Operad-algebras and homotopy categories

Robert G. Hank
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota 55454 USA

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Abstract

We provide background material necessary to understand Justin Noel’s paper $H_\infty \neq E_\infty$ [11]. We begin with an introduction to operads and algebras over operads. We describe classical examples of operads and their applications. In particular, we define $A_n$-structures and $E_n$-structures. We then introduce spectra and the notion of an $H_n$-structure in stable homotopy theory. We conclude by giving an overview of Noel’s paper equipped with this understanding, and our proposal for future work in this direction.

1 Introduction

Operads were developed in the late 1960’s by Boardman and Vogt [2] and in the early 1970’s by May [8]. These were originally developed as a way to recognize the structure of iterated loop spaces, but have since become far more ubiquitous for their use as a tool to more easily recognize structure on a space. In particular, operads are a way to generalize and organize collections of $n$-ary operations, operations which take $n$ inputs from a collection of spaces and give a coherent output in the collection. In Section 2 we give an intuitive example in the category of leaf-labeled trees for how such a collection can be organized. In Section 3 we abstract the information we observe from this example to define an operad and give several important examples.

One application of this machinery is to study algebras over an operad. Where an operad gives an abstract collection of $n$-ary operations, an algebra over an operad takes the abstract collection and gives concrete operations on the algebra. In Section 4 we study algebras over some of the important examples of operads. A natural question to ask at this point is if these concrete operations obey some usual desirable properties such as associativity or commutativity. These questions lead to the idea of $A_n$- and $E_n$-structures.
A set (or space, or $R$-module) $X$ admits an $A_n$-structure if it has a multiplication which is associative in a coherent way up to diagrams with $n$ elements in it. It admits an $E_n$-structure if it has a binary multiplication which makes all associativity diagrams commute in a coherent way, and is sufficiently commutative. The typical examples of operads which lead to these structures are the Stasheff associahedra operad $K$ in the $A_n$ case, and the little $n$-cubes operads $E_n$ in the $E_n$ case. In Section 5 we define $K$ and study its relationship to $A_n$-structures. In Section 6 we define $E_n$ and study its relationship to $E_n$-structures.

If we begin with a model category $\mathcal{C}$, a category where we can perform homotopy theory, we can pass to the homotopy category $\text{Ho}(\mathcal{C})$ of $\mathcal{C}$. Structures which existed in the original category still persist in the homotopy category. However, by passing to the homotopy category we may introduce new structure which was not originally observed. In particular, an $H_n$-structure is an $E_n$-structure, but where the appropriate diagrams only commute up to homotopy. Equivalently, we could say that the diagrams commute on the nose when we pass to the homotopy category, where homotopic maps become isomorphic.

In Section 7 we give an overview of operads in homotopy categories. We begin by giving a particular example in the category of spectra. Much of modern research in algebraic topology involves the study of stable homotopy theory, homotopy theory done over a “stable” range where computations become easier. The correct starting point for this study is the category of spectra. Since this category is a model category, we may pass to its homotopy category, the stable homotopy category, and study $H_n$-structures on it. Proving that an object of the stable homotopy category admits an $H_n$-structure provides a rich framework which can immediately be put to use. Several examples of such applications are given in [3, §2].

It is not the case that every $H_n$-structure arises by beginning with an $E_n$-structure and passing to the homotopy category, even though there is a strong relationship between the two notions and some reasons to believe it may be the case. Noel in [11] provides an explicit counterexample in the category of spectra. In our future work, we will attempt to find a counterexample in the category of differential graded $R$-modules.

2 A motivating example of an operad

We begin with a motivating example in the category of leaf-labeled trees. A full definition of this category may be found in [7, §1.5]. For our purposes a tree is a finite connected graph without any loops. However, we will delete the vertices which have only one adjacent edge. This means that some of the edges will have only one adjacent vertex, and will be called external edges. We will have a distinguished external edge $e$ called the root, and all other external edges will be called leaves. We will write our trees with the root on bottom and branches extending upwards. The leaves are then edges of the tree without a vertex at the top.

Let $L(T)$ denote the set of leaves of the tree $T$. An $X$-labeled tree $(T, l)$ is a tree $T$ together with a bijection $l : L(T) \to X$ from the set of leaves of $T$ to the set $X$. An $n$-labeled tree is an $X$-labeled tree with $X = [n] = \{1,2,\ldots,n\}$ for some positive integer $n$. An example of
a 7-labeled tree is shown in Figure 1.

![Diagram of a 7-labeled tree](image)

Figure 1: An example of \((T, l)\) of a 7-labeled tree. A tree with 7 leaves and a bijection \(L(T) \to [7]\).

There is some appropriate notion of isomorphism that allows us to consider two \(n\)-labeled trees to be the same if we can obtain one tree from the other by successively swapping two branches (along with the labelings of leaves contained in those branches). Let \(T(n)\) denote the collection of (isomorphism classes of) \(n\)-labeled trees. The reader should have in mind that a tree in \(T(n)\) represents an \(n\)-ary operation which takes in \(n\) inputs, one input for each leaf at the top of the tree, and gives one output at the root of the tree. The structure of the tree shows whether or not this \(n\)-ary operation is built up from compositions of smaller operations. In the example shown above, the given tree would represent a composition of a 2-ary operation \(F(x, y)\), a 3-ary operation \(G(x, y, z)\) and a 4-ary operation \(H(w, x, y, z)\) giving a single complete operation

\[
(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto H(x_1, F(x_2, x_3), x_4, G(x_5, x_6, x_7)),
\]

where we have used the outputs from \(F\) and \(G\), applied to the appropriate terms, as inputs into \(H\).

We now observe some properties for the collection of such trees. First, for each \(n\) there is a left \(\Sigma_n\)-action on the labelings of trees \(T \in T(n)\). If \(L(T) = \{l_1, \ldots, l_n\}\) are labeled via \(l : l_i \mapsto i\), then \(\sigma \in \Sigma_n\) acts on \(l\) by relabeling \(l_i\) with \(\sigma^{-1}(i)\). That is, \(\sigma l : l_i \mapsto \sigma^{-1}(i)\). This action induces a right action on the trees given by

\[
(T, l)\sigma = (T, \sigma^{-1}l).
\]

If \((T, l)\) is the example above and \(\sigma = (1\ 3\ 4)(2\ 5\ 7\ 6)\), then \((T, l)\sigma\) is the same tree but with the relabeling seen in Figure 2. For example, the leaf which was labeled 5 is now labeled 2 since \(\sigma : 2 \mapsto 5\).

There is also a sense in which we can compose trees by “grafting”. For convenience, we will drop the labeling \(l\) from the notation and write \(T\) instead of \((T, l)\). Suppose \(T \in T(3)\) is the corolla with 3 leaves as shown in Figure 3 and \((T_1, T_2, T_3) \in T(2) \times T(1) \times T(4)\) is the 3-tuple of trees in Figure 4. Then we can form the tree \(T \circ (T_1, T_2, T_3) \in T(2 + 1 + 4) = T(7)\) by...
Figure 2: The tree $T$ from Figure 1 after applying the action $(1 \ 3 \ 4)(2 \ 5 \ 7 \ 6)$.

Figure 3: The 3-corolla is an element of $T(3)$.

grafting the root of tree $T_i$ to the leaf of $T$ labeled $i$. The resulting tree, shown in Figure 5 has one leaf for each leaf of each tree in the 3-tuple. The labeling on this new tree is obtained as follows. Suppose $T \in T(n)$ and $T_m \in T(i_m)$ for $1 \leq m \leq n$. Suppose $l$ is a leaf in $T_k$ labeled with the number $j$. Find the leaf corresponding to $l$ in the grafted tree $T \circ (T_1, \ldots, T_n)$ and label it with the number $j + i_1 + \cdots + i_{k-1}$. Notice that every tree can be built up by successive compositions from the collection of $n$-corollas, shown in Figure 6 and appropriate applications of $\Sigma_n$ at each level of the composition.

Summarizing the above structure, we have a collection of spaces $T(n)$ for each $n \in \mathbb{N}$ consisting of (isomorphism classes of) trees with $n$ leaves. Each space $T(n)$ has a right $\Sigma_n$-action. Moreover, for each $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{N}$ we have a structure map

$$\gamma : T(n) \times (T(i_1) \times \cdots \times T(i_n)) \longrightarrow T(i_1 + \cdots + i_n)$$

depending on $n, i_1, \ldots, i_n$.

