Questions From Old Exams

1. Write the equation of a quadratic function whose graph has the following characteristics:
   - It opens down; it is stretched by a factor of 2; its axis of symmetry is the line x = 1;
   - it passes through the point (4, -2).
2. Express the area A of an equilateral triangle as a function of x where x is the length of one side of the triangle.
3. Find the Average Rate of Change of \( f(x) = -\frac{2}{3}x + 2 \) from 1 to 3.
4. Given the function \( f(x) = -(x + 1)^5 (x - 2)^2 \)
   a. At what point does the graph cross the x-axis?
   b. At what point does the function touch the x-axis?
   c. What is the behavior of the function near each of its zeros?
      That is, what function does \( f(x) = -(x + 1)^5 (x - 2)^2 \) look like near each zero?
5. Given the function \( f(x) = -2x^2 + 12x - 10 \):
   a. Write the function in vertex form.
   b. What are the coordinates of the vertex?
   c. What is the equation of the axis of symmetry.
   d. What are the coordinates of the x-intercepts?
   e. What are the coordinates of the y-intercept?
6. Given the function \( f(x) = \frac{2x^3 - 2x^2 - 12x}{3x^3 + 12x^2 + 9x} = \frac{2x(x - 3)(x + 2)}{3x(x + 1)(x + 3)} \):
   a. What are the equations of the vertical asymptotes?
   b. What are the coordinates of the x-intercepts?
   c. What are the coordinates of holes in the graph, if any?
   d. What is the equation of the horizontal or oblique asymptote?
7. Sketch the graph of the function \( f(x) = -\frac{x^3 + 5x^2 - 6x}{x^2 - 1} \). On your graph, be sure to clearly show asymptotes, holes, intercepts, and indicate where the graph crosses an asymptote if it does.
8. Solve: \( \frac{1}{x} + 2 < -x \) Write your answer using interval notation.
Solutions

1. Write the equation of a quadratic function whose graph has the following characteristics:
   It opens down; it is stretched by a factor of 2; it’s axis of symmetry is the line \( x = 1 \);
   it passes through the point \((4, -2)\).

   \[
   f(x) = a(x - h)^2 + k
   \]
   \[
   y = -2(x - 1)^2 + k
   \]
   When \( x = 4, y = -2 \)
   \[
   -2 = -2(4 - 1)^2 + k
   \]
   \[
   16 = k
   \]
   \[
   f(x) = -2(x - 1)^2 + 16
   \]

2. Express the area \( A \) of an equilateral triangle as a function of \( x \) where \( x \) is the length of one side of the triangle.

   \[
   A = \frac{1}{2}bh
   \]
   \[
   = \frac{1}{2}(x)(h)
   \]

   To find the height, use Pythagoras:
   \[
   c^2 = a^2 + b^2
   \]
   \[
   x^2 = \left(\frac{x}{2}\right)^2 + h^2
   \]
   \[
   x^2 - \frac{x^2}{4} = h^2
   \]
   \[
   \frac{3x^2}{4} = h^2
   \]
   \[
   \frac{\sqrt{3}x}{2} = h
   \]

3. Find the Average Rate of Change of \( f(x) = -\frac{2}{3}x + 2 \) from 1 to 3.

   The ARC of a linear function is the slope of the line. In this case, \( m = -\frac{2}{3} \).
4. Given the function \( f(x) = -(x + 1)^5 (x - 2)^2 \)
   
a. At what point does the graph cross the x-axis?
   
The zero at \( x = -1 \) has odd multiplicity so the graph crosses the x-axis at \((-1, 0)\).
   
b. At what point does the function touch the x-axis?
   
The zero at \( x = 2 \) has even multiplicity so the graph touches the x-axis at \((2, 0)\).
   
c. What is the behavior of the function near each of its zeros?
   
That is, what function does \( f(x) = -(x + 1)^5 (x - 2)^2 \) look like near each zero?

