Introduction to Rankin-Selberg Method

- History
- Basic example (holomorphic modular forms for $SL(2, \mathbb{Z})$)
- A more general case: $GL_n \times GL_m$
- Application
- Reference

History

Rankin-Selberg Method

Robert Alexander Rankin \(\uparrow\) Atle Selberg

is introduced independently in 1939 by Rankin, 1940 by Selberg.

This is also known as the theory of integral representations of $L$-functions. It is a technique for directly constructing and analytically continuing several important examples of automorphic $L$-functions.

- Standard $L$-function on $GL_n$ (Goddement-Jacquet)
- Standard $L$-function on classical groups (Piatetski-Shapiro-Rallis)
- Tensor product $L$-function on $GL_n \times GL_m$ (Jacquet, PS, Shalika)
  (\& reverse-engineered to establish the \textit{converse theorem})
- Exterior square on $GL_n$ (Jacquet-Shalika, Bump-Ginzburg)
Let $f, g$ be two holomorphic cusp forms on the upper half plane $\mathbb{H}$.

A modular form is a holomorphic function $f: \mathbb{H} \to \mathbb{C}$ that satisfies the transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for $(a, b, c, d) \in \text{SL}(2, \mathbb{Z})$, and is holomorphic at the cusps.

A modular form with $c(0) = 0$, $c(0)$ is the value of $f$ at $i\infty$.

A modular form of weight $k$ for $\text{SL}(2, \mathbb{Z})$, with Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

$$g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

Write $\Gamma = \text{SL}(2, \mathbb{Z})$, the modular group.

Let $P = \{ (x, x') \in \Gamma \}$, upper triangular group.

Then we can define Eisenstein series $E_k$ as

$$E_k(z) = \sum_{\gamma \in \Gamma \backslash \mathbb{H}} \frac{1}{|\gamma z|^k}$$

Here $\gamma$ acts on $z$ by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot z = \frac{a z + b}{c z + d}, \quad \text{Im}(z) = \frac{1}{|c z + d|^2}$$

Those are

One can verify: $E_k$ converges absolutely for $\text{Re}(z) > 1$.

We focus on

- $E_k(z) \in \text{SL}(2, \mathbb{Z})$ - invariant.

Analytic properties

- $E_k$ has an analytic continuation to $\mathbb{C}$.

Functional properties

Let $\zeta(s) = \pi^{-s/2} \Gamma(s/2) \zeta(1/2)$

with gamma factor $\Gamma \frac{s}{2}$.

Functional equation $\zeta(1-s) \zeta(s) = \pi^{-s/2} \Gamma(s/2) \zeta(1/2)$.

Petersson inner product for weight $k$ modular forms:

$$\langle f, g \rangle = \int f(z) \overline{g(z)} Y^k \text{ d}m$$

$\Gamma$-invariant $\text{SL}(2, \mathbb{R})$-invariant measure $\text{d}m$.
Rankin-Selberg integral:

\[
\langle f \cdot E_s, g \rangle := \int \frac{f(z) \overline{g(\tau z)}}{y^k} \, \frac{dxdy}{y^2}
\]

- converges for all \( s \in \mathbb{C} \) (if one of the forms is cuspidal)
- analytic continuation:

Thm: \( \langle f \cdot E_s, g \rangle = (4\pi)^{-\frac{s-k}{2}} \frac{\Gamma(s+k-1)}{\Gamma(s+1)} \sum \frac{\text{anbn}}{n^s} \left[ \frac{1}{\Gamma(2s)} \sum \frac{\text{anbn}}{n^s} \right] \).

\( \sum \frac{\text{anbn}}{n^s} \) has an analytic continuation to \( \mathbb{C} \), with poles at most at \( s = 0 \).

PF: For integral \( f \cdot \text{P-invariant function on } \mathbb{H} \), we have an identity similar to Farini's thm:

\[
\int \frac{f(z) \, dx \, dy}{y^k} = \int \frac{1}{2} \sum \frac{f(\frac{1}{2}z)}{y^{\frac{1}{2}}} \, dx \, dy
\]

Let \( f(z) = y^s \cdot \overline{f(z \, g(z)) \, y^k} \)

\[
\int \frac{y^s \cdot f(z \, g(z)) \, y^k \, dx \, dy}{y^2} = \langle f \cdot E_s, g \rangle
\]

For fundamental domain for \( \text{Pf} \), take

\( \bar{\mathbb{H}} = \{ z = x + iy : 0 \leq x \leq 1 \} \)

\[
\int \frac{y^s \cdot f(z \, g(z)) \, y^k \, dx \, dy}{y^2} = \sum_{\text{m}\,|\,2} \frac{\text{anbn}}{n^s} \sum_{y > 0} \int_{x > 0} e^{-\pi my^2} \, dx \, dy
\]

\[
= \sum_{n > 1} \frac{\text{anbn}}{n^s} \int_{y > 0} 2 \left( \frac{\text{anbn}}{n^s} \right) \sum_{m > 1} \frac{\text{anbn}}{m^s} \, dy
\]

This is true for \( \text{Re}(s) > 1 \).

And by the identity principle of Eisenstein series (analytically continuous), it is absolutely convergent for \( s \in \mathbb{C} \), away from...
Rankin's original purpose in considering the tensor product L-function was to approach Rankin's conjecture on the size of Hecke eigenvalues.

The poles of the Eisenstein series (i.e., the Eisenstein series is of moderate growth) and the cusp forms are of rapid decay.

- Functional Equation.

(Cogdell's paper) (Jacquet, P., Shalika)

$GL_n \times GL_m$ convolution

different between $n = m$ and $n \neq m$

$\uparrow$

involve with Eisenstein series.