There are various ways of combining the data contained in this structure. For example, we could graft trees at more than one level, using the structure maps several times. There are two different ways to apply the structure maps, each giving the same output. This naturally leads us to an associativity relation on the structure. The 1-corolla, which has only 1 vertex, acts as a unit under grafting. When we graft trees with the 1-corolla, we obtain a tree which is isomorphic to the tree we started with. Finally, we may also mix applications of the structure maps with the $\Sigma_n$-actions, leading to an equivariance condition on our structure.
Then if \( n, i_1, \ldots, i_n \in \mathbb{N} \), \( T \in T(n) \), and \( S_k \in T(i_k) \), let \( I = i_1 + \cdots + i_n \) and let \( S = (S_1, \ldots, S_n) \). Then if \( j_1, \ldots, j_I \in \mathbb{N} \), \( R_I \in T(j_I) \), and \( R = (R_1, \ldots, R_I) \) we can compose the maps \( \gamma \) in two different ways. We could first apply \( \gamma \) to \( (T, S_1, \ldots, S_n) \in T(n) \times T(i_1) \times \cdots \times T(i_n) \) and get a tree \( S \) in \( T(I) \). Then we can apply \( \gamma \) to \( (S, R_1, \ldots, R_I) \) and get a tree \( R \) in \( T(j_1, \ldots, j_I) \). That is, if \( J = j_1 + \cdots + j_I \), \( T(I) = T(i_1) \times \cdots \times T(i_1) \) and \( T(J) = T(j_1) \times \cdots \times T(j_I) \), we get the composition

\[
T(n) \times T(I) \times T(J) \xrightarrow{\gamma \times \text{id}^I} T(I) \times T(J) \xrightarrow{\gamma} T(J).
\]

We also get a composition by shuffling the ordered tuple so that the first \( i_1 \) trees \( R_1, \ldots, R_{i_1} \) lie immediately after \( S_1 \), the next \( i_2 \) trees \( R_{i_1+1}, \ldots, R_{i_1+i_2} \) lie immediately after \( S_2 \), and so forth. We could apply \( \gamma \) on each tuple \((S_k, R_{i_1+\cdots+i_{k-1}+1}, \ldots, R_{i_1+\cdots+i_k})\), obtaining \( n \) different trees \( Q_k \), each in \( T(j_{i_1+\cdots+i_{k-1}+1} + \cdots + j_{i_1+\cdots+i_k}) \). We could then apply \( \gamma \) to the tuple \((T, Q_1, \ldots, Q_n)\) and obtain a tree \( Q \in T(J) \). That is, if \( k = j_{i_1+\cdots+i_{k-1}+1} + \cdots + j_{i_1+\cdots+i_k} \) is the sum of the \( k \)-th block of dimensions of trees \( R_j \), and \( T(J_k) = T(j_{i_1+\cdots+i_{k-1}+1}) \times \cdots \times T(j_{i_1+\cdots+i_k}) \), we have the composition

\[
T(n) \times T(I) \times T(J) \xrightarrow{\text{shuffle}} T(n) \times (T(i_1) \times T(J_1)) \times \cdots \times (T(i_n) \times T(J_n)) \xrightarrow{\text{id} \times \gamma^n} T(n) \times T(1) \times \cdots \times T(n) \xrightarrow{\gamma} T(J).
\]

### 2.1 Associativity

We have an associativity relation for all the various ways of composing the maps \( \gamma \). If \( n, i_1, \ldots, i_n \in \mathbb{N} \), \( T \in T(n) \), and \( S_k \in T(i_k) \), let \( I = i_1 + \cdots + i_n \) and let \( S = (S_1, \ldots, S_n) \). Then if \( j_1, \ldots, j_I \in \mathbb{N} \), \( R_I \in T(j_I) \), and \( R = (R_1, \ldots, R_I) \) we can compose the maps \( \gamma \) in two different ways. We could first apply \( \gamma \) to \( (T, S_1, \ldots, S_n) \in T(n) \times T(i_1) \times \cdots \times T(i_n) \) and get a tree \( S \) in \( T(I) \). Then we can apply \( \gamma \) to \( (S, R_1, \ldots, R_I) \) and get a tree \( R \) in \( T(j_1, \ldots, j_I) \). That is, if \( J = j_1 + \cdots + j_I \), \( T(I) = T(i_1) \times \cdots \times T(i_1) \) and \( T(J) = T(j_1) \times \cdots \times T(j_I) \), we get the composition

\[
T(n) \times T(I) \times T(J) \xrightarrow{\gamma \times \text{id}^I} T(I) \times T(J) \xrightarrow{\gamma} T(J).
\]
since $J = 1 + \cdots + n$. The two resulting trees $R$ and $Q$ are the same. An example of this for $T(2) \times (T(2) \times T(1)) \times (T(2) \times T(1) \times T(3))$ is shown in Figure 7 below.

![Figure 6: The corolla with $n$ leaves.](image)

![Figure 7: Associativity of grafting trees.](image)

### 2.2 Unit

There is a unit object $\mathbb{1} \in T(1)$, the tree with no nodes:
The unit satisfies two unit relations, one for each side of a composition or “grafting”. For any tree $T \in T(n)$, we have the equality $\gamma(\mathbb{I}, T) = T$.

On the other side, we have $\gamma(T, \mathbb{I}^n) = T$, where $\mathbb{I}^n$ is the $n$-tuple with $\mathbb{I}$ in each component.

2.3 Equivariance

The structure maps are also equivariant under the action by the symmetric group $\Sigma_n$. These relations require a little more work to set up. Our goal is that if we have something like

$$T(n) \times T(i_1) \times \cdots \times T(i_n),$$

we can either allow a permutation $\sigma \in \Sigma_n$ to act on the first factor $T(n)$, or we could allow it to act on the last $n$ factors by permuting their ordering in some way based on $\sigma$. Recall that the way we allow $\Sigma_n$ to act on a tree of $T(n)$ is by permuting the labelings. If $\sigma \in \Sigma_n$, $T \in T(n)$ and $l$ is a leaf labeled with the number $i$, then in $T\sigma$ the labeling of $l$ becomes $\sigma^{-1}(i)$. In the example before we had a tree $T \in T(7)$ and applied $\sigma = (1 \ 3 \ 4)(2 \ 5 \ 7 \ 6)$. We now present a more simplified example to see what happens when we graft trees and apply actions in different orders.

Suppose $R$ and $S$ are 2-corollas and $T$ is a 3-corolla, with labelings in order from left to right. We have the structure map $\gamma : T(2) \times T(2) \times T(3) \to T(2 + 3) = T(5)$ sending $(R, S, T)$ to the “grafted” tree $R \circ (S, T)$ shown in Figure 8. On the resulting tree we have a $\Sigma_5$-action.
However, if we begin with an element of $\Sigma_2$, we have two different ways to induce an action on the resulting tree. In particular, let $\sigma = (1 2)$ be the generator of $\Sigma_2$. We could first allow $\sigma$ to act on $R$ and then compose, to obtain the grafted tree $R\sigma \circ (S, T)$ as shown in Figure 9. Here we must be careful to remember to graft the first tree to the leaf labeled with 1, and in the resulting tree give the leaves associated with the first tree the first labelings.

Figure 9: On the left is $(R\sigma, S, T) = (R, T, S)$ mapping to $R\sigma \circ (S, T)$ on the right.

