1. **Rep theory definitions**

2. **Motivation**

3. **Classification of reps & corresponding L-factors**

4. **Whittaker models & Casselman-Shalika formula**

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1. **Def.** A representation of a group $G$ is a homomorphism $\pi : G \rightarrow GL(V)$, $V$ a $\mathbb{C}$-vector space.
   - If we pick a basis for $V$, $GL(V) = GL_n(\mathbb{C})$, $n = \dim V$
   - Think of $\pi$ as an action of $G$ on $V$
   - If $G$ has a topology, usually add more adjectives ("smooth rep")
   - $\dim (\pi, V) := \dim_{\mathbb{C}} V$

   - A one-dim rep $\pi : G \rightarrow \mathbb{C}^*$ is called a character.
     - Given $(\pi, V)$, the func. $\chi(g) = \text{tr}(\pi(g))$ is also called the character of $\pi$

2. **Def.** A rep $(\pi, V)$ is **irreducible** if the only subreps are $\{0\}$ and $V$.

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3. **(Induction)**

   **Def.** Given $H \leq G$, and a rep $(\chi, W)$ of $H$, we can define a representation $\text{Ind}^G_H(\chi, V)$ of $G$, where
   $$ V = \left\{ f : G \rightarrow W \mid f(hg) = \chi(h)f(g) \quad \forall h \in H \right\} $$
   \[G \xrightarrow{(s \cdot f)} W \text{ by } (s \cdot f)(g) = f(gs).\]
Recall: Number Theorists & automorphic forms & L-functions -- (generating functions for arithmetic data)

\[ G(A) \text{ acts by right multipl.} \]

\[ \mathcal{L}^2(G(\mathbb{Q}) \backslash G(A)) \]

Given \( \phi \in \mathcal{L}^2(G(\mathbb{Q}) \backslash G(A)) \), get rep \((\Pi_\mathbb{Q}, V_\mathbb{Q})\) where \( V_\mathbb{Q} = G(A) \cdot \phi \). These are called \underline{automorphic representations}.

Given \( \pi \) of rep, have decomposition

\[ \pi = \bigotimes_{\mathbb{P}} \pi_p \]

into local representations of \( G(\mathbb{Q}_p) \) \& \( G(\mathbb{R}) \).

Understanding local reps helps us define local \( \mathcal{L} \)-functions w\text{t} the nice properties we like (analytic continuation, functional eqn) \( \mathcal{L} = \prod_{\mathbb{P}} \mathcal{L}_{\pi_p}(s, x) \), \( \mathcal{L}_{\pi_p} \) \text{L-factors for local reps}

Slight (at least for \( p \neq 2 \)): representations are parametrized by characters of \( \text{tor} \cong (\mathbb{Q}_p^\times)^2 \)

\[ B = \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p^\times \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{Q}_p^\times \end{pmatrix}, \quad T = \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p^\times \\ 0 & 1 \end{pmatrix}, \quad B : T U \]

\[ \text{Important subgroups of } G: \]

\[ \text{Classification} \]

\[ \text{a) 1-dimensional reps: } \chi \text{det for } \chi \text{ a character of } \mathbb{Q}_p^\times \text{ (don't care too much about these)} \]

\[ \text{b) Irreducible principal series } \pi(\chi_1, \chi_2): \]

*take character of \( T \), inflate to \( B \), induce to \( G \) ("parabolic induction")

Explicitly: Let \( \chi_1, \chi_2 \) be chars of \( \mathbb{Q}_p^\times \)

1. Define \( \chi \) on \( T \): \( \chi(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}) = \chi_1(a) \chi_2(b) \)

2. Inflate to \( B \) by acting trivially on \( U \): \( \chi(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}) = \chi_1(a) \chi_2(b) \)
\[ \pi(\chi_1, \chi_2) = \text{Ind}_B^G \chi \]

If irreducible, define the \( L \)-factor:

\[ L(s, \pi) = \frac{1}{(1 - \chi_1(p)p^{-s})(1 - \chi_2(p)p^{-s})} \]

*\( \chi_1, \chi_2 \) unramified - otherwise, replace \( \chi_i(p) \) with 0.

There are irreducibles, unless \( \chi_1 \chi_2^{-1} = \text{id}_p \), in which case a subquotient is irreducible. These look like:

\[ \pi(\chi_1 \cdot p^{1/2}, \chi_2 \cdot p^{-1/2}) \]

called special reps.

\[ L(s, \pi) = \frac{1}{1 - \alpha p^{-s}} \]

where \( \alpha = \chi(p)p^{1/2} = p^{1/2} \chi(p) \) (or 0 if \( \chi \) ramified).

