The Hecke algebra is an important object in rep theory that in the affine case gives us info. about the repn theory of reductive p-adic groups. But we're going to talk today about the finite Hecke algebra. Let's start by giving three different definitions of the finite Hecke algebra and talking about why each one is important.

(Generators and Relations) (Deformation) of Coxeter Group Alg.

$W$: finite Coxeter group, $W = \langle S \rangle$

$H_W := \langle T_s \mid s \in S \rangle$

Braid: $T_s T_t \overline{=} T_t T_s \overline{mst}$

Quad: $T_s^2 = (q_s - 1) T_s + q_s$

(often, we take the $q_s$ to all equal some cplx #, but here keep transcendental $\mathbb{Q}$)

$q_s = q_t \iff s \not\sim t$

For now, working over $\mathbb{C}[\{q_s\}]$

Basis: $\{$ $T_w \mid w \in W \}$

$q$-analogue to $W$ e.g. trivial character now = length function
2) Borel-bi-invariant functions of a reductive group $G$ and finite Chevalley group $G$. 

B.N.: 
\{ reps of $G$ \} \leftrightarrow \{ reps of $G$ with $B$-fixed vectors \}

3) Type A: Centralizer of quantum group $V_q(gl_n)$ (Jimbo, 1986) 
$V$: std repn of $V_q(gl_n)$, $n \geq k$
$H_{sk} = \text{End}_{V_q(gl_n)}(V_{sk})$
(don't get Hecke alg in other types)

Tmt: These three definitions are equivalent.

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**Character Theory**

(like for any alg. ob., want to study char. theory. We'll see later an application of this to knot theory.)

The first and most important tool is Tits' Deformation Thm.

Tits' Deformation Thm: Let $W$ be a finite Coxeter gp, $H_W$ its Hecke algebra over a "large enough" field $k$.

Then $H_W \cong k[W]$, and $H_W$ is semisimple.

(What this means is that the repn theory of $H$ and $W$ is "the same." Explicit isom. do exist, but used less often than Tits' Deformation Thm.)
How do we define a character table for $H$?

(3)

Specifically, need to define "std" elts. on which we can take the characters and compute from them the char. values of the rest of the Hecke alg.

Thm (Starkes, Ram, Geck-Pfeiffer): If $\lambda$ is a CL class of $W$, we can take the std. elt. corresp. to $\lambda$ to be $T_{w_{\lambda}}$ for any min'le length $w_{\lambda} \in \lambda$.

(Weighted orthog. rel's)

Computing the Character Table

1) "Inductive on Rank": $M-N$ rule (types $A, B, D$, Ariki-Koike)
2) "By deformation": Starkes Rule (type $A$)

Starkes Rule (1975):

$$\chi(T_{w_{\lambda}}) = \sum_{\nu \vdash \lambda} \chi(w_{\nu}) \prod_{\nu}$$

where

$$p_{\nu} = \frac{|C_{\nu} \cap S_{\lambda}|}{|S_{\lambda}|} \det(q-\text{id}_{\nu} - p_{\lambda}(w_{\nu}))$$

Ex: $W = A_{2} = S_{3}$

<table>
<thead>
<tr>
<th>$S_{3}$</th>
<th>$T_{s_{1}}$</th>
<th>$T_{s_{2}}$</th>
<th>$T_{s_{3}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{1}\tau_{s_{2}}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$s_{1}^{-1}\tau_{s_{2}}$</td>
<td>$1$</td>
<td>$-1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$s_{1}\tau_{s_{3}}$</td>
<td>$2$</td>
<td>$0$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

$\chi_{\text{ref}}(T_{s_{1}}) = \sum_{\nu} \chi_{\text{ref}}(w_{\nu}) \prod_{\nu} = p_{(3)}^{2} - p_{(2)}^{(3)}$

$p_{(3)}^{(1)} = \frac{1}{2} \det(q-p_{(3)}(w_{(1)})) = \frac{1}{2} (q-1)$

$p_{(21)}^{(21)} = 0$

So $\chi_{\text{ref}}(T_{s_{1}}) = 2 \cdot \frac{1}{2} (q-1) = q - 1$
Application: Ocneanu's Trace (used to construct $\text{HF} \bowtie \text{FLY}$ poly)

Starkey's Rule: computes the $\omega$'s

\[ \tau: H_w \to C \]

\[ \tau(h) = \sum_{x \in \omega} a_x \chi_A(x) \]

\[ x \vdash h \quad \omega \]

(These $\omega$'s are in terms of Schur functions, so

(These $\omega$'s give positivity properties related to the classification of

Von Neumann algebras).

(Type $B$, trace $\mathcal{E}$, $\omega$'s $\mathcal{E}$, but proof uses type $A$ $\omega$'s; would be

slicker if & easier computationally to go directly there).

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**Starkey's Rule Proof**

One of my thesis problems is to develop a Starkey's Rule for type $B$.

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**Step**

1) $\tau^2$ central in $H_w$ (Springer)

2) If $H_w = H^2$, $\mathcal{E}$ deformation

   formula for $\mathcal{X}(H_w)$ (Broué-Michler)

3) Coxeter elts satisfy this property

4) Using ref $\mathcal{E}$ repn, $\mathcal{E}$ det $\mathcal{X}(T_w)$

   formula for $\mathcal{X}(T_w)$

5) Can use (4) to prove Starkey's Rule

   for any $T_w$ w/ $\omega$: Coxeter elt. of std.

   parab. subgp.

6) Every $\mathcal{X}$ has such an elt

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**Extendability**

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general

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all types, can extend (eg. longest

elt)

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Types $A$ & $B^+$ *new

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general, so works in

Types $A$ & $B$

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Type $A$ only
Strategies

1) Expand elts in step 3) (eg. "good" elts, quasi-integer elts)
2) Expand std. parabolic subgp. to other subgp. (nonstandard parabolic, other refp subgps)
3) Work back wards from Oceana's trace
4) Extend Yau's construction
5) Use...