We can obtain the same tree in another way, using $\sigma$ to permute the ordering of our 2-tuple of trees. That is, we have an induced action $\overline{\sigma}(S, T) = (T, S)$. In general, given an $n$-tuple $(T_1, \ldots, T_n)$ and an element $\sigma \in \Sigma_n$, we have an induced left-action $\sigma(T_1, \ldots, T_n) = (T_{\sigma^{-1}(1)}, \ldots, T_{\sigma^{-1}(n)})$ which permutes the ordering. We can then graft the trees to obtain $R \circ \overline{\sigma}(S, T)$ as shown in Figure 10. This gives the correct tree, but with incorrect labelings. To solve this problem, we apply another permutation $\sigma_{2,3} \in \Sigma_5$ called a block permutation induced from $\sigma \in \Sigma_2$ on the final tree to get back the correct labeling. The permutation in this example would need to be $(1 \ 4 \ 2 \ 5 \ 3)$. The general pattern is more easily seen using the 2-row representation for a permutation

$$
\sigma_{2,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}.
$$
To determine the correct 2-row representation in general, given \( \sigma \in \Sigma_n, i_1, \ldots, i_n \in \mathbb{N} \) and \( i = i_1 + \cdots + i_n \), divide the set \([i] = \{1, 2, \ldots, i\}\) into \( n \) consecutive blocks, block \( j \) being size \( i_j \). So block \( j \) consists of the integers \( \{i_1 + \cdots + i_{j-1} + 1, \ldots, i_1 + \cdots + i_j\} \). Now divide \([i]\) into \( n \) blocks once again, but this time the size of block \( j \) is \( i_{\sigma^{-1}(j)} \). Permute these blocks by sending the numbers in block \( j \) to the corresponding numbers in block \( \sigma^{-1}(j) \), and line up the resulting ordering of \([n]\) in the second row.

## 3 Operads

The notion of an operad is a generalization of the properties observed for the example above. The most general setting this can take place is in a symmetric monoidal category \( \mathcal{C} \) with product \( \otimes \).

**Definition 3.1.** An operad \( \mathcal{O} \) in \( \mathcal{C} \) is a collection of objects \( \mathcal{O}(n) \) in \( \mathcal{C} \), \( n \geq 1 \), with a right \( \Sigma_n \)-action on \( \mathcal{O}(n) \) and structure morphisms

\[
\gamma : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n)
\]

for every \( n \geq 1, i_1, \ldots, i_n \geq 1 \) satisfying the following axioms:

(a) **Associativity.** Given \( n \in \mathbb{N} \) and \( m_i, l_{i,j} \in \mathbb{N} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m_i \), let

\[
\begin{align*}
l_i & = l_{i,1} + \cdots + l_{i,m_i}, \\
l & = l_1 + \cdots + l_n, \\
m & = m_1 + \cdots + m_n.
\end{align*}
\]

Then the following diagram commutes:
\[ O(n) \otimes \left( \bigotimes_{i=1}^{n} O(m_i) \right) \otimes \left( \bigotimes_{1 \leq i \leq n} O(l_{i,j}) \right) \xrightarrow{\gamma \otimes \text{id}^n} O(m) \otimes \left( \bigotimes_{1 \leq i \leq m_i} O(l_{i,j}) \right) \]

(b) Unit. Let \( I \) denote the unit object in \( \mathcal{C} \). Then there is a morphism \( \eta : I \rightarrow O(1) \) such that the composite morphisms

\[ O(n) \otimes \bigotimes_{i=1}^{n} O(m_i) \xrightarrow{\text{id} \otimes \gamma \otimes \text{id}^n} O(n) \otimes (\bigotimes_{1 \leq i \leq m_i} O(l_{i,j})) \]

and

\[ I \otimes O(n) \xrightarrow{\eta \otimes \text{id}} O(1) \otimes O(n) \xrightarrow{\gamma} O(n) \]

are the iterated right unit morphism (the right unit morphism applied \( n \) times) and the left unit morphism in \( \mathcal{C} \).

(c) Equivariance. Given \( n \in \mathbb{N}, \, m_i \in \mathbb{N} \) for \( 1 \leq i \leq n \), and a permutation \( \sigma \in \Sigma_n \), the following equivariance diagram commutes:

\[ O(n) \otimes \bigotimes_{i=1}^{n} O(m_i) \xrightarrow{\text{id} \otimes \sigma} O(n) \otimes \bigotimes_{i=1}^{n} O(m_{\sigma^{-1}(i)}) \]

where again \( m = m_1 + \ldots + m_n \). Recall that the block permutation \( \sigma_{m_1, \ldots, m_n} \) is required at the end of the top path as we discussed in Section 2.3.

Remark 3.2. If we drop everything in the definition related to the \( \Sigma_n \)-actions, we get a non-\( \Sigma \) operad (read “non-symmetric operad”).

Remark 3.3. In the definition we gave above, we have secretly been lazy and ignored some subtleties with units. We should allow \( O(0) \) to exist, but we will generally want it to be the unit under \( \otimes \). For example, in \( \mathcal{T} \text{op} \), the category of topological spaces, we will require that \( O(0) = * \), a one point set throughout this paper.
Example 3.4. The collection \( \{T(n)\} \) of (isomorphism classes of) \( n \)-labeled trees, \( n \geq 1 \), with structure maps given in the previous section, is an operad in \( \mathcal{S}et \), the category of sets. We obtain a right \( \Sigma_n \)-action on \( T(n) \) by defining \( [T, l] \sigma = [T, \sigma^{-1}l] \) for \( \sigma \in \Sigma_n \) acting on the tree \( T \) with labeling \( l \).

Example 3.5. The most important example of an operad in a symmetric monoidal category \( (\mathcal{C}, \otimes) \) is the endomorphism operad. Assume \( \mathcal{C} \) has an internal hom (in practice, our categories are typically topological categories where the hom sets are topological spaces). Given an object \( X \) in \( \mathcal{C} \), the endomorphism operad \( \mathcal{E}nd_X \) is given by

\[
\mathcal{E}nd_X(n) = \text{Hom}(X^{\otimes n}, X),
\]

the collection of all morphisms \( X^{\otimes n} \to X \). Given morphisms \( f : X^{\otimes n} \to X \in \mathcal{E}nd_X(n) \) and \( g_i : X^{\otimes m_i} \to X \in \mathcal{E}nd_X(m_i) \), the structure map is defined by

\[
\gamma(f, g_1, \ldots, g_n) = f \circ (g_1 \otimes \cdots \otimes g_n).
\]

There is a left action of \( \Sigma_n \) on \( X^{\otimes n} \) by permuting the coordinates:

\[
\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).
\]

This induces a right \( \Sigma_n \)-action on \( \mathcal{E}nd_X(n) \) by defining \( (f \sigma)(x_1, \ldots, x_n) = f \circ \sigma(x_1, \ldots, x_n) \). The proof that this satisfies the three operad axioms is straightforward. If \(*\) is the unit object in \( \mathcal{C} \), then the unit axiom is satisfied by the unique map \( \eta : * \to \text{id}_X \).

For example,

- In the category \( \mathcal{S}et \) of sets we can define the product to be the usual cartesian product. The unit object in this category is the one point set \( \{*\} \). Then \( \mathcal{E}nd_X(n) \) on a set \( X \) is just the set of all functions \( X^n \to X \).

- In the category \( \mathcal{T}op \) of (compactly generated weak Hausdorff) topological spaces we can define the product to be the normal cartesian product with the product topology. The unit object in this category is the one point space \(*\). In this case \( \mathcal{E}nd_X(n) \) is just the set of all continuous maps \( X^n \to X \).

- If \( \mathcal{C} = \mathcal{T}op_* \), the category of based (compactly generated weak Hausdorff) topological spaces, with product the smash product, the endomorphism operad is the set of all based continuous maps \( X^{\wedge n} \to X \).

- On the algebraic side, if \( R \) is a commutative ring and \( \mathcal{C} = \text{Mod}_R \), the category of two sided \( R \)-modules, the product is the usual tensor product \( \otimes \) over \( R \). The unit object in this category is \( R \). The endomorphism operad in this case is the set of all module homomorphisms \( X^{\otimes n} \to X \).

- Finally, if \( \mathcal{C} = \mathcal{C}h(R) \) is the category of differential graded \( R \)-modules (which we will also refer to as chain complexes over \( R \)) with unit the chain complex with \( R \) concentrated in degree zero and the usual tensor product

\[
(C \otimes D)_n = \sum_{p+q=n} C_p \otimes D_q
\]
and differential $\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|} c \otimes \partial d$, then the endomorphism operad on a chain module $C_*$ is the set of chain maps $C^\otimes n \to C$.

In the remainder of this paper, we will restrict our attention to the symmetric monoidal categories $\text{Top}$, $\text{Top}_*$ and $\text{Ch}(R)$. Much of the theory that has been developed can be generalized to sufficiently nice symmetric monoidal categories, but the technical definitions required would be too distracting for our purposes. In particular, the following examples of operads that will be useful in this paper have clear expressions in categories $\text{Top}$, $\text{Top}_*$ and $\text{Ch}(R)$. These definitions can be generalized, but require some work before we could do so.

