet \rangle \\
&\overset{\mathrm{IV}}{=} \mathrm{Pf}_{i,j=1}^n \langle \bullet | z_i z_j | \bullet \rangle
\end{aligned}$$

at I by the second part of the Schur lemma, at II by the anticommutator trick and the first part of the Schur Lemma, at III since as already noted above we have $\{y_1, z_j\} = \langle \bullet | z_1 z_j | \bullet \rangle$, and at IV by induction on n combined with the recursive form of the definition of the pfaffian. The proof of the Wick-Lieb identity is complete.

4.3. **Another trio of technical lemmas.** Let A and B again denote n by n matrices with entries in the complex numbers.

Lemma 4.3.1. $\mathrm{Pf} \left[\begin{array}{cc} 0 & A \\ B & 0 \end{array} \right] = (-1)^{\frac{n(n-1)}{2}} \det A = \det\limits_{i,j=1}^{n} A_{i,n-j+1}$.

Proof. The case $n = 1$ is clear, so assume that $n > 1$. By the definition of the pfaffian in its recursive form and induction on n, the pfaffian in question can be rewritten as a sum which, up to the sign $(-1)^{\frac{n(n-1)}{2}}$, coincides with the expansion of $\det A$ by minors of its first row. □

Lemma 4.3.2. *Let $I \subset \{1, \ldots, n\}$ be a subset such that $A_{ij} = 0$ if $i, j \in I$ or $i, j \in \{1, \ldots, n\} \setminus I$. If* $\mathrm{Pf}\, A \neq 0$*, then $\#I = n/2$.*

Proof. We may assume that n is even, for otherwise $\mathrm{Pf}\, A = 0$ and there is nothing to prove. Now consider the definition of the pfaffian in its nonrecursive form. Let σ be a permutation indexing a nonvanishing term in the sum defining $\mathrm{Pf}\, A$. By hypothesis we have

$$\#(\{\sigma(2i-1), \sigma(2i)\} \cap I) = 1$$

for $i = 1, \ldots, n/2$, whence the result. □

Lemma 4.3.3. *Assume that n is even. Let ℓ be a positive even integer not exceeding n. For each subset $I = \{i_1 < \cdots < i_p\} \subset \{1, \ldots, n\}$ put $A_I := \mathrm{Pf}_{\alpha,\beta=1}^p A_{i_\alpha, i_\beta}$. Assume that $A_{\{1,\ldots,n-\ell\}} \neq 0$. We have*

$$\frac{A_{\{1,\ldots,n\}}}{A_{\{1,\ldots,n-\ell\}}} = \mathrm{Pf}_{i,j=1}^{\ell} \frac{A_{\{1,\ldots,n-\ell\} \cup \{n-\ell+i, n-\ell+j\}}}{A_{\{1,\ldots,n-\ell\}}}.$$

We shall call this relation the *Pfaff-Plücker identity.*

Proof. Since for all $I \subset \{1,\dots,n\}$ the subpfaffian A_I depends only on the entries of A strictly above the diagonal, we may without loss of generality assume that A is antisymmetric. Consider now the block decomposition

$$A = \begin{bmatrix} a & b \\ -b^T & d \end{bmatrix}$$

where the block a is $n-\ell$ by $n-\ell$ and the other blocks are of the appropriate sizes. By hypothesis a is antisymmetric and we have $\mathrm{Pf}\, a \neq 0$. Consequently there exists an $n-\ell$ by $n-\ell$ matrix u with complex entries such that

$$uau^T = e = \begin{bmatrix} & 1 & & & \\ -1 & & & & \\ & & \ddots & & \\ & & & & 1 \\ & & & -1 & \end{bmatrix}, \quad \det u \cdot \mathrm{Pf}\, a = \mathrm{Pf}\, e = 1.$$

Put

$$B = \begin{bmatrix} u & 0 \\ b^T a^{-1} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ -b^T & d \end{bmatrix} \begin{bmatrix} u^T & -a^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & b^T a^{-1} b + d \end{bmatrix}.$$

Then the Pfaff-Plücker identity holds trivially for B and moreover we have $\det u \cdot A_I = B_I$ for all sets I between $\{1,\dots,n-\ell\}$ and $\{1,\dots,n\}$. Therefore the Pfaff-Plücker identity holds for A, too. □

4.4. **A variant and an enhancement.**

4.4.1. By combining the Wick-Lieb identity with Lemmas 4.3.1 and 4.3.2, the reader can easily verify that

$$\langle \bullet | f_1^\sharp \cdots f_p^\sharp g_q^\flat \cdots g_1^\flat | \bullet \rangle = \begin{cases} \det\limits_{i,j=1}^{p} \mathrm{Res}(f_i\,(g_j)_{\geq 0}) & \text{if } p = q, \\ 0 & \text{if } p \neq q \end{cases}$$

for all finite sequences $f_1,\dots,f_p$ and $g_1,\dots,g_q$ in $\mathbb{C}((1/t))$. Abusing language slightly, we refer to this relation also as the Wick-Lieb identity. Note that this continues to hold with roles of sharps and flats reversed.

4.4.2. From Lemma 4.3.3 we obtain a significantly enhanced version of the Wick-Lieb identity, namely

$$\frac{\langle \bullet | (f_1^\sharp + g_1^\flat) \cdots (f_{k+\ell}^\sharp + g_{k+\ell}^\flat) | \bullet \rangle}{\langle \bullet | (f_1^\sharp + g_1^\flat) \cdots (f_k^\sharp + g_k^\flat) | \bullet \rangle}$$

$$= \mathrm{Pf}_{i,j=1}^{\ell} \frac{\langle \bullet | (f_1^\sharp + g_1^\flat) \cdots (f_k^\sharp + g_k^\flat)(f_{k+i}^\sharp + g_{k+i}^\flat)(f_{k+j}^\sharp + g_{k+j}^\flat) | \bullet \rangle}{\langle \bullet | (f_1^\sharp + g_1^\flat) \cdots (f_k^\sharp + g_k^\flat) | \bullet \rangle}$$

where k and ℓ are any even positive integers and $f_1, \dots, f_{k+\ell}$ and $g_1, \dots, g_{k+\ell}$ are any finite sequences in $\mathbb{C}((1/t))$, provided of course that the denominator $\langle \bullet | (f_1^\sharp + g_1^\flat) \cdots (f_k^\sharp + g_k^\flat) | \bullet \rangle$ does not vanish. This enhancement is not needed for proving our main results, but it is worth mentioning because it is essential for more advanced applications of the Wick-Lieb circle of ideas, e. g., to random matrix theory and to abelian function theory.

5. Operators and commutation relations: key examples

5.1. **An exceedingly brief review of freshman calculus.** The rules for combining the operations in $\mathbb{C}((1/t))$ defined in §2.1.3 are more or less those of freshman calculus. For example, we have

$$(fg)' = f'g + fg', \quad (f_{\geq 0})' = (f')_{\geq 0}, \quad (f_{<0})' = (f')_{<0},$$

$$(f \circ v)' = (f' \circ v)v', \quad (fg) \circ v = (f \circ v)(g \circ v),$$

$$\mathrm{Res}(f') = 0, \quad \deg v \cdot \mathrm{Res}(f) = \mathrm{Res}((f \circ v)v')$$

for all $f, g, v \in \mathbb{C}((1/t))$ with $\deg v > 0$. The reader should keep these basic facts in mind as we proceed.

5.2. **Operators.** We exploit the lifting and Lie-lifting constructions in order to define the suite of operators on fermionic Fock space we need to do our work.

5.2.1. Given any $u \in \mathbb{C}((1/t))$ such that $\deg u = 0$, the lifting construction yields a unique invertible linear operator σ_u on $\mathcal{H}$ such that

$$\sigma_u f^\sharp = (uf)^\sharp \sigma_u, \quad \sigma_u g^\flat = (u^{-1}g)^\flat \sigma_u, \quad \sigma_u |\bullet\rangle = |\bullet\rangle$$

for all $f, g \in \mathbb{C}((1/t))$.

5.2.2. The lifting construction yields a unique invertible linear operator σ_t on $\mathcal{H}$ such that

$$\sigma_t f^\sharp = (tf)^\sharp \sigma_t, \quad \sigma_t g^\flat = (t^{-1}g)^\flat \sigma_t, \quad \sigma_t |\bullet\rangle = (t^0)^\sharp |\bullet\rangle$$

for all $f, g \in \mathbb{C}((1/t))$.

5.2.3. Given any $v \in \mathbb{C}((1/t))$ such that $\deg v = 1$, the lifting construction yields a unique invertible linear operator ρ_v on $\mathcal{H}$ such that

$$\rho_v f^\sharp = (v' \cdot (f \circ v))^\sharp \rho_v, \quad \rho_v g^\flat = (g \circ v)^\flat \rho_v, \quad \rho_v |\bullet\rangle = |\bullet\rangle$$

for all $f, g \in \mathbb{C}((1/t))$.

