Iwahori - Hecke Algebras in Multiple Contexts

1) Hecke algebras for a reductive group
2) Presentation of spherical/finite/affine Hecke algebras
3) Quantum Schur - Weyl duality

1) Reductive Groups

Definitions

\( G \): reductive gp \( /F \): monarch, local field
\( O \): ring of ints. of \( F \)
\( P \): max’l ideal of \( O \)

\( B = \text{Borel subgp.} \)

\( K^o = \text{max’l compact subgp} \)

\( J = \text{Iwahori subgp.} \)

Favorite example

\( G = GL_n(\mathbb{Q}_p) \)
\( \mathfrak{O} = \mathbb{Z}_p \)
\( \mathfrak{P} = \langle p \rangle \)

\( B = \begin{pmatrix} * & \cdots & * \\ O & \ddots & \\ & & * \end{pmatrix} \)

\( K^o = \begin{pmatrix} \mathfrak{O} & \cdots & \mathfrak{O} \\ \vdots & \ddots & \vdots \\ \mathfrak{O} & \cdots & \mathfrak{O} \end{pmatrix} \)

\( J = \begin{pmatrix} \mathfrak{O} & \mathfrak{O} \\ p & \mathfrak{O} \end{pmatrix} \)
Let \( k \) be a compact open subgp. of \( G \). The Hecke alg. of \( G \) relative to \( k \) is the set of smooth compactly supp. \( k \)-biinvariant funs on \( G \):

\[
H_k := \left\{ \phi : G \rightarrow \mathbb{C}, \text{smooth cpt. supp.} \right\}
\]

\[
\phi(kgk) = \phi(g) \quad \forall k, k' \in K, \ g \in G \exists,
\]

w/ mult. defined as convolution.

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**Remarks**

1) Reductive groups are hard

2) Hecke alg. are relatively simple: often finite (ish) dim.

3) Borel-Matsumoto: \( \exists \) corresp. btwn irreps of \( H_k \) and "admissible" irreps of \( G \) w/ a \( K \)-fixed vector \( v \) (\( k \cdot v = v \ \forall k \in K \)).

4) So Hecke algebras are a tool to understand reps of red. gps.

But what do Hecke algebras look like?
2) Presentations (Iwahori')

For this section, $G = GL_n$, (but can be done for any Cartan type).

$H_{k_0} = X_\ast(T) \cong \mathbb{Z}^n$ (spherical Hecke alg.)

\[ H_B = \langle T_i, i=1, \ldots, n-1 \mid \begin{align*}
T_i T_i+1 &= T_i+1 T_i, \\
T_i T_j &= T_j T_i, & i \neq j \pm 1 \\
T_i^2 &= (q-1) T_i + q
\end{align*} \rangle \]

(finite Hecke alg.)

\[ H_J = \langle T_i, i=0, \ldots, n-1 \mid \text{same reln's as for } H_B, \text{ but indices taken mod } n \rangle \]

(affine Hecke alg.)

Remarks

1) Not guaranteed a simple presentation of $H_k$ for other subgps. $K$, but

2) $H_{k_0}$ is commutative!
3) $H_b$ is a deformation of the gp. alg of $S_n$:
   If $q \mapsto 1$
   
   $$H_b \mapsto \mathbb{C}[S_n].$$

So repn theory of finite Hecke algebras relate to repn. theory of $S_n$.

4) Exact sequences:
   
   $$1 \rightarrow P K^o \rightarrow J \rightarrow B(\mathcal{H}_b) \rightarrow 1$$

   $$\downarrow$$

   $$0 \rightarrow H_k^o \rightarrow H_j \rightarrow H_b \rightarrow 0$$

So to understand $H_j$, want to understand $H_k^o, H_b$.

3) Quantum Schur-Weyl Duality

First, classical S-W duality:

Let $V = \mathbb{C}^n$ be std. repn of $G = GL_n(\mathbb{C})$.

Now, take $V \otimes^k$, for $k \leq n$, and let $G$ act diagonally:

$$g \cdot (V \otimes \cdots \otimes V_k) := g \cdot V_1 \otimes \cdots \otimes g \cdot V_k.$$
Let $S_k$ act on $V \otimes k$ by permuting the factors:

$$(V \otimes \cdots \otimes V_k) \cdot \sigma = V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(k)}$$

These actions commute, and in fact are mutual centralizers.

**Schur–Weyl Duality:** As a $(GL_n, S_k)$-bimod, $V \otimes k$ decomposes as

$$V \otimes k = \bigoplus_{\lambda \vdash k} L^\lambda \otimes S^\lambda,$$

where the $L^\lambda$ are (distinct) highest wt. modules, and the $S^\lambda$ are (distinct) Specht mods.

Now, let $V$ be the std. repn. of the quantum gp. $V := V_q(GL_n), q \neq \text{root of unity}$, and let $U$ act on $V \otimes k$ via the coproduct map.

Since $V$ not cocommut., we can't just permute the factors. Instead, we use the Yang–Baxter eqn. to define isomorphisms:

$$R_i : V_i \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_k \cong V_j \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_k.$$
Thm (Jimbo, '86): The alg. gen'd by the Ri is isom. to $H_B$ (for $GL_k$), and the $V$ and $H_B$ actions are mutual centralizers, so we the decomp.

$$V^g = \bigoplus_{\lambda \in k} L^\lambda \otimes S^\lambda,$$

where $L^\lambda, S^\lambda$ are irreduc. and deformations of the $L^\lambda, S^\lambda$.

**Remarks**

1) Jimbo's results helped kick-start huge breakthrough. One notable example: Jones' Field Medal work on knot invariants.

2) This section only holds for $GL_n$, not a reductive group of any other type.

3) Not surprising that $U_q(gln)$ is in S-W duality w/ a deformation of $CE_{5n}$, but it is remarkable that this deformation turned out to be the Hecke alg.
4) I am not aware of any "natural" reason for remark 3, and in light of remark 2, might be hard to have a general result. Would be very interesting if such a result existed.

\[
U(\mathbb{R}_2) = \langle e, f, h \mid [e, f] = 2h, \quad \square \rangle
\]

\[\exists k, k^{-1} \quad \square\]