**Example 3.6.** The operad $\Sigma$ is given by $\Sigma(n) = \Sigma_n$ with the obvious right-action of $\Sigma_n$ on itself. The structure maps are given via the induced block permutation described earlier. That is, if $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{m_i}$, then

$$\gamma(\sigma, \tau_1, \ldots, \tau_n) = \sigma_{m_1, \ldots, m_n} \circ (\tau_1 \otimes \cdots \otimes \tau_n),$$

where $\sigma_{m_1, \ldots, m_n} \in \Sigma_{m_1 + \cdots + m_n}$ is the block permutation induced by $\sigma$, defined in Section 2.3. Note that this operad makes sense as an operad in any of the categories mentioned above. We can think of $\Sigma_n$ as a set or as a discrete topological space. In each of these cases, we will call $\Sigma$ the **associative operad**, and denote it $\text{Ass}$.

For a space $X$, let $X^+$ denote the space obtained from $X$ by adding a disjoint basepoint. Then in $\text{Top}_*$, the associativity operad is given in degree $n$ by $\Sigma_n^+$, where the basepoint is fixed by the $\Sigma_n$-action. We must add the disjoint basepoint because the identity permutation is not fixed by the $\Sigma_n$-action.

We will soon see that the operad $\text{Ass}$ characterizes associative multiplication on a topological space. There is a corresponding notion of associativity in the category of differential graded $R$-modules.

**Example 3.7.** In $\text{Mod}_R$, the associative operad $\text{Ass}$ is defined by letting its degree $n$ component $\text{Ass}(n)$ be the free $R$-module $R[\Sigma_n]$ on the basis $\Sigma_n$. In $\text{Ch}(R)$, the associative operad is defined by letting $\text{Ass}(n)$ be the chain complex which consists of $R[\Sigma_n]$ concentrated in degree zero.

In general, if $O$ is any operad in the category $\text{Set}$ we can associate to it an operad $R[O]$ in the category $\text{Mod}_R$ by defining the degree $n$ component to be the free $R$-module over $O(n)$. That is,

$$R[O](n) = R[O(n)].$$

We also get an induced operad in the category $\text{Ch}(R)$ by defining the degree $n$ component to be chain complex consisting of $O(n)$ concentrated in degree zero. With this perspective, and by abusing notation, the associative operad in $\text{Mod}_R$ or $\text{Ch}(R)$ is just $\text{Ass} = R[\Sigma] = R[\text{Ass}]$ for the operad $\Sigma = \text{Ass}$ in $\text{Set}$.

**Example 3.8.** There is a corresponding operad which characterizes commutative multiplication. In the categories $\text{Set}$, $\text{Top}$, or $\text{Top}_*$, define the **commutative operad** $\text{Com}$ to be the one point set or space in every degree. That is,

$$\text{Com}(n) = \ast$$

12
for every \( n \geq 1 \). In \( \text{Mod}_R \) or \( \text{Ch}(R) \), using the same abuse of notation as in the previous example, define the commutative operad to be \( \text{Com} = R[\text{Com}] \). Note that as a differential graded \( R \)-module, \( \text{Com}(n) = R[\text{Com}(n)] = R[\ast] \) is isomorphic to the chain complex \( R \) with \( R \) concentrated in degree zero.

4 Algebras over operads

Now we make the connection between the abstract definition of an operad and the intuition that an operad parametrizes \( n \)-ary operations in a category. We begin by describing various categories of operads.

Definition 4.1. A morphism \( f : \mathcal{O} \to \mathcal{O}' \) of operads is a collection of \( \Sigma_n \)-equivariant maps \( f(n) : \mathcal{O}(n) \to \mathcal{O}'(n) \) which commute with the structure maps of the operads so that the diagram

\[
\begin{array}{ccc}
\mathcal{O}(n) \otimes \bigotimes_{i=1}^{n} \mathcal{O}(m_i) & \xrightarrow{\gamma} & \mathcal{O}(m) \\
\downarrow f(n) \otimes \bigotimes_{i=1}^{n} f(m_i) & & \downarrow f(m) \\
\mathcal{O}'(n) \otimes \bigotimes_{i=1}^{n} \mathcal{O}'(m_i) & \xrightarrow{\gamma'} & \mathcal{O}'(m)
\end{array}
\]

commutes for \( n \geq 1 \) and \( m_i \geq 0 \) with \( m = \sum m_i \).

This gives a category \( \text{Oper}_C \) of operads in \( C \). The equivariance condition is encoded in the statement \( f(n)(X \cdot \sigma) = f(n)(X) \cdot \sigma \), where \( X \in \mathcal{O}(n) \), \( \sigma \in \Sigma_n \), and \( X \cdot \sigma \) represents the action of \( \Sigma_n \) on \( \mathcal{O}(n) \).

Definition 4.2. Given an operad \( \mathcal{O} \) in a category \( C \), an \( \mathcal{O} \)-algebra is a morphism of operads \( \theta : \mathcal{O} \to \mathcal{E}\text{nd}_X \) for some object \( X \in C \).

Remark 4.3. If the \( C \) has an internal hom which is right adjoint to the product \( \otimes \), it is usually more practical to understand this definition by using the adjoint relation

\[
\text{Hom}_C(\mathcal{O}(n), \text{Hom}_C(X^{\otimes n}, X)) \cong \text{Hom}_C(\mathcal{O}(n) \otimes X^{\otimes n}, X).
\]

Then \( \theta \) can be associated to a collection of maps

\[
\theta(j) : \mathcal{O}(j) \otimes X^{\otimes j} \to X
\]

which satisfy the necessary associativity, unit, and equivariance conditions as follows.

(a) For \( n \geq 1 \), \( m_i \geq 0 \) and \( m = \sum m_i \), the following associativity diagram commutes:
The shuffle map decomposes $X^m$ as $X^m_i \otimes \cdots \otimes X^m_n$ and then groups each term $X^m_i$ with $O(m_i)$.

(b) If $I$ is the unit object of $C$, the following unit diagram commutes:

(c) $\theta$ is equivariant under the $\Sigma_n$ action. If $\sigma \in \Sigma_n$, the following diagram commutes:

Remark 4.4. If $C$ has finite colimits, we can combine the equivariance with the definition of the maps $\theta(n)$ by requiring the maps to be

$$\theta(n) : O(n) \otimes_{\Sigma_n} X^{\otimes n} \to X,$$

where $O(n) \otimes_{\Sigma_n} X^{\otimes n}$ is the coequalizer of the maps

$$\sigma^{-1} \otimes \sigma : O(n) \otimes X^{\otimes n} \to O(n) \otimes X^{\otimes n}$$

for $\sigma \in \Sigma_n$.

The definition of a morphism of $O$-algebras $f : X \to Y$ is a morphism in $C$ such that $f$ commutes with the algebra maps $\theta_X$ and $\theta_Y$ in the obvious way. This gives a subcategory $O-Alg$ of $C$ for every operad $O$ in $C$.

**Proposition 4.5.** In the category $Set$, Ass-algebras are monoids and Com-algebras are commutative monoids.

**Proof.** Suppose $X$ is an Ass-algebra. Since $Ass(n) = \Sigma_n$ there is a map $\Sigma_2 \to End_X(2) = \{X \times X \to X\}$. This gives two different multiplication maps $m$ and $m'$ for the identity permutation and $\tau = (1 \ 2)$, respectively. Given $x, y \in X$ the two maps correspond to the
two different orders in which we can multiply $x$ and $y$. That is, we get two different elements $m(x, y) = x \cdot y$ and $m'(x, y) = y \cdot x$. The equivariance condition makes the phrase “switching the order of $x$ and $y$” precise. Therefore $\text{Ass}(2)$ gives a (not necessarily commutative) multiplication. The map $\text{Ass}(0) \times X^0 = * \rightarrow X$ provides a 2-sided unit of the multiplication. The two compositions $m \circ (m \times \text{id})$ and $m \circ (\text{id} \times m)$ are both elements of $\text{End}_X(3)$. Since $m$ is the image of the identity permutation of $\Sigma_2$, both of these maps are the image of the identity permutation of $\Sigma_3$ and are therefore equal. Therefore the multiplication $m$ is associative.