Near \( x = -1 \) the function looks like \( f(x) \approx -(-1 - 2)^2 (x + 1)^5 = -9(x + 1)^5 \).

This is a very steep reflected cubic shape.

Near \( x = 2 \) the function looks like \( f(x) \approx -(x - 2)^2 (x + 1)^5 = -243(x - 2)^2 \).

This is a very steep upside down parabolic shape.
5. Given the function \( f(x) = -2x^2 + 12x - 10 \):

a. Write the function in vertex form.

\[
\begin{align*}
f(x) &= -2x^2 + 12x - 10 \\
&= -2(x^2 - 6x) - 10 \\
&= -2(x^2 - 6x + 2 - 2) - 10 \\
&= -2(x^2 - 6x + 9 - 9) - 10 \\
&= -2(x^2 - 6x + 9) - 2(-9) - 10 \\
&= -2(x - 3)^2 + 8
\end{align*}
\]

b. What are the coordinates of the vertex?

We can read the vertex directly from the equation: \((3, 8)\).

Or, the x-coordinate is given by \( x_{vertex} = \frac{-b}{2a} = \frac{-12}{2(-2)} = 3 \)

To find the y-coordinate, calculate \( f(x_{vertex}) = f(3) = -2(3)^2 + 12(3) - 10 = -18 + 36 - 10 = 8 \)

c. What is the equation of the axis of symmetry.

The axis of symmetry is the vertical line that passes through the vertex. The equation of any vertical line is \( x = k \), where \( k \) is a constant. In this case \( k \) is the value of \( x_{vertex} \). So, the equation of the axis of symmetry is \( x = 3 \).

d. What are the coordinates of the x-intercepts? \((1,0), (5,0)\)

The x-intercepts are the values of \( x \) when \( y = 0 \).

\[
\begin{align*}
0 &= -2x^2 + 12x - 10 \\
0 &= -2(x^2 - 6x + 5) \\
0 &= -2(x - 1)(x - 5) \\
x &= 1 \quad \text{or} \quad x = 5
\end{align*}
\]

e. What are the coordinates of the y-intercept?

The y-intercept is the value of \( y \) when \( x = 0 \):

\[
\begin{align*}
f(x) &= -2x^2 + 12x - 10 \\
y &= -2(0)^2 + 12(0) - 10 \\
y &= -10
\end{align*}
\]
6. Given the function \( f(x) = \frac{2x^3 - 2x^2 - 12x}{3x^3 + 12x^2 + 9x} = \frac{2x(x - 3)(x + 2)}{3x(x + 1)(x + 3)} \):

a. What are the equations of the vertical asymptotes?

The function reduces to \( g(x) = \frac{2(x - 3)(x + 2)}{3(x + 1)(x + 3)} \). The denominator is 0 when \( x = -1 \) and \( x = -3 \).

So, the vertical asymptotes are the lines \( x = -1 \) and \( x = -3 \).

b. What are the coordinates of the x-intercepts?

These are the zeros of the numerator. They are \((3, 0)\) and \((-2, 0)\).

c. What are the coordinates of any holes in the graph, if any?

The common factor of \( x \) cancels so there is a hole when \( x = 0 \). The y-coordinate is \( g(0) \).

\[
g(0) = \frac{2(0 - 3)(0 + 2)}{3(0 + 1)(0 + 3)} = \frac{2(-3)(2)}{3(1)(3)} = -\frac{4}{3}
\]

So, the hole is at \((0, -\frac{4}{3})\).

d. What is the equation of the horizontal or oblique asymptote?

The degree of the numerator and denominator is the same so there is a horizontal asymptote whose value is the ratio of the coefficients of the dominant terms. This is the line \( y = \frac{2}{3} \).
7. Sketch the graph of the function \( f(x) = \frac{-x^3 + 5x^2 - 6x}{x^2 - 1} \).