5.2.4. Given $h \in \mathbb{C}((1/t))$, we claim that there exists a unique linear operator T_h on $\mathcal{H}$ such that

$$[T_h, f^\sharp] = (hf)^\sharp, \quad [T_h, g^\flat] = (-hg)^\flat,$$

for all $f, g \in \mathbb{C}((1/t))$ and

$$T_h|\bullet\rangle = \sum_{\alpha \geq 0} \sum_{\beta \geq 0} \operatorname{Res}(t^{-2-\alpha-\beta} h)(t^\alpha)^\sharp (t^\beta)^\flat |\bullet\rangle.$$

Note that the sum on the right makes sense because only finitely many nonzero terms appear in it. The main point of the proof of the claim is the following calculation. We have

$$\begin{array}{rl} & (f_{<0})^\sharp \sum_{\alpha\geq 0}\sum_{\beta\geq 0} \operatorname{Res}(t^{-2-\alpha-\beta}h)(t^\alpha)^\sharp(t^\beta)^\flat|\bullet\rangle \\ = & -\sum_{\alpha\geq 0}\sum_{\beta\geq 0} \operatorname{Res}(t^{-2-\alpha-\beta}h)(t^\alpha)^\sharp \operatorname{Res}(f_{<0}t^\beta)|\bullet\rangle \\ = & -\sum_{\alpha\geq 0} \operatorname{Res}(t^{-1-\alpha}f_{<0}h)(t^\alpha)^\sharp|\bullet\rangle \\ = & -((f_{<0}h)_{\geq 0})^\sharp|\bullet\rangle = -(f_{<0}h)^\sharp|\bullet\rangle, \end{array}$$

$$\begin{array}{rl} & (g_{<0})^\flat \sum_{\alpha\geq 0}\sum_{\beta\geq 0} \operatorname{Res}(t^{-2-\alpha-\beta}h)(t^\alpha)^\sharp(t^\beta)^\flat|\bullet\rangle \\ = & \sum_{\alpha\geq 0}\sum_{\beta\geq 0} \operatorname{Res}(t^{-2-\alpha-\beta}h) \operatorname{Res}(g_{<0}t^\alpha)(t^\beta)^\flat|\bullet\rangle \\ = & \sum_{\beta\geq 0} \operatorname{Res}(t^{-1-\beta}g_{<0}h)(t^\beta)^\flat|\bullet\rangle \\ = & ((g_{<0}h)_{\geq 0})^\flat|\bullet\rangle = (g_{<0}h)^\flat|\bullet\rangle \end{array}$$

for all $f, g \in \mathbb{C}((1/t))$. We can safely leave the remaining details of the verification of the claim to the reader.

5.2.5. We continue in the setting of the preceding paragraph. We claim that

$$T_{h_{<0}}|\bullet\rangle = 0, \quad \langle\bullet|T_h|\bullet\rangle = 0, \quad \langle\bullet|T_{h_{\geq 0}} = 0.$$

The first relation follows immediately from the definitions. The second relation follows from definitions combined with the second part of the Schur Lemma. To prove the third relation, we begin by observing that

$$\begin{array}{rl} & \langle\bullet|T_{h_{\geq 0}}((f_{\geq 0})^\sharp + (g_{\geq 0})^\flat) \\ = & \langle\bullet|((((f_{\geq 0})^\sharp + (g_{\geq 0})^\flat)T_{h_{\geq 0}} + [T_{h_{\geq 0}}, (f_{\geq 0})^\sharp + (g_{\geq 0})^\flat]) \\ = & \langle\bullet|((((f_{\geq 0})^\sharp + (g_{\geq 0})^\flat)T_{h_{\geq 0}} + ((h_{\geq 0}f_{\geq 0})^\sharp - (h_{\geq 0}g_{\geq 0})^\flat) = 0 \end{array}$$

for all $f, g \in \mathbb{C}((1/t))$ by the Leibniz identity, the definition of $T_{h_{\geq 0}}$, and the second part of the Schur Lemma, respectively. It follows by a further application of the second part of the Schur Lemma that we have

$$\langle\bullet|T_{h_{\geq 0}} = c\langle\bullet|$$

for some scalar c. Finally, we have

$$c = \langle\bullet|T_{h_{\geq 0}}|\bullet\rangle = 0,$$

and hence the third relation holds. The claim is proved.

5.2.6. For each integer i we write

$$T_{t^i} = T_i$$

in order to abbreviate notation conveniently. In this notation we have

$$T_i|\bullet\rangle = \sum_{\substack{\alpha,\beta\geq 0\\ \alpha+\beta=i-1}} (t^\alpha)^\sharp (t^\beta)^\flat |\bullet\rangle$$

and

$$\langle\bullet|T_i|\bullet\rangle = 0, \quad i \geq 0 \Rightarrow \langle\bullet|T_i = 0, \quad i \leq 0 \Rightarrow T_i|\bullet\rangle = 0$$

for all i.

5.2.7. We claim that for each positive integer n we have

$$\sigma_t^n|\bullet\rangle = (t^{n-1})^\sharp \cdots (t^0)^\sharp|\bullet\rangle, \quad \sigma_t^{-n}|\bullet\rangle = (t^{n-1})^\flat \cdots (t^0)^\flat|\bullet\rangle,$$

$$\langle\bullet|\sigma_t^n = \langle\bullet|(t^{-1})^\sharp \cdots (t^{-n})^\sharp, \quad \langle\bullet|\sigma_t^{-n} = \langle\bullet|(t^{-1})^\flat \cdots (t^{-n})^\flat.$$

By induction on n we may assume without loss of generality that $n = 1$. Of course the first relation holds by definition of σ_t. We have

$$\sigma_t(t^0)^\flat|\bullet\rangle = (t^{-1})^\flat(t^0)^\sharp|\bullet\rangle = (1 - (t^0)^\sharp(t^{-1})^\flat)|\bullet\rangle = |\bullet\rangle$$

and hence the second relation holds. We have

$$\langle\bullet|(t^{-1})^\sharp\sigma_t^{-1}((f_{\geq 0})^\sharp + (g_{\geq 0})^\flat) = \langle\bullet|(t^{-1})^\sharp((t^{-1}f_{\geq 0})^\sharp + (tg_{\geq 0})^\flat)\sigma_t^{-1} = 0$$

for all $f, g \in \mathbb{C}((1/t))$ by the definition of σ_t, the trivial case of the anticommutator trick and the second part of the Schur Lemma, respectively. Consequently, by a further application of the second part of the Schur Lemma we have

$$\langle\bullet|(t^{-1})^\sharp\sigma_t^{-1} = c\langle\bullet|$$

for some scalar c; but then we must have

$$c = \langle\bullet|(t^{-1})^\sharp\sigma_t^{-1}|\bullet\rangle = \langle\bullet|(t^{-1})^\sharp(t^0)^\flat|\bullet\rangle = \langle\bullet|(1 - (t^0)^\flat(t^{-1})^\sharp)|\bullet\rangle = 1,$$

and hence the third relation holds. The fourth relation is proved similarly. The proof of the claim is complete.

5.3. **Commutation relations.** We exploit the Schur Lemma and the Wick-Lieb identity to determine the commutation relations standing among the operators introduced above.

5.3.1. Fix $h_1, h_2 \in \mathbb{C}((1/t))$. We claim that

$$[T_{h_1}, T_{h_2}] = \mathrm{Res}(h_1 h_2').$$

This relation is the point of contact between our *ad hoc* theory of fermionic Fock space and the interpretation of residues given in [Tate 1968]. We carry out the proof of this crucially important commutation relation in a fairly detailed manner in order to set the pattern; we then carry out subsequent proofs of this nature at a brisker pace. At any rate, we have

$$[[T_{h_1}, T_{h_2}], (f^\sharp + g^\flat)] = [[T_{h_1}, (f^\sharp + g^\flat)], T_{h_2}] + [T_{h_1}, [T_{h_2}, (f^\sharp + g^\flat)]] = 0$$

by the Jacobi identity and the definitions for all $f, g \in \mathbb{C}((1/t))$. By the third part of the Schur Lemma it follows that we have