If $X$ is a $\text{Com}$-algebra, then the two maps $m$ and $m'$ in the previous paragraph are identical since $\text{Com}(2) = \{\ast\}$, and the multiplication $m$ is commutative.

This result has analogs in more general symmetric monoidal categories. The following results in the categories we care about in this paper are straightforward generalizations.

Example 4.6. In the category $\text{Mod}_R$, $\text{Ass}$-algebras are $R$-modules with an associative multiplication, i.e. an associative algebra. $\text{Com}$-algebras are associative commutative algebras.

Example 4.7. In $\text{Top}$, $\text{Ass}$-algebras are topological monoids and $\text{Com}$-algebras are commutative topological monoids. In $\text{Top}_*$ we have the same the same result, but with the added structure that the basepoint serves as a zero for the multiplication. Suppose that $X$ is an $\text{Ass}$-algebra. Recall that the associative operad in $\text{Top}_*$ is $\Sigma_n^+$, where we must add a disjoint basepoint so the basepoint is fixed by the action of $\Sigma_n$ on $\text{Ass}(n)$. Then $\text{Ass}(1)$ is a two point set with the identity permutation being sent to the unit of $X$ and the disjoint basepoint being sent to the basepoint of $X$.

Example 4.8. In the category $\text{Ch}(R)$ of differential graded $R$-modules, $\text{Ass}$-algebras are associative differential graded $R$-algebras. $\text{Com}$-algebras are graded commutative $R$-modules, that is $R$-modules with a multiplication which is commutative up to sign:

$$xy = (-1)^{|x||y|}yx.$$

5 Stasheff associahedra and $A_n$-operads

In this section, we describe the standard operad which characterizes associative multiplication. We begin by describing the Stasheff associahedra pioneered in [14] which are used to describe appropriate levels of associativity, and associate to this the notion of $A_n$-operads and $A_n$-structures. Another development of $A_n$-structure from the tree perspective we discussed in Section 2 can be found in [15].

The Stasheff associahedra are cell complexes $K_n$ which fit together to give a non-$\Sigma$ operad $\mathcal{K}$. The original description of the construction can be found in [13]. The intuition behind the technical definition is that the cell complex $K_n$ represents all the possible meaningful ways of inserting pairs of parantheses into a word of $n$ letters. For example, the word $abc$ has two meaningful ways of inserting pairs of parantheses, $(ab)c$ and $a(bc)$, representing the usual two quantities which are required to be equal in an associativity axiom. The insertions $a(b)c$ and $((ab))c$ are considered equivalent to $abc$ and $(ab)c$, respectively.
The vertices of $K_n$ consist of all the ways to insert $n - 2$ meaningful pairs of parentheses. Two vertices are connected by a 1-cell if you can obtain one word from the other by dropping one pair of parentheses and inserting another. The 1-cell connecting these two vertices is labeled by the word in between these two words which has only $n - 3$ meaningful pairs of parentheses. For example, $(ab)c$ can be obtained from $a(bc)$ by first dropping the pairs of parentheses around $b$ and $c$, giving the 1-cell $abc$, and then adding pairs of parentheses around $a$ and $b$. This gives the structure of $K_3$ as

$$(ab)c \xrightarrow{abc} a(bc)$$

Next, a 2-cell connects all those 1-cells which can be obtained from another by dropping one pair of parentheses and adding another. Equivalently, a 2-cell spans vertices which can be obtained from another by dropping two pairs of parentheses and adding two pairs. And in general, a $k$-cell of $K_n$ spans vertices which can be obtained from another by dropping $k$ pairs of parentheses and adding $k$ pairs. The complex $K_4$ is shown in Figure 11.

![Figure 11: The cell complex $K_4$.](image)

**Definition 5.1** ([13], p. 278). Let $K_2 = \ast$, a point. Define $K_i$ inductively as follows. Assume $K_2, \ldots, K_{i-1}$ are defined. For $r + s = i + 1$, associate via the map $\partial_k(r, s)$ the cell complex $K_r \times K_s$ to the word $x_1 \cdots x_{k_1}(x_k \cdots x_{k+s-1})x_{k+s} \cdots x_i$ with one pair of parentheses around $s$ letters beginning at $x_k$. Let $L_i$ consist of all possible such cells $K_r \times K_s$ glued together subject to the attaching conditions

1. $\partial_j(r, s + t - 1)(1 \times \partial_k(s, t)) = \partial_{j+k-1}(r + s - 1, t)(\partial_j(r, s) \times 1)$, and
2. $\partial_{j+k-1}(r + s - 1, t)(\partial_k(r, s) \times 1) = \partial_k(r + t - 1, s)(\partial_j(r, t) \times 1)(1 \times \tau)$, where $\tau : K_s \times K_t \to K_i \times K_s$ is a twisting permutation.

Let $K_i = CL_i$ be the cone over $L_i$. That is, $L_i$ is the topological boundary of the cell complex $K_i$.  


The conditions above are quite opaque. The first condition is equivalent to the statement that
\[ x_1 \cdots x_j \cdot x_{j+k-2} (x_{j+k-1} \cdots x_{j+k+t-2}) x_{j+k+t-1} \cdots x_i \]
can be obtained by either first inserting the inner parentheses then the outer, or vice versa. That is, when one pair of parentheses contains another, the order of inserting parentheses does not matter. The second condition has a similar interpretation for two non-containing pairs of parentheses.

In other words, \( K_n \) is homeomorphic to the disk \( I^{n-2} \) or the standard simplex \( \Delta^{n-2} \) but with different boundary conditions. The boundary of an \( i \)-dimensional cell consists of all those words which can be obtained by inserting exactly one pair of meaningful parantheses.

We can now define the non-\( \Sigma \) operad \( K \) in \( Top \) by letting \( K(n) = K_n \) and defining the structure maps
\[ \gamma : K(n) \times \prod_{i=1}^{n} K(m_i) \to K(m), \]
for \( m = \sum m_i \), to be associated to the process of taking a word \( y_1 \cdots y_n \) of \( n \) letters, and inserting in words of length \( m_i \) in spot \( y_i \) with parantheses around the inserted word. This gives a word with \( m \) letters and \( n \) parantheses, i.e. a \( m-n-2 \) dimensional cell of \( K_m \).

We now set about giving a definition of an \( A_n \)-operad. We will not go into all the technicalities, because the exact definition is not all that important. What will be important for us is that if \( O \) is an \( A_n \) operad, then the structure of an \( O \)-algebra has a simple expression in terms of generators and relations. This alternate formulation is especially practical in the world of chain complexes, where computation is desirable.

Given an operad \( O \) there is a notion of its homology \( H(O) \), and if \( \theta : O \to P \) is a morphism of operads there is an induced morphism \( H(\theta) : H(O) \to H(P) \) on homology.

**Definition 5.2** ([7], p. 187). A quasi-isomorphism is a morphism \( \theta : O \to P \) of operads which is an isomorphism on homology. Two operads \( O \) and \( Q \) are weakly equivalent if they are connected by a zigzag of quasi-isomorphisms

\[ O \to P_1 \to P_2 \to \cdots \to P_{s-1} \to Q \]

We will loosely define \( K_n \) to be the suboperad of \( K \) generated by \( K_i \) for \( i \leq n \). An \( A_n \)-operad in \( Top \) is any operad weakly equivalent to \( K_n \).

**Definition 5.3.** We say a based space \( X \) with basepoint \( e \) admits an \( A_n \)-structure if there are multiplication maps \( m_i : K_i \times X^i \to X \) for \( i \leq n \), such that

1. \( m_2 \) with \( e \) in either spot is the identity. That is, \( m_2(\ast, e, x) = m_2(\ast, x, e) = x \) for all \( x \in X \), and

17
2. If \( r + s = i + 1 \), \( \rho \in K_r \), \( \sigma \in K_s \), and \( (\rho, \sigma)_k \in K_i \) is the cell associated to inserting a pair of parentheses around \( s \) letters starting at the \( k^{th} \) spot, then

\[
m_i((\rho, \sigma)_k, x_1, \ldots, x_i) = m_r(\rho, x_1, \ldots, x_{k-1}, m_s(\sigma, x_k, \ldots, x_{k+s-1}), x_{k+s}, \ldots, x_i).
\]

This is equivalent to a collection of \( n \)-ary multiplication maps \( X^n \to X \) for every \( \sigma \in K_n \). Since \( K_2 = * \) the first condition implies the existence of a 2-ary multiplication map \( X^2 \to X \) with the basepoint \( e \) as its identity. The second condition says that the multiplication maps behave well with inserting parentheses. For example, if we rewrite \( m_2(\ast, x, y) \) as \( x \cdot y \), then this condition says that for \( \rho \in K_{i-1} \) we have

\[
m_i((\rho, \ast)_k, x_1, \ldots, x_i) = m_{i-1}(\rho, x_1, \ldots, x_{k-1}, x_k \cdot x_{k+1}, \ldots, x_i).
\]

That is, we can obtain a multiplication on \( i \) elements with respect to \( \rho \) by taking two successive elements and multiplying them together, and then using \( \rho \) to multiply the remaining \( i - 1 \) elements.