(1) Supercuspidal reps — basically defined to be "all other reps." They are more complicated to describe.

The Jacquet module associated to a rep \((\pi, V)\) is

\[ J_\pi = \left\langle \pi(u)v - v \mid u \in U, v \in V \right\rangle \]

An rep is supercuspidal if \( J_\pi = 0 \).

(Idea: principal series — \( U \) acts trivially; if \( V \) supercuspidal, no elt of \( U \) acts trivially. Turns out these are the only options)

- Obtained from chaos on non-split tori: \((\text{Ad} a)\) where \( a \in \mathbb{Q}_p \)

\[ L(s, \pi) = 1 \]

Local Langlands:

\[
\begin{align*}
\{ \text{smooth} \} & \begin{cases} \text{irreducible} \end{cases} \rightarrow \{ \text{rep of } \text{Gal}_L(\mathbb{Q}_p) \} \\
\{ \text{2-D, semisimple} \} & \begin{cases} \text{Weil group} \end{cases} \rightarrow \{ \text{rep of } \text{Gal}(L/\mathbb{Q}_p) \}
\end{align*}
\]

& this bijection respects \( L \)-factors

& \( E \)-factors ← (come up in functional eqn \( \Lambda(m, s) = \zeta(m, s) \cdot \Lambda(m, 1 - s) \))
* In the case of $GL_2(\mathbb{F}_p)$ or $GL_2(\mathbb{Q}_p)$, this bijection is described explicitly in the books of Piatetski-Shapiro and Bushnell & Henniart, respectively.

Un the bijection, special & supercuspidal \[ \longrightarrow \] irreps of Weil gp

principal series \[ \longleftrightarrow \] reducible reps

(4) Whittaker models & Casselman-Shalika

Let $\psi$ be a character of $\mathbb{Q}_p$. Define $\psi_u : \mathbb{V} \rightarrow \mathbb{C}$ by $\psi_u(1 x) = \psi(x)$.

**Def** A **Whittaker model** of a rep $(\pi, \mathbb{V})$ of $G$ is an embedding $\mathbb{V} \hookrightarrow \text{Ind}_u^G \psi_u$.

In other words, it is a space $W(\pi)$ of functions $W : G \rightarrow \mathbb{C}$ s.t.

$W(1 x) g = \psi(x) W(g)$.

Why do we care about Whittaker models??

* $\text{Ind}_u^G \psi_u$ is multiplicity-free (if they exist, W. models are unique)

* Whittaker functions give "Fourier decompositions" of automorphic forms

* Useful in proving analytic continuation & fnc'd eqn for $L$-functs (by equating $L$'s w/ "zeta integrals")
Casselman-Chalika formula:
\[ \pi(x_1, x_2)[x_1, x_2 \text{ unramified}] \text{ admits a \( \text{Whittaker} \) model} \]

C-S formula computes the (spherical) Whittaker fun explicitly:
\[ W_\lambda((p_{m_0}^0)) = (\star) \cdot \frac{\alpha_1^{m_0} - \alpha_2^{m_0}}{\alpha_1 - \alpha_2} \quad \text{where} \quad \alpha_1 = \text{char}(p) \quad \alpha_2 = \text{char}(C) \quad m > 0 \]

Some stuff

\[ = \text{Schur poly} \ S_\lambda(\alpha_1, \alpha_2), \lambda = m \]

= value of character of irreps of
\[ \text{GL}_2(C) \text{ on } (\alpha_1, \alpha_2) \]

For more general \( p \)-adic groups \( G \):
\[ \text{value of} \]
\[ \text{sph. Whitt. \ fun} \]
\[ \text{of Langlands} \]
\[ \text{dual group} \text{ } \left( \alpha_1 \alpha_2 \right) \]

Some references:

* Bump, "Automorphic Forms and Representations"

* Piatetski-Shapiro, "Complex Representations of \( \text{GL}(2, K) \) for finite fields \( K \)"

* Bushnell & Henniart, "The Local Langlands Conjecture for \( \text{GL}(2) \)"

* Kimball Martin, Automorphic representations course notes

* Emily's talk from summer rep theory seminar - see Claire's website &
  (good summary of P-S)