$$[T_{h_1}, T_{h_2}] = c$$

for some scalar c. Now write

$$h_1 = \sum_i a_i t^i, \quad h_2 = \sum_j b_j t^j.$$

We have

$$\begin{array}{rcl}
\langle \bullet | T_{h_1} T_{h_2} | \bullet \rangle & \stackrel{\mathrm{I}}{=} & \sum_{\alpha \geq 0} \sum_{\beta \geq 0} b_{1+\alpha+\beta} \langle \bullet | T_{h_1} (t^\alpha)^\sharp (t^\beta)^\flat | \bullet \rangle \\
& \stackrel{\mathrm{II}}{=} & \sum_{\alpha \geq 0} \sum_{\beta \geq 0} b_{1+\alpha+\beta} \langle \bullet | (h_1 t^\alpha)^\sharp (t^\beta)^\flat | \bullet \rangle \\
& + & \sum_{\alpha \geq 0} \sum_{\beta \geq 0} b_{1+\alpha+\beta} \langle \bullet | (t^\alpha)^\sharp (-h t^\beta)^\flat | \bullet \rangle \\
& + & \sum_{\alpha \geq 0} \sum_{\beta \geq 0} b_{1+\alpha+\beta} \langle \bullet | (t^\alpha)^\sharp (t^\beta)^\flat T_{h_2} | \bullet \rangle \\
& \stackrel{\mathrm{III}}{=} & \sum_{\alpha \geq 0} \sum_{\beta \geq 0} b_{1+\alpha+\beta} \langle \bullet | (h_1 t^\alpha)^\sharp (t^\beta)^\flat | \bullet \rangle \\
& \stackrel{\mathrm{IV}}{=} & \sum_{\alpha \geq 0} \sum_{\beta \geq 0} b_{1+\alpha+\beta} a_{-1-\alpha-\beta} = \sum_{\ell=1}^{\infty} \ell a_{-\ell} b_\ell
\end{array}$$

at I by the definition of T_{h_2}, at II by the definition of T_{h_1} and the Leibniz identity, at III by the second part of the Schur Lemma and at IV by the Wick-Lieb identity. It follows that

$$c = \langle \bullet | [T_{h_1}, T_{h_2}] | \bullet \rangle = \sum_{\ell=1}^{\infty} \ell (a_{-\ell} b_\ell - b_{-\ell} a_\ell) = \sum_\ell \ell a_{-\ell} b_\ell = \mathrm{Res}(h_1 h_2').$$

The claim is proved.

5.3.2. Fix $u \in \mathbb{C}((1/t))$ such that $\deg u = 0$. We claim that

$$\sigma_u \sigma_t = \mathrm{Res}(t^{-1} u) \sigma_t \sigma_u.$$

This relation is the point of contact between our *ad hoc* theory of fermionic Fock space and the interpretation of the tame symbol given in [AdCK 1987]. In any case, the operators on both sides of the claimed identity agree up to a nonzero scalar factor by the third part of the Schur Lemma. But since we also have

$$\sigma_u\sigma_t|\bullet\rangle = \sigma_u(t^0)^\sharp|\bullet\rangle = u^\sharp|\bullet\rangle = \mathrm{Res}(t^{-1}u)(t^0)^\sharp|\bullet\rangle = \mathrm{Res}(t^{-1}u)\sigma_t\sigma_u|\bullet\rangle,$$

the claim must hold exactly as stated.

5.3.3. For all

$$u_1, u_2, u, v, v_1, v_2 \in \mathbb{C}((1/t))$$

such that

$$\deg u_i = 0, \quad \deg u = 0, \quad \deg v = 1, \quad \deg v_i = 1,$$

we claim that

$$\sigma_{u_1}\sigma_{u_2} = \sigma_{u_1u_2}, \quad \rho_v\sigma_u = \sigma_{u\circ v}\rho_v, \quad \rho_{v_1}\rho_{v_2} = \rho_{v_2\circ v_1}.$$

By the third part of the Schur Lemma each of the claimed relations holds up to a nonzero scalar factor. But all the operators appearing here fix the vector $|\bullet\rangle$. Therefore all the claimed relations hold exactly as stated.

5.3.4. Fix $u, h \in \mathbb{C}((1/t))$ with $\deg u = 0$. We claim that

$$\sigma_u T_h \sigma_u^{-1} = T_h - \mathrm{Res}(hu'/u).$$

In any case we have

$$\sigma_u T_h \sigma_u^{-1} - T_h = c$$

for some scalar c by the third part of the Schur Lemma. But we also have

$$\begin{array}{rcl} c & = & \langle\bullet|(\sigma_u T_h\sigma_u^{-1} - T_h)|\bullet\rangle = \langle\bullet|\sigma_u T_h|\bullet\rangle \\ & = & \sum_{\alpha\geq 0}\sum_{\beta\geq 0}\mathrm{Res}(t^{-2-\alpha-\beta}h)\langle\bullet|\sigma_u(t^\alpha)^\sharp(t^\beta)^\flat|\bullet\rangle \\ & = & \sum_{\alpha\geq 0}\langle\bullet|\sigma_u(t^\alpha)^\sharp((t^{-1-\alpha}h)_{\geq 0})^\flat|\bullet\rangle \\ & = & \sum_{\alpha\geq 0}\langle\bullet|\sigma_u(t^\alpha)^\sharp(t^{-1-\alpha}h)^\flat|\bullet\rangle \\ & = & \sum_{\alpha\geq 0}\langle\bullet|(t^\alpha u)^\sharp(t^{-1-\alpha}h/u)^\flat|\bullet\rangle = \sum_{\alpha\geq 0}\mathrm{Res}((t^\alpha u)_{<0}t^{-1-\alpha}h/u) \\ & = & \mathrm{Res}\left(\left(\sum_{\alpha\geq 0}(t^\alpha u)_{<0}t^{-1-\alpha}\right)h/u\right) = -\mathrm{Res}(u'h/u). \end{array}$$

This proves the claim.

5.3.5. Fix $h \in \mathbb{C}((1/t))$. We claim that

$$\sigma_t T_h \sigma_t^{-1} = T_h - \mathrm{Res}(t^{-1}h)$$

for all $h \in \mathbb{C}((1/t))$. In any case we have

$$\sigma_t T_h \sigma_t^{-1} - T_h = c$$

for some scalar c by the third part of the Schur Lemma. But we also have

$$\begin{aligned} c &= \langle \bullet | \sigma_t T_h \sigma_t^{-1} - T_h | \bullet \rangle = \langle \bullet | (t^{-1})^\sharp T_h (t^0)^\flat | \bullet \rangle \\ &= \langle \bullet | (t^{-1})^\sharp (-ht^0)^\flat + (t^{-1})^\sharp (t^0)^\flat T_h | \bullet \rangle \\ &= \langle \bullet | (t^{-1})^\sharp (-ht^0)^\flat + (1 - (t^0)^\flat (t^{-1})^\sharp) T_h | \bullet \rangle \\ &= \langle \bullet | (t^{-1})^\sharp (-ht^0)^\flat | \bullet \rangle = -\operatorname{Res}(t^{-1}h). \end{aligned}$$

This proves the claim.

5.3.6. Fix $h, v, \bar{v} \in \mathbb{C}((1/t))$ with

$$\deg v = \deg \bar{v} = 1, \quad v \circ \bar{v} = \bar{v} \circ v = t.$$

We claim that

$$\rho_v T_h \rho_v^{-1} = T_{h \circ v} + \frac{1}{2} \operatorname{Res}(h \bar{v}'' / \bar{v}').$$

A square root $\sqrt{v'}$ of v' in $\mathbb{C}((1/t))$ exists since $\deg v' = 0$. By the commutation relations worked out in §5.3.4, and in view of the identity

$$-\operatorname{Res}\left((h \circ v) \left(\sqrt{v'} \right)' / \sqrt{v'} \right) = \frac{1}{2} \operatorname{Res}(h \bar{v}'' / \bar{v}')$$

the verification of which we leave to the reader, it suffices to prove that

$$\sigma_{\sqrt{v'}}^{-1} \rho_v T_h \rho_v^{-1} \sigma_{\sqrt{v'}} = T_{h \circ v}.$$

Now in any case we have

$$\sigma_{\sqrt{v'}}^{-1} \rho_v T_h \rho_v^{-1} \sigma_{\sqrt{v'}} - T_{h \circ v} = c$$

for some scalar c by the third part of the Schur Lemma. Further, for all $\alpha, \beta \geq 0$ we have

$$\operatorname{Res}(v^\alpha \sqrt{v'}\, (v^\beta \sqrt{v'})_{\geq 0}) = \operatorname{Res}((v^\alpha \sqrt{v'})_{<0}\, v^\beta \sqrt{v'}),$$

$$\operatorname{Res}((v^\alpha \sqrt{v'})_{<0}\, v^\beta \sqrt{v'}) + \operatorname{Res}((v^\alpha \sqrt{v'})_{\geq 0}\, v^\beta \sqrt{v'}) = \operatorname{Res}(v^{\alpha+\beta} v') = 0,$$

and hence the matrix

$$\{\operatorname{Res}(v^\alpha \sqrt{v'}\, (v^\beta \sqrt{v'})_{\geq 0})\}_{\alpha, \beta \geq 0}$$

is antisymmetric. Finally, we have

$$\begin{aligned} c &= \langle \bullet | \sigma_{\sqrt{v'}}^{-1} \rho_v T_h \rho_v^{-1} \sigma_{\sqrt{v'}} - T_{h \circ v} | \bullet \rangle = \langle \bullet | \sigma_{\sqrt{v'}}^{-1} \rho_v T_h | \bullet \rangle \\ &= \textstyle\sum_{\alpha, \beta \geq 0} \operatorname{Res}(t^{-2-\alpha-\beta} h) \langle \bullet | (v^\alpha \sqrt{v'})^\sharp (v^\beta \sqrt{v'})^\flat | \bullet \rangle \\ &= \textstyle\sum_{\alpha, \beta \geq 0} \operatorname{Res}(t^{-2-\alpha-\beta} h) \operatorname{Res}(v^\alpha \sqrt{v'}\, (v^\beta \sqrt{v'})_{\geq 0}) = 0, \end{aligned}$$

which finishes the proof of the claim.