Notice that \( X \) admits an \( A_2 \)-structure if it is an \( H \)-space with multiplication \( m_2 \). \( X \) admits an \( A_3 \)-structure if there is a homotopy \( m_3 \) between the two possible ways \( m_2(m_2 \times 1) \) and \( m_2(1 \times m_2) \) of multiplying three elements.

Recasting the language of [13, pp. 276-278] we can state the following theorem.

**Theorem 5.4.** A based space \( X \) admits an \( A_n \)-structure if and only if it is an \( O \)-algebra for some \( A_n \)-operad \( O \).

We are particularly interested in a nice expression of the above structure in the category \( Ch(R) \). In fact, the original purpose of the Stasheff associahedra was to encode the structure of an \( A_\infty \)-algebra in a more concise way.

Roughly speaking, an \( A_\infty \)-algebra \( A \) is an algebra over \( R \) which has a chain homotopy associative multiplication \( m : A \otimes_R A \to A \). An \( A_n \)-algebra is an algebra over \( R \) which a multiplication which is chain homotopy associative only up to the \( n^{th} \) level.

The definitions in this section are all consistent: in \( Ch(R) \), \( A_n \)-algebras are exactly equivalent to \( R \)-algebras which admit \( A_n \)-structures, and so are \( O \)-algebras for some \( A_n \)-operad \( O \). We will use these various terminologies interchangeable throughout the rest of the paper.

The classical representation of an \( A_n \)-algebra is typically preferred for its simplicity and computational usefulness.

**Definition 5.5.** An \( A_n \)-algebra \( A \) in \( Ch(R) \) is a differential graded \( R \)-algebra \( A \) with multiplication maps

\[
m_k : A^\otimes k \to A
\]

for \( k \leq n \), of degree \( k - 2 \), which satisfy the following relation whenever \( k = p + 1 + r \) and \( n \geq 1 \):

\[
\sum_{p+q+r=n} (-1)^{p+qr} m_k \circ (\text{id}^\otimes p \otimes m_q \otimes \text{id}^\otimes r) = 0.
\]
This definition encodes the differential $d$ of $A$ as $m_1$. Some small examples of the relations, for $n = 1, 2, 3$ are shown explicitly below. To better exhibit the homotopy relations we write $m_1$ as $d$ and rewrite the equation to have $d \circ m_n$ on one side:

\[
\begin{align*}
    d \circ d &= 0 \\
    d \circ m_2 &= m_2(d \otimes \text{id}) + m_2(\text{id} \otimes d) \\
    d \circ m_3 &= -m_2(m_2 \otimes \text{id}) + m_2(\text{id} \otimes m_2) \\
    &\quad -m_3(d \otimes \text{id}^2) - m_3(\text{id} \otimes d \otimes \text{id}) - m_3(\text{id}^2 \otimes d).
\end{align*}
\]

6 Little $n$-cubes operad and $E_n$-operads

In this section we discuss the visually intuitive and classically important family of operads, the little $n$-cubes operads. Analogous to the operad $K$ from the previous section which in some sense generate universal examples of $A_n$-operads and yield algebras with associative multiplications, the little $n$-cubes operads are universal examples of $E_n$-operads which yield algebras with associative and commutative multiplications.

The notational conventions in the definitions $A_n$ and $E_n$ have a mnemonic appeal. The $A$ stands for “associativity” and the $E$ stands for “everything”, or associativity plus commutativity. Both $A_n$-spaces and $E_n$-spaces were originally studied in relation to loop spaces. The ultimate connection between operads and loop spaces was made in May’s work [8].

Definition 6.1. We follow the standard constructions which can be found in [7] or [8]. Let $I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \text{ for every } 1 \leq i \leq n\}$ denote the $n$-dimensional unit cube in $\mathbb{R}^n$, or $n$-cube for short. A little $n$-cube is a linear embedding $c : I^n \to I^n$ which preserves the axes. This is equivalent to requiring the existence of real numbers $0 < a_i \leq 1$ and $0 \leq b_i \leq 1 - a_i$ such that $c$ sends the $i^{th}$ coordinate $x_i$ of $x \in I^n$ to $a_i x_i + b_i$. A graphical representation of a little 2-cube is given in Figure 12.

Definition 6.2. The little $n$-cubes operad $\mathcal{E}_n$ in the category $\mathcal{T}op$ is defined by letting $\mathcal{E}_n(j)$ be the collection of all $j$-tuples $(c_1, \ldots, c_j)$ of little $n$-cubes with disjoint interiors of the images. Note that $n$ represents the dimension of the cubes, while $j$ represents the number of little cubes. Elements of the various spaces $\mathcal{E}_n(j)$ are usually represented geometrically by graphing the images $c_i(I^n)$ all at once inside $I^n$. An example of an element of $\mathcal{E}_2(3)$ can be found in Figure 13.

$\Sigma_j$ acts on $\mathcal{E}_n(j)$ by permuting the labelings appropriately. The structure maps

\[ \gamma : \mathcal{E}_n(k) \times \mathcal{E}_n(l_1) \times \cdots \mathcal{E}_n(l_i) \to \mathcal{E}_n(l_1 + \cdots + l_i) \]

are given by shrinking down the cube $I^n$ representing $\mathcal{E}_n(l_i)$ and inserting it in the $i^{th}$ slot of the cube representing $\mathcal{E}_n(k)$, relabeling the very little $n$-cubes as appropriate. An example of how this composition works can be found in Figure 14.
Figure 12: An example of a little $n$-cube $c : I^n \to I^n$. The shaded region is the image $c(I^n)$ inside $I^n$.

Figure 13: An example of an element of $\mathcal{E}_2(3)$.

An iterated loop space is a space $X$ which is homotopy equivalent to $\Omega^k Y$ for some space $Y$, where $\Omega^k Y \cong \pi_k(Y) \cong \{ f : (I^k, \partial I^k) \to (Y, *) \}$. If $X \cong \Omega^k Y$, there are obvious maps $\mathcal{E}_k(n) \times X^n \to X$ defined by letting $(c_1, \ldots, c_n) \times f_1 \times \cdots \times f_n$ be the map $(I^k, \partial I^k) \to (Y, *)$ which is $f_i \circ c_i^{-1}$ on the image of each $c_i$, and $*$ everywhere else. This turns every iterated loop space into an $\mathcal{E}_k$-algebra for some $k$. The impressive result that was proven by May, called a recognition principle, is that the converse holds as well.

**Theorem 6.3** ([8]). If $X$ is a connected $\mathcal{E}_k$-algebra there is a space $Y$ such that $X \simeq \Omega^k Y$.

**Remark 6.4.** This theorem also holds for $k = \infty$ by letting $\mathcal{E}_\infty(n)$ be the colimit

$$\lim_{\leftarrow} \mathcal{E}_k(n)$$

under the sequence of maps induced by the inclusions $I^k \hookrightarrow I^{k+1}$ given by

$$(t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_k, 0).$$
Definition 6.5. An $E_n$-operad in $Top_*$ is any operad which is weakly equivalent to the little $n$-cubes operad $E_n$. This definition holds for any extended positive integer $1 \leq n \leq \infty$. An $E_n$-space is a topological space which is an algebra over an $E_n$-operad.

Remark 6.6. In the category $Ch(R)$ we define the analog of the little $n$-cubes operad to be the chain complex $C_*(E_n(i); R)$ in degree $i$. An $E_n$-operad is any operad which is weakly equivalent to this operad, and an $E_n$-algebra is any chain complex which is an algebra over an $E_n$-operad.