**Step 1.** Factor, state domain, and THEN reduce.

\[
f(x) = -\frac{x^3 + 5x^2 - 6x}{x^2 - 1} = -\frac{x(x^2 + 5x - 6)}{x^2 - 1} = -\frac{x(x - 1)(x + 6)}{(x - 1)(x + 1)}
\]

\[
g(x) = -\frac{x(x + 6)}{x + 1}
\]

There is a hole when \( x - 1 = 0 \), which means when \( x = 1 \).

The y-coordinate of the hole is \( g(1) = -\frac{1(1 + 6)}{1 + 1} = -\frac{7}{2} \) So, the hole is at \( \left( 1, -\frac{7}{2} \right) \).

**Step 2.** Plot intercepts. For x-intercepts, state whether graph touches (multiplicity is even) or crosses (multiplicity is odd) x-axis.

x-intercepts at \( x = 0 \) and \( x = -6 \). Both have multiplicity odd so graph crosses at \((0, 0)\) and \((-6, 0)\).

**Step 3.** Draw vertical asymptotes (the zeros of the denominator).

The denominator of the reduced fraction is 0 when \( x = -1 \) so \( x = -1 \) is a VA.

**Step 4.** Draw horizontal asymptotes: \( \text{deg num} = \text{deg denom} + 1 \), so use long division to find oblique asymptote, \( y = mx + b \)

\[
g(x) = -\frac{x(x + 6)}{x + 1} = -\frac{x^2 + 6x}{x + 1} = -\frac{-x^2 - 6x}{x + 1}
\]

\[
x + 1 \overline{\underline{-x^2 - 6x}}
\]

\[
-x - 5
\]

\[
-x^2 - x
\]

\[
-5x
\]

\[
-(-5x - 5)
\]

\[
5
\]

So, the OA is \( y = -x - 5 \).

**Step 5.** Plot where the graph crosses an HA or OA..

\[
g(x) = y
\]

\[
x^2 + 6x
\]

\[
-x^2 - x
\]

\[
x + 1
\]

\[
-x - 5
\]

\[
x^2 + 6x = x^2 + 6x + 5
\]

\[
0 = 5
\]

This has no solution so the graph does not cross the OA.

**Step 6.** If needed, plot a few extra points.

**Step 7.** Connect the dots.
8. Solve: \( \frac{1}{x} + 2 < -x \) Write your answer using interval notation.

**Step 1.** Write inequality with 0 on one side.

\[
\frac{1}{x} + 2 < -x \\
\frac{1}{x} + 2 + x < 0 \\
\frac{1}{x} + 2 \cdot \frac{x}{x} + x \cdot \frac{x}{x} < 0 \\
\frac{1 + 2x + x^2}{x} < 0 \\
\frac{(x + 1)(x + 1)}{x} < 0
\]

**Step 2.** Find the real zeros of the numerator and denominator.

\( x = -1 \) and \( x = 0 \)

**Step 3.** Use the real zeros to break up the number line into intervals.

\[
\begin{array}{cccc}
\text{Interval} : & -\infty & -1 & 0 & \infty \\
\end{array}
\]

**Step 4.** Use a test point in each interval to see if the function is pos or neg.

\[
\begin{array}{cccc}
\text{Interval} : & -\infty & -1 & 0 & \infty \\
\text{Try } x = & -2 & -1/2 & 1 \\
\text{Get } f = & \frac{(-2 + 1)(-2 + 1)}{-2} & \frac{(-1/2 + 1)(-1/2 + 1)}{-1/2} & \frac{(1 + 1)(1 + 1)}{1} \\
\text{Simplify} : & \frac{1}{2} & \frac{1}{2} & 4 \\
\text{ } f (x) < 0 & \text{yes} & \text{yes} & \text{no} \\
\end{array}
\]

We want \( f (x) < 0 \) so \( (-\infty, -1) \cup (-1, 0) \). Note that \( f(-1) = 0 \) so we must exclude -1.