5.3.7. Fix $v \in \mathbb{C}((1/t))$ such that $\deg v = 1$. We claim that

$$\rho_v \sigma_t = \mathrm{Res}(t^{-1}v')\sigma_t \sigma_{v/t} \rho_v.$$

In any case, by the third part of the Schur Lemma, the claimed identity holds up to a nonzero scalar factor. But we have

$$\begin{aligned} \rho_v \sigma_t |\bullet\rangle &= \rho_v (t^0)^\sharp |\bullet\rangle = (v')^\sharp |\bullet\rangle \\ &= \mathrm{Res}(t^{-1}v')(t^0)^\sharp |\bullet\rangle = \mathrm{Res}(t^{-1}v')\sigma_t \sigma_{v/t} \rho_v |\bullet\rangle, \end{aligned}$$

and hence the claim holds exactly as stated.

6. Bridge to symmetric function theory

In building our bridge we pursue a line of thought suggested to us by [Segal-Wilson 1985, §8], and due ultimately to Sato. But in our treatment of this circle of ideas the details are different and simpler because, more in line with [DJKM 1981], we emphasize operator identities instead of geometry. We follow [Macdonald, Chap. I] on the symmetric function side.

6.1. **Partitions.** We briefly recall definitions from [Macdonald, I, 1].

6.1.1. A *partition* λ is a nonincreasing sequence $\{\lambda_i\}_{i=1}^\infty$ of nonnegative integers such that $\lambda_i = 0$ for $i \gg 0$.

6.1.2. Given an ordered pair (i, j) of positive integers and a partition λ, we call (i, j) a *node* of λ if $j \leq \lambda_i$. By definition the *diagram* of a partition is the set of its nodes. Following the conventions of matrix algebra rather than those of coordinate geometry, we think of (i, j) as being located in the i^{th} row and j^{th} column. We say, for example, that there are λ_i nodes in the i^{th} row of the diagram of λ.

6.1.3. Given a partition λ, the *length* $\ell(\lambda)$ is the number of rows of the diagram of λ, the *size* $|\lambda|$ is the number of nodes of the diagram of λ, and the partition λ' *conjugate* to λ is the unique partition with diagram equal to the transpose of the diagram of λ.

6.1.4. Given a partition λ, we write

$$\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r)$$

under these conditions:

- There are exactly r nodes of the diagram of λ along the main diagonal.
- For $i = 1, \dots, r$ there are exactly α_i nodes of the diagram of λ in the i^{th} row strictly to the right of the main diagonal.
- For $j = 1, \dots, r$ there are exactly β_j nodes of the diagram of λ in the j^{th} column strictly below the main diagonal.

Note that in this situation we have

$$|\lambda| = r + \sum_i \alpha_i + \sum_j \beta_j, \quad \lambda' = (\beta_1 \dots \beta_r | \alpha_1 \dots \alpha_r).$$

We call $(\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r)$ the *Frobenius notation* for the partition λ.

6.2. **Symmetric functions.** We briefly recall definitions and basic facts from [Macdonald, I, 2-3].

6.2.1. Let $\{x_i\}_{i=1}^\infty$ be a family of independent variables. Let Λ be the ring of symmetric functions in the x_i as defined in [Macdonald, I,2], but with complex rather than integral coefficients. Let ϵ be a variable independent of the x_i. The *complete symmetric functions* h_i, the *elementary symmetric functions* e_i and the *power sum symmetric functions* p_i are the elements of Λ characterized by the identities

$$\begin{aligned} \textstyle\prod_{i=1}^\infty (1 - x_i\epsilon) &= \textstyle\sum_i (-1)^i e_i \epsilon^i, \\ \textstyle\prod_{i=1}^\infty (1 - x_i\epsilon)^{-1} &= \textstyle\sum_i h_i \epsilon^i = \exp\left(\sum_{i=1}^\infty \frac{p_i \epsilon^i}{i}\right) \end{aligned}$$

holding in the ring of power series in Λ with coefficients in ϵ. Note that by convention $h_0 = 1 = e_0$ and $h_i = 0 = e_i$ for $i < 0$, whereas p_i is not defined for $i \leq 0$.

6.2.2. Given a finite sequence $z_1, \dots, z_n$ of complex numbers and a symmetric function $f \in \Lambda$, let $f(z_1, \dots, z_n)$ denote the result of making in f the substitutions $x_i = z_i$ for $i = 1, \dots, n$ and $x_i = 0$ for $i > n$. Note that symmetric functions with complex coefficients are uniquely determined by their values on all finite sequences of complex numbers.

6.2.3. Let $z_1, \dots, z_n$ be any finite sequence of complex numbers. To make first contact with our *ad hoc* theory of fermionic Fock space, we remark that if

$$u = \prod_{i=1}^n (1 - z_i t^{-1})^{-1},$$

then

$$h_i(z_1, \dots, z_n) = \operatorname{Res}(t^{i-1}u), \quad (-1)^i e_i(z_1, \dots, z_n) = \operatorname{Res}(t^{i-1}u^{-1})$$

for all i and

$$p_i(z_1, \dots, z_n) = -\operatorname{Res}(t^i u'/u)$$

for all $i > 0$, as follows immediately by specialization of the power series identities defining h_i, e_i and p_i.

6.2.4. For each partition λ, let

$$p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}, \quad s_\lambda = \det_{i,j=1}^{\ell(\lambda)} h_{\lambda_i - i + j} = \det_{i,j=1}^{\ell(\lambda')} e_{\lambda'_i - i + j}$$

be the power sum symmetric function and the S-function (Schur function) indexed by λ, respectively. The family $\{p_\lambda\}$ indexed by partitions λ is a basis for Λ and so is the family $\{s_\lambda\}$.

6.3. Bridge definitions.

6.3.1. Since the family $\{p_\lambda\}$ is a basis for Λ, we are permitted to view Λ as the ring of polynomials in independent variables p_i for $i > 0$, with complex coefficients. Moreover, by the commutation relations worked out in §5.3.1, the members of the family $\{T_i\}_{i=1}^\infty$ of linear operators on $\mathcal{H}$ commute among themselves. Accordingly, given any $f \in \Lambda$, it makes sense to substitute T_i for p_i in f for all $i > 0$, thus obtaining from f a well-defined linear operator $f(T)$ on $\mathcal{H}$. The map

$$(f \mapsto f(T)) : \Lambda \to (\text{linear operators on } \mathcal{H})$$

thus defined is linear and satisfies

$$(fg)(T) = f(T)g(T)$$

for all $f, g \in \Lambda$.

6.3.2. For each partition

$$\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r)$$

put

$$|\lambda\rangle = (-1)^{\beta_1 + \cdots + \beta_r} (t^{\alpha_1})^\sharp \cdots (t^{\alpha_r})^\sharp (t^{\beta_r})^\flat \cdots (t^{\beta_1})^\flat |\bullet\rangle \in \mathcal{H},$$

cf. the remarks of §2.5.3 and §2.5.5. Note that we have

$$|\lambda\rangle = \pm |I, J\rangle$$

where

$$I = \{\alpha_1, \dots, \alpha_r\}, \quad J = \{\beta_1, \dots, \beta_r\},$$

and $|I, J\rangle$ is defined as in Lemma 2.7.3. Put

$$\mathcal{H}_0 = (\text{span of the family } \{|\lambda\rangle\}) \subset \mathcal{H}.$$

Since the family $\{|\lambda\rangle\}$ up to signs is part of the basis $\{|I, J\rangle\}$ for $\mathcal{H}$ provided by Lemma 2.7.3, the family $\{|\lambda\rangle\}$ is linearly independent, and hence a basis for $\mathcal{H}_0$.

Theorem 6.4. *(i) The map*

$$(f \mapsto f(T)|\bullet\rangle) : \Lambda \to \mathcal{H}_0$$

is one-to-one and onto. (ii) Moreover, we have

$$s_\lambda(T)|\bullet\rangle = |\lambda\rangle$$

for all partitions λ.

The basic idea behind Theorem 6.4 is physics folklore. On the physics side it is known as the boson-fermion correspondence, cf. [Pressley-Segal, pp. 214-215]. The theorem efficiently bridges the gap between classical symmetric function theory and the *ad hoc* theory of fermionic Fock space developed here. The proof of the theorem requires some preparation and is not going to be completed until §6.6.