$n = m$


$P$: standard maximal parabolic subgroup of $GL_n$ with Levi factors $GL_{n-1} \times GL_1$.

$\xi: P(A) \to \mathbb{C}^*$ modular quasi-character

$\xi(h^{-1} a^{-1})$, $h \in GL_{n-1}(A), a \in \mathbb{A}^r$.

For $s \in \mathbb{C}$, let $f_s \in \text{Ind}_{P(A)}^{GL_n(A)} (Sp^s)$

- $f_s$ smooth function.

- $f_sp_g = Sp(p)^s f_s(g)$

Also assume

- $f_s$ restricted to a standard maximal compact subgroup of $GL_{n-1}(A)$ is independent of $s$.

WLOG, we can write

Resticted product

$\Rightarrow f_s(g) = \prod f_{s,v}(g_v)$, where $f_{s,v} \in \text{Ind}_{P_v}^{GL_n(A_v)} (Sp^s)$.

We have an Eisenstein series.
\( E(g,s) = \Phi(n s) \sum_{P \in \text{GL}(n,F)} \text{Sp}(g r_g)^5 \)

- It is convergent for \( \text{Re}(s) \) suff. large
- It has meromorphic continuation to all \( s \).

Let \( \phi_1, \phi_2 \) be \( \text{GL}(n) \) cusp forms in automorphic repn of \( \text{GL}(n) \).

Consider

\[ \int_{\text{center of } \text{GL}(n)} \phi_1(g) \phi_2(g) E(g,s) \, dg \]

After “unfolding”,

\[ \int_{\text{center of } \text{GL}(n)} \prod_{i=1}^n W_i(g) W_i(g) f_i(s) \, dg \]

where \( W_1, W_2 \) are Whittaker functions defined as

\[ \psi : A/F \to \mathbb{C} \text{ nontrivial additive character} \]

\[ N : \text{algebraic subgroup of upper-triangular unipotent matrices in } \text{GL}(n) \]

define a character \( \psi_N : NA \to \mathbb{C} \)

\[ \psi_N(n) = \psi \left( \prod_{i=1}^n \zeta_i \right) \]

Then \( W_1(g) = \int_{N_1(g) \text{adj }} \phi_1(n) \psi(n) \, dn \in W(\pi_1, \psi) \)

\[ W_2(g) = \int_{N_2(g) \text{adj }} \phi_2(n) \psi(n) \, dn \in W(\pi_2, \psi) \]

Because of the uniqueness of Whittaker model (\( \dim W \leq 1 \))

these functions are Euler products, i.e., if we assume \( W(\pi) \)

\[ \phi_i = \prod \phi_{i,v} \quad \phi = \prod \phi_{\text{adj},v}, \text{ in } \pi = \prod \pi_{i,v} \]
\[
\text{local integral:} \quad \int F \left( \frac{g}{2}, \frac{\xi}{2} \right) \text{d}g
\]

\[
\text{global integral:} \quad \int G \left( \frac{m}{2}, \frac{m}{2} \right) \text{d}g
\]

Since the Rankin-Selberg integrals of Hecke type are fairly rare,

\[
\text{it looks like a Eisenstein series.}
\]

\[
\text{global integral is \textit{of Hecke type}}.
\]

\[
\text{local integral:}
\]

\[
\text{local integral:}
\]

\[
W_i(a) = \int W_i(f) w_i(g) \text{d}g
\]

\[
\text{Whittaker Function on } GL(2, \mathbb{F})
\]

\[
\text{Redefine zero func.}
\]

\[
\text{To be continued...}
\]
Main Results

Thm: $\mathcal{W}$, $\mathcal{T}_x$ repn of $Gln$, $Glm$ of Whittaker type.

Let $W_1, W_2, W_3, W_4$, then:

1. Each of the integrals $\mathcal{W}(s,W_1,W_2,fs)$ $(n=m)$
   & $\mathcal{T}_x(s,W_1,W_2,fs)$ is absolutely convergent
   for Re(s) large.

2. They are rational functions of $q^{-s}$.

   More precisely, $n=m$, $\mathcal{W}(s,W_1,W_2,fs)$ spans a fractional
   ideal $c(q^{-s}, q^s) L(s, \pi_1 \pi_2)$ of the ring $c(q^{-s}, q^s)$.

   The factor $L(s, \pi_1 \pi_2)$ has the form $P(q^{-s})$, where $P(0)$,
   and $P(0)=1$.

(Similar result for $n \neq m$)

(iii) Functional equation $n=m$.

For factor $c(s, \pi_1 \pi_2, fs)$ of the form $c(q^{-s}, fs)$

$\mathcal{W}(1-s, W_1, W_2, fs) / L(1-s, \pi_1 \pi_2) = W_1 \cdot (-1)^{m-1} \cdot c(s, \pi_1 \pi_2, fs) \cdot c(s, W_1, W_2, fs) / L(s, \pi_1 \pi_2)$

(Similar result for $n \neq m$).

Application:

- Eisenstein series theory

  $\implies$ the analytic properties of the constant term of the Eisenstein
  series can be derived from our theory.

- Characterization of automorphic repns.

  If $r$ repn of $Gnl(F_A)$ with some auxiliary conditions,

  it is cuspidal & automorphic iff a automorphic repn $\pi$ of
  $Glnl(F_A)$, the corresponding $L$-fun has the appropriate
analytic behavior.

References:

- The Rankin-Selberg Method: An Introduction and Survey (Daniel Bump)
- Rankin-Selberg Convolutions (Jacquet, P.S., Shalika)
- Basic Rankin-Selberg (Paul Garrett)