Recall from algebraic topology the definition of the product $F \ast G : (I^n, \partial I^n) \to (X, *)$ of two maps $F, G : (I^n, \partial I^n) \to (X, *)$, given by running the first time coordinate of each map twice as fast, as exhibited in Figure 15.

![Figure 15: The product $F \ast G$ of $F$ and $G$.](image)

Using the homeomorphism $I^n \cong S^n$, this allows us to define a product in $\pi_n(X)$. This
product is non-commutative in dimension 1, but is commutative in all dimensions \( n \geq 2 \). The proof of this commutativity is given by a homotopy visualized for dimension 2 in Figure 16.

![Figure 16: A visualization of the homotopy which shows \( F \ast G \cong G \ast F \) in dimensions \( n \geq 2 \). The shaded parts inside \( I^n \) represent the constant map to the basepoint.](image)

Notice the similarity of this visualization in dimension two with the earlier example of an element of \( \mathcal{E}_2(k) \). In particular, all of the pictures represent elements of \( \mathcal{E}_2(2) \). For the exact same reasons that this homotopy leads to an associative and commutative product on homotopy classes of maps, an \( E_2 \)-space (and in general an \( E_n \)-space, for \( n \geq 2 \)) \( X \) is a topological space \( X \) with a product which is homotopy associative and commutative. In particular, \( \mathcal{E}_2(2) \) provides a homotopy between the products \( m(x, y) = xy \) and \( m'(x, y) = yx \) of elements of \( X \).

Recall that an \( A_3 \)-space has an associative product, but higher associativity diagrams do not necessarily commute in a coherent way. Similarly, an \( E_2 \)-space has a homotopy commutative product, but the homotopy connecting the two products \( m \) and \( m' \) may not be unique. An \( E_3 \)-space provides the additional structure that any two homotopies \( h \) and \( h' \) from \( m \) to \( m' \) are homotopic by a 2-cell filling in the following diagram:

\[
\begin{array}{c}
m \downarrow \\
\quad \downarrow h = h' \\
m' \end{array}
\]

In general, an \( E_n \)-space provides the structure of a commutative product with higher diagrams filled in up to \( n - 1 \) cells. In particular, an \( E_\infty \)-space has a multiplication which is associative and commutative, with all possible diagrams commuting up to homotopy in a coherent way.

### 7 \( H_n \)-algebras

In this section, we describe the structure we obtain by taking conditions for an \( E_n \)-structure in some model category (such as the categories \( \text{Set} \), \( \text{Top}_* \), \( \text{Mod}_R \), or \( \text{Ch}(R) \)) and passing to the homotopy category. This is called an \( H_n \)-structure and is a loosening of the conditions of an \( E_n \)-structure. Noel in [11] gives a proof that this is actually a strict loosening. Not every \( H_n \)-structure in the homotopy category is obtained by starting with an \( E_n \)-structure.
in the base category and passing to the homotopy category. In order to understand Noel’s result, we will need to give a brief overview of the category $\mathcal{S}pec$ of spectra. While Noel proves his result in the category $\mathcal{S}pec$ requiring a very technical result from [5], we believe there should be examples in other categories such as $Ch(R)$.

As we mentioned in the introduction, the point of studying $H_n$-structures is that they combine the organizational efficiency of operads with homotopy theoretical properties. There are homotopy operations on $H_\infty$ ring spectra which have strong ties to the classical Steenrod operations and the Adams spectral sequence [3, §IV,VI]. May proves in [9] and [3] the result of Nishida in [10] that the ring $\pi_*^S$ of stable homotopy groups of spheres is nilpotent. May’s proof is only several pages long and follows after proving the sphere spectrum has an $H_\infty$-structure. In general our motivation is that if we can show that an object in some category (a space in $Top_*$ or a chain complex in $Ch(R)$, for example) is an $H_n$-algebra, we have a powerful tool at our disposal.

7.1 Spectra

If we begin with a based topological space $X$, we can generate spaces $\Sigma^n X$ by taking iterated suspensions of $X$. Setting $X_0 = X$ and $X_i = \Sigma X_{i-1} = \Sigma^i X$, we obtain a sequence of based topological spaces $\{X_i\}_{i \geq 0}$. By the definition we have (equality) maps $\Sigma X_i \to X_{i+1}$ from the suspension of one space in the sequence to the next space. The reader who is not familiar with suspension should think of it in the case of CW-complexes where the topological space is built by disks $D^n$ of various dimensions being glued inductively starting with points $D^0$, gluing on intervals $D^1$, then disks $D^2$, etc. The suspension takes all the disks (except for one vertex considered to be the basepoint) and increases their dimension by one. A spectrum is a generalization of the structure we observed above. A full treatment of spectra can be found in [4], [12], or [1, §III.2].

Definition 7.1. A spectrum $X$ is a collection of topological spaces $X_0, X_1, X_2, \ldots$ together with structure maps $\Sigma X_i \to X_{i+1}$. A morphism $f : X \to Y$ of spectra is a collection of maps $f_n : X_n \to Y_n$ such that the following diagrams commute for $n \geq 0$:

$$
\begin{array}{ccc}
\Sigma X_n & \longrightarrow & X_{n+1} \\
\Sigma f_n & \downarrow & \downarrow f_{n+1} \\
\Sigma Y_n & \longrightarrow & Y_{n+1}
\end{array}
$$

This definition gives a category $\mathcal{S}pec$ of spectra. A homotopy theory on the category is built in the following way. First, given spaces $X$ and $Y$ and a map $f : X \to Y$, there is an induced map $\Sigma f : \Sigma X \to \Sigma Y$. By passing to homotopy classes, we obtain a map $[X,Y] \to [\Sigma X, \Sigma Y]$. In particular, if we apply this to the homotopy group $\pi_k(X_i) = [S^k, X_i]$, we obtain a map $\pi_k(X_i) \to \pi_{k+1}(X_{i+1})$ for any $i, k$. Applying $\pi_k$ to the structure maps $\Sigma X_i \to X_{i+1}$ contained in the data of a spectrum we get maps $\pi_k(\Sigma X_i) \to \pi_k(X_{i+1})$. Composing these maps in a chain gives a sequence of maps
The Freudenthal Suspension Theorem implies that the maps along the top row eventually become isomorphisms. In particular, the colimit of the sequence of maps in the top row exists and we define

\[ \pi^S_k X = \text{colim}_i \pi_k(X_i). \]

A map \( f : X \to Y \) of spectra is defined to be a weak equivalence if the induced map \( \pi^S_0 X \to \pi^S_0 Y \) is an isomorphism.

**Example 7.2.** The sphere spectrum \( S \) consists of the collection of \( n \)-spheres \( X_n = S^n \). Since the reduced suspension of \( S^n \) is homeomorphic to \( S^{n+1} \) for any \( n \), we have a collection of maps \( \Sigma S^n \to S^{n+1} \).

**Example 7.3.** For any based space \( X \), the collection \( X_i = \Sigma^i X \) defined at the beginning of this section is a spectrum \( \Sigma^\infty X \), called the suspension spectrum of \( X \).

Spectra were originally studied because of their relation to homology and cohomology theories. If \([X, Y]\) denotes the set of homotopy classes of maps \( f : X \to Y \), then the normal homotopy groups of a based space \( X \) are \( \pi_n(X) = [S^n, X] = [\Sigma^n S^0, X] \), where \( S^0 \) is the unit in the symmetric monoidal category of based topological spaces with the smash product as its multiplication.

There is a way to define the smash product \( E \land F \) of two spectra \( E \) and \( F \), with the sphere spectrum \( S \) as the unit, giving a symmetric monoidal structure on \( \text{Spec} \). We can also define a homology theory on \( \text{Spec} \) by letting \( \pi_n(E) = [\Sigma^n S, E] \). A version of the Brown representability theorem says that for every reduced cohomology theory \( \tilde{h} \), there is a spectrum \( E \) such that \( \tilde{h}^n(X) = [\Sigma^n X, E_n] \) for every based space \( X \) and every \( n \geq 0 \). For example, the spectrum associated to usual cohomology with coefficients in \( G \) is the Eilenberg-MacLane spectrum \( HG \).