6.5. **Yet another trio of technical lemmas.**

Lemma 6.5.1. *Fix a partition* λ *and an integer* $\ell > 0$. *The vector* $T_\ell|\lambda\rangle$ *is a linear combination of vectors of the form* $|\mu\rangle$ *with* $|\mu| = |\lambda| + \ell$. *In particular,* $T_\ell|\lambda\rangle = 0$ *if* $|\lambda| + \ell < 0$.

Proof. Write

$$\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r), \quad I = \{\alpha_1, \dots, \alpha_r\}, \quad J = \{\beta_1, \dots, \beta_r\}.$$

Recall that we have

$$|\lambda\rangle = \pm|I, J\rangle, \quad T_\ell|\bullet\rangle = \sum_{\substack{i,j \geq 0 \\ i+j=\ell-1}} (t^i)^\sharp (t^j)^\flat |\bullet\rangle$$

by the definitions. By straightforward exploitation of the Leibniz identity, the anticommutator trick and the definitions, it can be verified that

$$\begin{aligned} & T_\ell|\lambda\rangle \\ = \quad & \sum_{\substack{i \in I \\ i+\ell \notin I}} \pm|I \setminus \{i\} \cup \{i+\ell\}, J\rangle + \sum_{\substack{j \in J \\ j+\ell \notin J}} \pm|I, J \setminus \{j\} \cup \{j+\ell\}\rangle \\ & + \sum_{\substack{i,j \geq 0 \\ i+j=\ell-1 \\ i \notin I \\ j \notin J}} \pm|I \cup \{i\}, J \cup \{j\}\rangle. \end{aligned}$$

We leave the details to the reader. The last identity after a moment's reflection about Frobenius notation proves what we want. $\square$

Lemma 6.5.2. *We have*

$$(t^{n-1+\lambda_1})^\sharp \cdots (t^{n-n+\lambda_n})^\sharp |\bullet\rangle = \sigma_t^n |\lambda\rangle$$

for all partitions λ and integers $n \geq \ell(\lambda)$.

Proof. Put

$$\begin{array}{rcll} S & = & \{n-i+\lambda_i | i = 1,2,3,\dots\} & \supset \{i \in \mathbb{Z} | i < 0\}, \\ S' & = & \{-1+n+i-\lambda'_i | i = 1,2,3,\dots\} & \subset \{i \in \mathbb{Z} | i \geq 0\}. \end{array}$$

By [Macdonald, I, 1, (1.7)] the set $\mathbb{Z}$ is the disjoint union of the sets S and S'. Write

$$\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r).$$

To abbreviate notation conveniently, put

$$\alpha'_i = \alpha_i + n, \quad \beta'_j = \beta_j - n.$$

We have

$$\begin{array}{rcl} \{s \in S | s \geq n\} & = & \{\alpha'_1 > \cdots > \alpha'_r\}, \\ \{s \in S' | s < n\} & = & \{-1-\beta'_r > \cdots > -1-\beta'_1\} \\ & = & \{n-1 > \cdots > 0\} \setminus S, \end{array}$$

and we also have

$$\sigma_t^n |\lambda\rangle = (-1)^{\beta_1+\cdots+\beta_r} (t^{\alpha'_1})^\sharp \cdots (t^{\alpha'_r})^\sharp (t^{\beta'_r})^\flat \cdots (t^{\beta'_1})^\flat (t^{n-1})^\sharp \cdots (t^0)^\sharp |\bullet\rangle$$

by the definitions and the calculation of §5.2.7. The result follows now after a straightforward application of the anticommutator trick. □

Lemma 6.5.3. *Fix a finite sequence $z_1, \dots, z_n$ of complex numbers. Put*

$$u = \prod_{i=1}^{n} (1 - z_i t^{-1})^{-1} \in \mathbb{C}((1/t)).$$

(i) We have

$$\langle\bullet|\sigma_u f(T)|\bullet\rangle = f(z_1, \dots, z_n)$$

for all $f \in \Lambda$. (ii) We have

$$\langle\bullet|\sigma_u|\lambda\rangle = s_\lambda(z_1, \dots, z_n)$$

for all partitions λ.

Proof. (i) Without loss of generality we may assume that $f = p_\lambda$ for some partition λ. We have

$$\begin{array}{rcl} \langle\bullet|\sigma_u p_\lambda(T)|\bullet\rangle & \overset{\text{I}}{=} & \langle\bullet|\sigma_u p_\lambda(T)\sigma_u^{-1}|\bullet\rangle \\ & \overset{\text{II}}{=} & \langle\bullet|\prod_{i=1}^{\ell(\lambda)}\sigma_u T_{\lambda_i}\sigma_u^{-1}|\bullet\rangle \\ & \overset{\text{III}}{=} & \langle\bullet|\prod_{i=1}^{\ell(\lambda)}\left(T_{\lambda_i}-\mathrm{Res}(t^{\lambda_i}u'/u)\right)|\bullet\rangle \\ & \overset{\text{IV}}{=} & \prod_{i=1}^{\ell(\lambda)}\left(-\mathrm{Res}(t^{\lambda_i}u'/u)\right) \\ & \overset{\text{V}}{=} & p_\lambda(z_1,\dots,z_n), \end{array}$$

at I by definition of σ_u, at II because the intermediate factors of σ_u and σ_u^{-1} cancel, at III by the commutation relations worked out in §5.3.4, at IV by the fact noted in §5.2.4 that $\langle\bullet|T_i = 0$ for $i \geq 0$, and at V by the remark of §6.2.3. (ii) We have

$$\begin{array}{rcl} \langle\bullet|\sigma_u|\lambda\rangle & = & \langle\bullet|\sigma_t^{-n}\sigma_t^n\sigma_u|\lambda\rangle \\ & \overset{\text{I}}{=} & \langle\bullet|\sigma_t^{-n}\sigma_u\sigma_t^n|\lambda\rangle \\ & \overset{\text{II}}{=} & \langle\bullet|(t^{-1})^\flat\cdots(t^{-n})^\flat\sigma_u(t^{n+\lambda_1-1})^\sharp\cdots(t^{n+\lambda_n-n})^\sharp|\bullet\rangle \\ & \overset{\text{III}}{=} & \langle\bullet|(t^{-1})^\flat\cdots(t^{-n})^\flat(t^{n+\lambda_1-1}u)^\sharp\cdots(t^{n+\lambda_n-n}u)^\sharp|\bullet\rangle \\ & \overset{\text{IV}}{=} & \det_{i,j=1}^n\mathrm{Res}(t^{i-n-1}(t^{n+\lambda_j-j}u)_{\geq 0}) \\ & = & \det_{i,j=1}^n\mathrm{Res}((t^{i-n-1})_{<0}t^{n+\lambda_j-j}u) \\ & = & \det_{i,j=1}^n\mathrm{Res}(t^{i+\lambda_j-j-1}u) \\ & \overset{\text{V}}{=} & s_\lambda(z_1,\dots,z_n), \end{array}$$

at I by the commutation relation worked out §5.3.2, at II by the calculation of §5.2.7 and Lemma 6.5.2, at III by the definition of σ_u, at IV by the variant of the Wick-Lieb identity in §4.4.1 (with sharps and flats reversed) and at V by the remarks of §6.2.3 and the definitions. □

6.6. **Proof of Theorem 6.4.**

(i) For each integer $n \geq 0$, let $\mathrm{gr}_n\,\Lambda$ (resp., $\mathrm{gr}_n\,\mathcal{H}_0$) be the subspace of Λ spanned by symmetric functions of the form p_λ (resp., vectors of the form $|\lambda\rangle$) with λ a partition such that $|\lambda| = n$. Then we have direct sum decompositions

$$\Lambda = \bigoplus_{n=0}^\infty \mathrm{gr}_n\,\Lambda, \quad \mathcal{H}_0 = \bigoplus_{n=0}^\infty \mathrm{gr}_n\,\mathcal{H}_0.$$

The map in question sends $\mathrm{gr}_n\,\Lambda$ to $\mathrm{gr}_n\,\mathcal{H}_0$ for all n by Lemma 6.5.1. Further, this map is one-to-one by Lemma 6.5.3(i). Finally, since the families $\{p_\lambda\}$ and $\{|\lambda\rangle\}$ indexed by partitions λ are linearly independent, we have

$$\dim \mathrm{gr}_n\,\Lambda = \dim \mathrm{gr}_n\,\mathcal{H}_0 < \infty$$

for all n, and hence the map in question is onto.

(ii) By what we have already proved, we have

$$|\lambda\rangle = f(T)|\bullet\rangle$$

for a uniquely determined symmetric function $f \in \Lambda$. But then by Lemma 6.5.3 we must have

$$s_\lambda(z_1, \ldots, z_n) = f(z_1, \ldots, z_n)$$

for all finite sequences $z_1, \ldots, z_n$ of complex numbers. The latter is possible only if $f = s_\lambda$. □

Corollary 6.7. *Let ϵ be a variable independent of the variables x_i. Let $z_1, \ldots, z_n$ be any finite sequence of complex numbers. Write*

$$\exp\left(\sum_{i=1}^{\infty} \frac{p_i(z_1, \ldots, z_n) p_i \epsilon^i}{i}\right) = \sum_{i=0}^{\infty} f_i \epsilon^i \qquad (f_i \in \Lambda).$$

Then we have

$$f_i(T)|\bullet\rangle = \sum_{|\lambda|=i} s_\lambda(z_1, \ldots, z_n)|\lambda\rangle$$

for all $i \geq 0$.

Proof. By suitably specializing the orthogonality relations discussed in [Macdonald, I,4], we obtain the relations

$$f_i = \sum_{|\lambda|=i} s_\lambda(z_1, \ldots, z_n) s_\lambda.$$

The result now follows via Theorem 6.4(ii). □

7. Generalization of Weyl's identity

We continue in the setting of §6.

Theorem 7.1. *Fix a sequence $\{a_\ell\}_{\ell=1}^{\infty}$ of complex numbers arbitrarily. Let x and y be independent variables. Let complex coefficients $M_{\alpha\beta}$ be defined by the two-variable power series identity*

$$\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} M_{\alpha\beta} x^\alpha y^\beta = -\frac{y}{x} \frac{\partial}{\partial y} \log\left(1 - \sum_{\ell>0} a_\ell x y \frac{x^\ell - y^\ell}{x - y}\right).$$

Let ϵ be a variable independent of the x_i. Put

$$W = \prod_{i<j} \left(1 - \sum_{\ell>0} a_\ell x_i x_j \frac{x_i^\ell - x_j^\ell}{x_i - x_j} \epsilon^{\ell+1}\right),$$

thus defining a power series in ϵ with coefficients in Λ. Then we have identities

$$\begin{aligned} W &= \sum_{\lambda=(\alpha_1\ldots\alpha_r|\beta_1\ldots\beta_r)} (-1)^{\beta_1+\cdots+\beta_r} \det_{i,j=1}^{r} M_{\alpha_i\beta_j} \cdot s_\lambda \epsilon^{|\lambda|} \\ &= \exp\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{i-1,j} \frac{p_{i+j}-p_ip_j}{i+j}\epsilon^{i+j}\right) \end{aligned}$$

holding in the ring of power series in ϵ with coefficients in Λ.

The proof of the theorem takes up the rest of §7.

7.2. Interpretation and analysis of the coefficients $M_{\alpha\beta}$.

Lemma 7.2.1. *Put*

$$\bar{v} = t + \sum_{\ell=1}^{\infty} a_\ell t^{-\ell} \in \mathbb{C}((1/t))$$

and let $v \in \mathbb{C}((1/t))$ be the formal Laurent series inverse of $\bar{v}$, i. e., the unique Laurent series such that

$$v \circ \bar{v} = \bar{v} \circ v = t = v + \sum_{\ell=1}^{\infty} a_\ell v^{-\ell}.$$

We have

$$(v^\beta)_{\geq 0} \circ \bar{v} = (t^\beta \bar{v}')_{\geq 0} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i+j-1}((v^{\beta-i})_{\geq 0} \circ \bar{v})t^{-j}$$

for all integers β.

The sum on the right makes sense because for any fixed integer N the number of terms f of the sum such that $\deg f \geq N$ is finite.

Proof. We proceed by induction on β. The claimed identity holds for $\beta < 0$ because in that case both sides vanish identically. Given the identity for some β, we have

$$\begin{array}{rl} & (v^{\beta+1})_{\geq 0}\circ\bar{v}\\ = & (v^\beta t)_{\geq 0}\circ\bar{v}-\sum_{\ell>0}a_\ell(v^{\beta-\ell})_{\geq 0}\circ\bar{v}\\ = & \mathrm{Res}(v^\beta)+((v^\beta)_{\geq 0}t)\circ\bar{v}-\sum_{\ell>0}a_\ell(v^{\beta-\ell})_{\geq 0}\circ\bar{v}\\ = & \mathrm{Res}(t^\beta\bar{v}')+((v^\beta)_{\geq 0}\circ\bar{v})t+\sum_{\ell>0}a_\ell((v^\beta)_{\geq 0}\circ\bar{v})t^{-\ell}\\ & -\sum_{\ell>0}a_\ell(v^{\beta-\ell})_{\geq 0}\circ\bar{v}\\ \stackrel{\star}{=} & \mathrm{Res}(t^\beta\bar{v}')+(t^\beta\bar{v}')_{\geq 0}t+\sum_{i,j>0}a_{i+j-1}((v^{\beta-i})_{\geq 0}\circ\bar{v})t^{1-j}\\ & +\sum_{\ell>0}a_\ell((v^\beta)_{\geq 0}\circ\bar{v})t^{-\ell}-\sum_{\ell>0}a_\ell(v^{\beta-\ell})_{\geq 0}\circ\bar{v}\\ = & (t^{\beta+1}\bar{v}')_{\geq 0}+\sum_{i,j>0}a_{i+j-1}((v^{\beta-i})_{\geq 0}\circ\bar{v})t^{1-j}\\ & +\sum_{\ell>0}a_\ell((v^\beta)_{\geq 0}\circ\bar{v})t^{-\ell}-\sum_{\ell>0}a_\ell(v^{\beta-\ell})_{\geq 0}\circ\bar{v}\\ = & (t^{\beta+1}\bar{v}')_{\geq 0}+\sum_{i,j>0}a_{i+j}((v^{\beta-i})_{\geq 0}\circ\bar{v})t^{-j}\\ & +\sum_{\ell>0}a_\ell((v^\beta)_{\geq 0}\circ\bar{v})t^{-\ell}\\ = & (t^{\beta+1}\bar{v}')_{\geq 0}+\sum_{i,j>0}a_{i+j-1}((v^{\beta+1-i})_{\geq 0}\circ\bar{v})t^{-j}\end{array}$$

at $\star$ by the induction hypothesis and elsewhere by routine algebraic manipulations, and hence the identity holds for $\beta+1$. Therefore the identity holds in general. □

Lemma 7.2.2. *With v as in the preceding lemma, we have*

$$M_{\alpha\beta}=\mathrm{Res}(v'v^\alpha(v^\beta)_{\geq 0})$$

for all integers $\alpha,\beta\geq 0$.

Proof. To abbreviate notation put

$$\begin{array}{rcl} Q_{\alpha\beta} & = & \mathrm{Res}(t^\alpha((v^\beta)_{\geq 0}\circ\bar{v}))=\mathrm{Res}(v'v^\alpha(v^\beta)_{\geq 0})=\mathrm{Res}((v'v^\alpha)_{<0}v^\beta),\\ R_{\alpha\beta} & = & \mathrm{Res}(t^\alpha(t^\beta\bar{v}')_{\geq 0})=\mathrm{Res}((t^\alpha)_{<0}t^\beta\bar{v}')\end{array}$$

for all integers α and β. We have

$$\beta<0\ \Rightarrow\ \left\{\begin{array}{rcl} Q_{\alpha\beta} & = & 0,\\ R_{\alpha\beta} & = & 0,\end{array}\right.$$

$$\alpha<0\ \Rightarrow\ \left\{\begin{array}{rcl} Q_{\alpha\beta} & = & \mathrm{Res}(t^{\alpha+\beta}),\\ R_{\alpha\beta} & = & \mathrm{Res}(t^{\alpha+\beta}\bar{v}'),\end{array}\right.$$

$$\alpha\geq 0\ \Rightarrow\ \quad R_{\alpha\beta}=0.$$

Moreover, we have

$$Q_{\alpha\beta}=R_{\alpha\beta}+\sum_{i=1}^\infty\sum_{j=1}^\infty a_{i+j-1}Q_{\alpha-i,\beta-j}$$

by Lemma 7.2.1; the sum on the right makes sense since only finitely many nonzero terms appear in it. Now put

$$\begin{array}{rcl} f(x) & = & \sum_{\ell>0} a_\ell x^\ell, \\ F(x,y) & = & \sum_{\alpha,\beta\geq 0} Q_{\alpha\beta} x^\alpha y^\beta, \\ G(x,y) & = & \sum_{\alpha,\beta} (Q_{\alpha-1,\beta} - Q_{\alpha,\beta-1}) x^\alpha y^\beta, \\ H(x,y) & = & \sum_{\alpha,\beta} (R_{\alpha-1,\beta} - R_{\alpha,\beta-1}) x^\alpha y^\beta. \end{array}$$