**Definition 7.4.** A ring spectrum is a spectrum \( E \) with a ring structure with respect to the smash product. There is a unit map \( \eta : S \to E \), a multiplication \( m : E \land E \to E \), and a transposition \( \tau : E \land E \to E \land E \) so that the following diagrams commute:

1. Unit

\[ E \land S \xrightarrow{id \land \eta} E \land E \xrightarrow{\eta \land id} S \land E \xrightarrow{m} E \]

2. Associativity
3. Commutativity

For example, the sphere spectrum is a ring spectrum. The suspension spectrum $\Sigma^\infty X$ of any topological monoid $X$ is also a ring spectrum. We could define an $E_\infty$-operad in the category of spectra which carries with it the same associativity and commutativity conditions that it did in the categories considered in previous sections. A spectrum $E$ with an $E_\infty$-structure produces, as a part of its structure, a ring structure.

7.2 $H_\infty$ ring spectra

In the classical references [9] and [3, pp. 6, 11-12], May defines extended powers of spectra

$$D_j F = E\Sigma_n \wedge_{\Sigma_n} F^{\wedge n} = \frac{E\Sigma_n \times \Sigma_n F^{\wedge n}}{E\Sigma_n \times \Sigma_n \{\ast\}},$$

where $E\Sigma_n$ is some contractible space with a free action by the symmetric group $\Sigma_n$ with appropriate functoriality conditions for differing $n$. The reader should think of it as some analog to the degree $n$ component of the commutative operad $Com(n)$ described in Section 4. The subscript of $\wedge_{\Sigma_n}$ is usually understood and will be dropped in the remainder of the paper. Such extended powers have a collection of natural maps

$$\iota_n : F^{\wedge n} \to D_j F = E\Sigma_n \wedge F^{\wedge n},$$

$$\alpha_{m,n} : D_m F \wedge D_n F = \left( E\Sigma_m \wedge F^{\wedge m} \right) \wedge \left( E\Sigma_n \wedge F^{\wedge n} \right) \to E\Sigma_{m+n} \wedge F^{\wedge (m+n)} = D_{m+n} F,$$

$$\beta_{m,n} : D_m D_n F = E\Sigma_m \wedge \left( E\Sigma_n \wedge F^{\wedge n} \right) \wedge_m \to E\Sigma_{mn} \wedge F^{\wedge mn} = D_{mn} F,$$

$$\delta_n : D_n (E \wedge F) = E\Sigma_n \wedge (E \wedge F)^{\wedge n} \to \left( E\Sigma_n \wedge E^{\wedge n} \right) \wedge \left( E\Sigma_n \wedge F^{\wedge n} \right) = D_n E \wedge D_n F.$$

The first map is just taking the smash product with $E\Sigma_n$. The second and third maps involve combinations of shuffling the terms with the maps $E\Sigma_m \times E\Sigma_n \to E\Sigma_{m+n}$ induced by block permutations in the second case, and a wreath product $E\Sigma_m \times (E\Sigma_n)^{\wedge m} \to E\Sigma_{mn}$ in the third case. The last map combines a diagonal map $E\Sigma_n \to E\Sigma_n \times E\Sigma_n$ with a shuffle. The classical definition of an $H_\infty$ ring spectrum follows.
Definition 7.5. An $H_\infty$ ring spectrum is a spectrum $F$ with structure maps

$$\gamma_n : D_j F = E \Sigma_n \wedge F^\wedge n \to F$$

for $n \geq 0$ such that $\gamma_1$ is the identity map and the following diagrams commute for $m, n \geq 0$.

Just like with an $E_\infty$-structure on spectra, all $H_\infty$ ring spectra are ring spectra by loss of structure. This warrants the inclusion of the word ring in the definition to denote that the spectrum has a multiplicative structure.

7.3 General $H_n$-operads

The classical definition above is suited to one specific symmetric monoidal category and one particular class of operads on that category. We would like a more general definition which includes the classical definition as a particular case.

Definition 7.6. Given any operad $O$, we gave the definition of an $O$-structure on an object $X$ (i.e. an $O$-algebra) in Section 4. A homotopy $O$-structure is the same thing as an $O$-structure, except that the diagrams in the definition are only required to commute up to homotopy.

Using this definition, we say that $X$ has an $H_n$-structure if it has a homotopy $E_n$-structure for some $E_n$-operad $E$. Note that this is slightly stronger than saying that $X$ has an $E_n$-structure in the homotopy category. We require the operad to exist in the base category $C$ (as opposed to just the homotopy category of $C$) before passing to the homotopy category. An $H_\infty$ ring spectrum in the classical sense is the same thing as a homotopy $E_\infty$-spectrum.

At this point, the natural question to ask is whether every $H_\infty$ ring spectrum arises by starting with an $E_\infty$ ring spectrum and passing to the homotopy category. Indeed, the multiplicative structure on $H_\infty$ ring spectra is so rich, it may lead one to conjecture that no information has been lost by loosening the $E_\infty$-algebra conditions to only require commutativity up to homotopy. This is exactly the question addressed by Noel in [11].

Noel first uses a technical result of Kraines and Lada. In [5], Kraines and Lada define $L(n)$ structures which are defined in roughly the same way we defined an $A_n$-structure. The
difference is that the Stasheff associahedra operad \( K \) is replaced with the linear isometries operad \( L \) in the definition.

**Theorem 7.7 ([5]).** There exists a fibration sequence

\[
F \xrightarrow{i} E \to B
\]

with \( i \) a map of \( L(2) \) spaces. However, the \( L(2) \) structure on \( F \) does not lift to an \( E_3 \)-structure. In particular, \( E \) does not admit an \( E_\infty \)-structure compatible with this \( L(2) \) structure.

The total space \( E \) is \( BU(2) \), the classifying space of the unitary group \( U \) localized at the prime 2. The base space \( B \) is the Eilenberg-MacLane space \( K(\mathbb{Z}_2; 30) \). The fiber map from \( E \) to \( B \) is \( 4s \), where \( s \) a primitive generator of the homology group \( H^{30}(BU; \mathbb{Z}_2) \).

Noel translates this theorem into the language of \( H_\infty \) ring spectra. He first proves that the functor \( \Sigma^\infty_+ \) which takes a topological space \( X \), attaches a disjoint basepoint \( X_+ = X \amalg \{*\} \), and then takes the suspension spectrum \( \Sigma^\infty(X_+) \) takes \( L(2) \) spaces to \( H_\infty \) ring spectra. This implies that the spectra \( \Sigma^\infty_+ F \) and \( \Sigma^\infty_+ E \) are both \( H_\infty \) ring spectra and the following theorem is true:

**Theorem 7.8 ([11]).** The map

\[
\Sigma^\infty_+ F \xrightarrow{\Sigma^\infty i} \Sigma^\infty_+ E
\]

is a map of \( H_\infty \) ring spectra, but the \( H_\infty \)-structure on \( \Sigma^\infty_+ F \) does not lift to an \( E_3 \)-structure. In particular, \( \Sigma^\infty_+ F \) does not admit a compatible \( E_\infty \) ring structure.

This theorem gives an explicit example of some \( H_\infty \)-structure in some category which does not admit a compatible \( E_\infty \)-structure. We could conjecture that there should be examples of such situations in other categories. In particular, a future project of mine will be to prove the following:

**Question 7.9.** Does there exists a chain complex \( A \in Ch(R) \) with an \( H_\infty \)-structure but no compatible \( A_4 \)-structure?

The appeal of working in the category \( Ch(R) \) is that we have a lot of computational machinery at our fingertips. This result would give a more intuitive and approachable counterexample than the one provided by Noel.

We note a few observations that can already be made about this question. First, every \( H_\infty \)-structure does provide us with an honest-to-goodness multiplication \( m \) and a coherent homotopy between the 3-ary multiplications \( m(m \otimes id) \) and \( m(id \otimes m) \). Therefore every \( H_\infty \)-structure does provide an \( A_3 \)-structure.

Also, we can complete an \( A_3 \)-structure to an \( A_4 \)-structure whenever we can find a homotopy filling the pentagon \( K_4 \) from the Stasheff associahedra. Obstructions to such fillings typically arise as elements of Ext or Tor. If our base ring is actually a field, then all the Ext and Tor terms vanish, and we have no obstructions to filling any of the Stasheff associahedra or the homotopy commutativity diagrams and getting an \( E_\infty \)-structure. We will likely need some torsion in our base ring to find such an obstruction.
References