The properties of the coefficients $Q_{\alpha\beta}$ and $R_{\alpha\beta}$ noted above imply the power series identities

$$\begin{array}{rcl} G(x,y) & = & 1 + (x-y)F(x,y), \\ H(x,y) & = & 1 - y^2 f'(y), \\ G(x,y) & = & H(x,y) + xy\frac{f(x)-f(y)}{x-y} G(x,y). \end{array}$$

Solving for F, we find that

$$\begin{array}{rcl} F & = & \dfrac{xy\frac{f(x)-f(y)}{x-y} - y^2 f'(y)}{x-y} \Big/ \left(1 - xy\dfrac{f(x)-f(y)}{x-y}\right) \\ & = & -\dfrac{y}{x}\dfrac{\partial}{\partial y} \log\left(1 - xy\dfrac{f(x)-f(y)}{x-y}\right), \end{array}$$

which proves the result. □

Lemma 7.2.3. *For all partitions* $\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r)$ *we have*

$$\langle \bullet | \rho_v | \lambda \rangle = (-1)^{\beta_1 + \cdots + \beta_r} \det_{i,j=1}^{r} M_{\alpha_i \beta_j}.$$

Proof. We have

$$\begin{array}{rcl} \langle \bullet | \rho_v | \lambda \rangle & \stackrel{\text{I}}{=} & (-1)^{\beta_1+\cdots+\beta_r} \langle \bullet | \rho_v (t^{\alpha_1})^\sharp \cdots (t^{\alpha_r})^\sharp (t^{\beta_r})^\flat \cdots (t^{\beta_1})^\flat | \bullet \rangle \\ & \stackrel{\text{II}}{=} & (-1)^{\beta_1+\cdots+\beta_r} \langle \bullet | (v' v^{\alpha_1})^\sharp \cdots (v' v^{\alpha_r})^\sharp (v^{\beta_r})^\flat \cdots (v^{\beta_1})^\flat | \bullet \rangle \\ & \stackrel{\text{III}}{=} & (-1)^{\beta_1+\cdots+\beta_r} \det_{i,j=1}^r \operatorname{Res}(v^{\alpha_i} v' (v^{\beta_j})_{\geq 0}) \\ & \stackrel{\text{IV}}{=} & (-1)^{\beta_1+\cdots+\beta_r} \det_{i,j=1}^r M_{\alpha_i \beta_j} \end{array}$$

at I by definition of $|\lambda\rangle$, at II by definition of ρ_v, at III by the Wick-Lieb identity and at IV by Lemma 7.2.2. □

Lemma 7.2.4. *Let* $\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r)$ *be a partition. Assume* $|\lambda| > 0$ *and hence that* $r > 0$. *Fix complex numbers* z_1 *and* z_2. *Put*

$$\Delta = (-1)^{\beta_1 + \cdots + \beta_r} \det_{i,j=1}^{r} M_{\alpha_i, \beta_j}.$$

We have $\Delta \cdot s_\lambda(z_1, z_2) = 0$ *unless* $\lambda = (\alpha|1)$ *for some integer* $\alpha \geq 0$*, in which case*

$$\Delta \cdot s_\lambda(z_1, z_2) = -a_{\alpha+1} \sum_{i=0}^{\alpha} z_1^{\alpha-i+1} z_2^{i+1}.$$

Proof. Recall that we have

$$s_\lambda = \det_{i,j=1}^{\ell(\lambda)} h_{i-j+\lambda_j} = \det_{i,j=1}^{\ell(\lambda')} e_{i-j+\lambda'_j}.$$

Clearly we have

$$i > 2 \Rightarrow e_i(z_1, z_2) = 0.$$

Therefore we have

$$2 < \lambda'_1 = \ell(\lambda) \Rightarrow s_\lambda(z_1, z_2) = 0,$$

and hence the lemma holds in the case $\ell(\lambda) > 2$. Assume for the rest of the proof that $\ell(\lambda) \leq 2$. Three mutually exclusive cases exhaust the remaining possibilities, namely

$$(\ell(\lambda), r) = (1, 1), (2, 2), (2, 1).$$

In the first case we have $\lambda = (\alpha|0)$ for some nonnegative integer α, hence

$$\Delta = M_{\alpha 0} \stackrel{\star}{=} \operatorname{Res}(v' v^\alpha (v^0)_{\geq 0}) = \operatorname{Res}(v' v^\alpha) = 0,$$

at $\star$ by Lemma 7.2.2, and elsewhere clearly, and hence the lemma holds in the first case. In the second case we have $\lambda = (\alpha_1 \alpha_2|10)$ for some nonnegative integers $\alpha_1 > \alpha_2$, hence by what we have already proved $-\Delta$ is the determinant of a square matrix with an identically vanishing second row, hence $\Delta = 0$, and hence the lemma holds in the second case. In the third case we have $\lambda = (\alpha|1)$ for some integer $\alpha \geq 0$, hence

$$\Delta = -M_{\alpha 1} \stackrel{\star}{=} -\operatorname{Res}(v' v^\alpha (v_{\geq 0})) = -\operatorname{Res}(v' v^\alpha t) = -\operatorname{Res}(\bar{v} t^\alpha) = -a_{\alpha+1},$$

at $\star$ by Lemma 7.2.2 and elsewhere clearly, and since

$$s_\lambda(z_1, z_2) = \begin{vmatrix} h_{\alpha+1} & h_{\alpha+2} \\ 1 & h_1 \end{vmatrix} (z_1, z_2) = \sum_{i=0}^{\alpha} z_1^{\alpha-i+1} z_2^{i+1},$$

the lemma holds in the third and final case. □

7.3. **Analysis of commutation relations.**

Lemma 7.3.1. *Let* $\{X_i\}_{i=1}^\infty$ *and* $\{Y_i\}_{i=1}^\infty$ *be families of linear operators on* $\mathcal{H}$. *Let* $\{c_{ij}\}_{i,j=1}^\infty$ *be a family of scalars. Assume that the* X_i *commute among themselves, the* Y_i *commute among themselves, the sums* X_i+Y_i *commute among themselves, and* $[X_i, Y_j] = c_{ij}$ *for all indices* i *and* j. *As above, let* ϵ *be a variable. Then we have an identity*

$$\begin{aligned} &\exp\left(\textstyle\sum_{i=1}^\infty (X_i + Y_i)\frac{\epsilon^i}{i}\right) \\ = \ &\exp\left(\textstyle\sum_{i=1}^\infty X_i\frac{\epsilon^i}{i}\right)\exp\left(\textstyle\sum_{i=1}^\infty Y_i\frac{\epsilon^i}{i}\right)\exp\left(-\textstyle\sum_{i=1}^\infty\sum_{j=1}^\infty c_{ij}\frac{\epsilon^{i+j}}{j(i+j)}\right)\end{aligned}$$

in the ring of power series in ϵ *with coefficients in the ring of linear operators on* $\mathcal{H}$.

Proof. For any sequence $\{Z_i\}_{i=1}^\infty$ of linear operators on $\mathcal{H}$, the initial value problem

$$(\star) \qquad \frac{d}{d\epsilon}\Phi = \Phi\sum_{i=1}^\infty Z_i\epsilon^{i-1}, \quad \Phi|_{\epsilon=0} = 1$$

has a unique solution in the ring of power series in ϵ with coefficients in the ring of linear operators on $\mathcal{H}$, as can be verified by a straightforward calculation with undetermined coefficients. Moreover, if the Z_i commute among themselves, then $\Phi = \exp(\sum_{i=1}^\infty Z_i\frac{\epsilon^i}{i})$ is the unique solution of $(\star)$. In particular, $\exp(\sum_{i=1}^\infty (X_i+Y_i)\frac{\epsilon^i}{i})$ (resp., $\exp(\sum_{i=1}^\infty X_i\frac{\epsilon^i}{i})$, $\exp(\sum_{i=1}^\infty Y_i\frac{\epsilon^i}{i})$) solves $(\star)$ with $Z_i = X_i + Y_i$ (resp., $Z_i = X_i$, $Z_i = Y_i$). By hypothesis we have

$$\left[\sum_{i=1}^\infty X_i\epsilon^{i-1}, \sum_{j=1}^\infty Y_j\frac{\epsilon^j}{j}\right] = \sum_{i=1}^\infty\sum_{j=1}^\infty c_{ij}\frac{\epsilon^{i+j-1}}{j}.$$

Note that the power series on the right is central in the ring of power series in ϵ with coefficients in the ring of linear operators on $\mathcal{H}$. It follows by the Leibniz identity that we have

$$\left[\sum_{i=1}^\infty X_i\epsilon^{i-1}, \left(\sum_{j=1}^\infty Y_j\frac{\epsilon^j}{j}\right)^\ell\right] = \ell\left(\sum_{j=1}^\infty Y_j\frac{\epsilon^j}{j}\right)^{\ell-1}\sum_{i=1}^\infty\sum_{j=1}^\infty c_{ij}\frac{\epsilon^{i+j-1}}{j}$$

for all positive integers ℓ and in turn that

$$\left[\sum_{i=1}^\infty X_i\epsilon^{i-1}, \exp\left(\sum_{j=1}^\infty Y_j\frac{\epsilon^j}{j}\right)\right] = \exp\left(\sum_{j=1}^\infty Y_j\frac{\epsilon^j}{j}\right)\sum_{i=1}^\infty\sum_{j=1}^\infty c_{ij}\frac{\epsilon^{i+j-1}}{j}.$$

Therefore not only does the left side of the claimed identity solve $(\star)$ with $Z_i = X_i + Y_i$ but so does the right side. $\square$

Lemma 7.3.2. *Let* $\{b_i\}_{i=1}^\infty$ *be any sequence of complex numbers. We have an identity*

$$\begin{array}{rl} & \rho_v \exp\left(\sum_{i=1}^\infty \frac{b_i T_i \epsilon^i}{i}\right) \\ = & \exp\left(\sum_{i=1}^\infty \frac{b_i \operatorname{Res}(t^i \bar{v}''/\bar{v}')\epsilon^i}{2i}\right) \exp\left(-\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{M_{i-1,j} b_i b_j \epsilon^{i+j}}{i+j}\right) \\ \times & \exp\left(\sum_{i=1}^\infty \frac{b_i T_{(v^i)_{\geq 0}} \epsilon^i}{i}\right) \exp\left(\sum_{j=1}^\infty \frac{b_j T_{(v^j)_{<0}} \epsilon^j}{j}\right) \rho_v \end{array}$$

in the ring of power series in ϵ *with coefficients in the ring of linear operators on* $\mathcal{H}$.

Proof. We have

$$\begin{array}{rl} & \rho_v \exp\left(\sum_{i=1}^\infty \frac{b_i T_i \epsilon^i}{i}\right) \\ = & \exp\left(\sum_{i=1}^\infty \frac{b_i \rho_v T_i \rho_v^{-1} \epsilon^i}{i}\right) \rho_v \\ = & \exp\left(\sum_{i=1}^\infty \frac{b_i (T_{v^i} + \operatorname{Res}(t^i \bar{v}''/\bar{v}')/2)\epsilon^i}{i}\right) \rho_v \\ = & \exp\left(\sum_{i=1}^\infty \frac{b_i \operatorname{Res}(t^i \bar{v}''/\bar{v}')\epsilon^i}{2i}\right) \exp\left(\sum_{i=1}^\infty \frac{b_i T_{v^i} \epsilon^i}{i}\right) \rho_v \end{array}$$

by the commutation relations worked out in §5.3.6. Now put

$$X_i = b_i T_{(v^i)_{\geq 0}}, \quad Y_i = b_i T_{(v^i)_{<0}}, \quad c_{ij} = b_i b_j \operatorname{Res}((v^i)_{\geq 0}((v^j)_{<0})')$$

for all i and j. By the commutation relations worked out in §5.3.1, the hypotheses of Lemma 7.3.1 are fulfilled by this choice of X_i, Y_i and c_{ij}. Moreover, we have

$$\begin{array}{rcl} c_{ij} & = & b_i b_j \operatorname{Res}((v^i)_{\geq 0}((v^j)')_{<0}) \\ & = & b_i b_j \operatorname{Res}((v^i)_{\geq 0}(v^j)') \\ & = & j b_i b_j \operatorname{Res}((v^i)_{\geq 0} v^{j-1} v') \\ & = & j b_i b_j M_{j-1,i} \end{array}$$

for all i and j. So the desired result follows from Lemma 7.3.1. □

7.3.3. In order to state the final lemma of the paper we introduce a convenient abuse of notation. Given a power series

$$\sum_{i=0}^\infty X_i \epsilon^i$$

in ϵ with coefficients in the ring of linear operators on $\mathcal{H}$, we write

$$\langle \bullet | \sum_{i=0}^\infty X_i \epsilon^i | \bullet \rangle = \sum_{i=0}^\infty \langle \bullet | X_i | \bullet \rangle \epsilon^i,$$

thus defining a power series in ϵ with complex coefficients.

Lemma 7.3.4. *For every finite sequence $z_1, \dots, z_n$ of complex numbers we have an identity*

$$\begin{aligned} & \langle \bullet | \rho_v \exp\left(\textstyle\sum_{i=1}^\infty \frac{p_i(z_1,\dots,z_n) T_i \epsilon^i}{i}\right) | \bullet \rangle \\ = \quad & \exp\left(\textstyle\sum_{i=1}^\infty \frac{p_i(z_1,\dots,z_n)\operatorname{Res}(t^i \bar{v}''/\bar{v}')\epsilon^i}{2i}\right) \\ \times \quad & \exp\left(-\textstyle\sum_{i=1}^\infty \sum_{j=1}^\infty \frac{M_{i-1,j} p_i(z_1,\dots,z_n) p_j(z_1,\dots,z_n)\epsilon^{i+j}}{i+j}\right) \end{aligned}$$

in the ring of power series in ϵ with coefficients in the complex numbers.

Proof. Recall from §5.2.5 that

$$\langle \bullet | T_{h_{\geq 0}} = 0, \quad T_{h_{<0}} | \bullet \rangle = 0$$

for all $h \in \mathbb{C}((1/t))$. Recall also that $\rho_v | \bullet \rangle = | \bullet \rangle$. It follows that

$$\langle \bullet | \exp\left(\sum_{i=1}^\infty \frac{b_i T_{(v^i)_{\geq 0}} \epsilon^i}{i}\right) \exp\left(\sum_{j=1}^\infty \frac{b_j T_{(v^j)_{<0}} \epsilon^j}{j}\right) \rho_v | \bullet \rangle = 1.$$

The result now follows by Lemma 7.3.2. □

7.4. **Completion of the proof.** Fix a finite sequence $z_1, \dots, z_n$ of complex numbers arbitrarily. Put

$$U(z_1, \dots, z_n) = \langle \bullet | \rho_v \exp\left(\sum_{i=1}^\infty \frac{p_i(z_1, \dots, z_n) T_i \epsilon^i}{i}\right) | \bullet \rangle.$$

thus defining a power series in ϵ with complex coefficients. We have

$$U(z_1, \dots, z_n) = \sum_{\lambda = (\alpha_1 \dots \alpha_r | \beta_1 \dots \beta_r)} (-1)^{\beta_1 + \dots + \beta_r} \det_{i,j=1}^{r} M_{\alpha_i \beta_j} \cdot s_\lambda(z_1, \dots, z_n) \epsilon^{|\lambda|}.$$

by Corollary 6.7 and Lemma 7.2.3. We have

$$U(1, 0) = U(1) = 1$$

by Lemma 7.2.4, hence

$$\sum_{i=1}^\infty \frac{\operatorname{Res}(t^i \bar{v}''/\bar{v}')\epsilon^i}{2i} = \sum_{i=1}^\infty \sum_{j=1}^\infty \frac{M_{i-1,j}\epsilon^{i+j}}{i+j},$$

by the corresponding special case of Lemma 7.3.4 (alternatively: check the power series identity

$$\frac{1}{2} \sum_{i=1}^\infty \operatorname{Res}(t^i \bar{v}''/\bar{v}')\epsilon^{i-1} = \sum_{i=1}^\infty \sum_{j=1}^\infty M_{i-1,j}\epsilon^{i+j-1} = \sum_{i=0}^\infty \sum_{j=0}^\infty M_{ij}\epsilon^{i+j}$$

by direct calculation) and hence

$$U(z_1,\ldots,z_n)=\exp\left(\sum_{i=1}^\infty\sum_{j=1}^\infty M_{i-1,j}\frac{(p_{i+j}-p_ip_j)(z_1,\ldots,z_n)}{i+j}\epsilon^{i+j}\right)$$

by the general case of Lemma 7.3.4. Clearly we have

$$(p_{i+j}-p_ip_j)(z_1,\ldots,z_n)=\sum_{1\leq k<\ell\leq n}(p_{i+j}-p_ip_j)(z_k,z_\ell)$$

for all $i,j>0$ and hence

$$U(z_1,\ldots,z_n)=\prod_{1\leq i<j\leq n}U(z_i,z_j).$$

By Lemma 7.2.4 we have

$$U(z_1,z_2)=1-\sum_{\alpha=0}^\infty a_{\alpha+1}\left(\sum_{i=0}^\alpha z_1^{\alpha-i+1}z_2^{i+1}\right)\epsilon^{\alpha+2}.$$

We have proved enough now to verify that the identities asserted in Theorem 7.1 hold after we make the substitutions $x_i=z_i$ for $i=1,\ldots,n$ and $x_i=0$ for $i>n$. But the finite sequence $z_1,\ldots,z_n$ of complex numbers is arbitrary. Therefore the identities in question hold in general. The proof of Theorem 7.1 is complete. □

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