Casselman-Shalika Formula for $GL_2$:

**Outline**
1. Whittaker models/functionals
2. Spherical representations
3. Spherical Whittaker function, the $C$-$S$ formula
4. Significance

**0. Automorphic representations decompose**

$$\pi \cong \otimes \pi_v$$

as $GL_2 \otimes \pi_v$

*Our formula lives here in the local theory*

**Notation**

- $F$: non-Archimedean local field
- $\mathcal{O}$: ring of integers
- $\mathfrak{p}$: maximal ideal of $\mathcal{O}$
- $q = |\mathfrak{p}|$
- $\omega$: uniformizer

$$G = GL_2(F), \quad B = \left( \begin{array}{cc} 1 & \star \\ 0 & 1 \end{array} \right), \quad N = \left( \begin{array}{cc} 1 & \star \\ 0 & 1 \end{array} \right), \quad T = \left( \begin{array}{cc} \star & \star \\ 0 & \star \end{array} \right)$$

*Fix a nontrivial additive character $\psi: F \to \mathbb{C}^\times$.

**1. Whittaker models/functionals**

**def:** Let $(\pi, \nu)$ be any irrep of $G$. A **Whittaker functional** is a linear functional $L: \nu \to \mathbb{C}$ s.t.

$$L(\pi(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) \nu) = \psi(x) L(\nu)$$

**Frobenius reciprocity**

$$\text{Whittaker functionals} \longleftrightarrow \text{G-mod} \text{ homs}$$

$$\nu \leftrightarrow \text{Space of functions } f: G \to \mathbb{C}^\times$$

$$\text{Ind}_P^G f(ng) = \nu(n) f(g) \quad \forall n \in \mathbb{N}$$

A Whittaker model is the image of such a hom.
existence? All smooth \( \infty \)-dim'\( l \) irreps have a Whittaker model.

\( \square \) not true for \( \text{GL}_n \) in general

uniqueness? The space of Whittaker models is at most 1-dim'

has nice consequences — will play a role in proof of the C-S formula

2. Spherical Representations:

Iwasawa decomposition: \( G \) has a maximal compact subgroup \( K := \text{GL}_n(\mathbb{O}) \) and \( G = B \cdot K. \)

def: An irreducible admissible rep \( \pi \) is spherical if it contains a \( K \)-fixed vector.

[In \( \pi \equiv \oplus \pi_i \), almost all are spherical]

eexistence? Any \( \infty \)-dim' spherical irreducible admissible rep is a nonramified principal series rep

1. \( \chi_1, \chi_2 \) nonramified quasi-characters of \( F \)

1. Inflated: \( \chi'(y_1, y_2) := \chi_1(y_1) \chi_2(y_2) \)

2. Induce: \( \Pi(\chi_1, \chi_2) := \text{Ind}_{B}^{G} \chi' \)

The reps irreducible after this are nonramified principal series.

Here, \( \gamma_k \) is modular quasi-char

\( \gamma_k(bk) := \delta^{1/2} \chi(k) \)

is the normalized spherical vector in \( \Pi(\chi_1, \chi_2) \).

uniqueness? The space of \( K \)-fixed vectors is one-dim.
C.S. formula:
def: Let \( \pi(x_1, x_2) \) be an unramified Principal series rep w/ \( \alpha_i, x_i(\mathbb{C}) \). The spherical Whittaker function is the spherical vector in the Whittaker model:

\[
W_0(\varrho) := \Lambda(\pi(\varrho) \varphi_k) \quad \text{not really a def}
\]

\[
\Lambda(f) = \int_C f(\varpi(x))(\varpi(x)) \varphi_k(x) \, dx
\]
makes \( W_0(\varrho) = 1 - q \alpha_i \alpha_j z^2 \) —
comes from Fourier expansion of Eisenstein series

Note: If we fix \( \varrho \), \( W_0(\varrho) \) is a holomorphic function of \( \alpha_i \) and \( \alpha_j \).

Goal: Compute this.

B/c of transformivity properties of the components of \( W_0(\varrho) \), we only need to compute \( \sim \) cosets

\[
\mathbb{Z} \setminus G/K
\]

this + Iwahara decimp \( \Rightarrow \) only need to find \( W_0(\varpi^m \varrho) \) for \( m \in \mathbb{Z} \).

The Casselman-Shalika formula:

\[
(1 - q^{-\alpha_i \alpha_j} z^2) W_0(\varpi^m \varrho) = \sum_{m > 0} Q^{-m \alpha_i \alpha_j} \alpha_i \alpha_j z^{m+2} \cdot \varpi^m \varrho, \quad m > 0
\]

\[
(1 - q^{-\alpha_i \alpha_j} z^2) W_0(\varpi^m \varrho) = 0 \quad \text{for } m < 0.
\]

Appears (in more generality) in papers of C-S in 1980, not first proven by them, but by Kato and Shintani in different settings

Sketch of Casselman's method:

1. Can "easily" get \( W_0(\varpi^m \varrho) = 0, \) for \( m < 0 \).
2. Uniqueness of Whittaker functionals helps \( \Rightarrow \)
3. (1 - \( q^{-\alpha_i \alpha_j} z^2 \))\( W_0(\varrho) \) is inv under \( q_i \leftrightarrow q_2 \)
4. Out of \( \varphi_k \), make

\[
F_m(\varrho) = \int_C \varphi_k(\varpi(x)(\varrho^m \varrho)) \, dx
\]
is fixed by the Iwahori subgroup \( K_0(p) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K \mod p \).

Express \( F_m \) in the Casselman basis \( \psi_0, \psi_2 \) of \( V^{K_0(p)} \).

iv. Turns out

\[
W_0(0, 0, i) = \int_{\text{Int}(G^0)} \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \psi(-x) \, dx
\]

Sub in \( F_m \) in terms of Casselman basis

get

\[
W_0(0, 0, i) = C_1 q^{-m/2} \alpha_1^m + C_2 q^{-m/2} \alpha_2^m
\]

integrals involving \( \psi_0, \psi_2 \).

v. Compute \( C_0 \) and apply FE from ii

4. Significance & History:

- \( S_{(\alpha_1, \alpha_2)} \) is the value of a character of an irrept of \( GL_2(\mathbb{C}) \) on the conjugacy class of \( (\alpha_1, \alpha_2) \)

\( \Rightarrow GL_2(\mathbb{C}) \) is the "L-group" or Langlands dual group of \( GL_2 \).

- Another example: \( GL_3 \), \( Sp_4 \sim SO_5 \)

- More generally, values of spherical Whittaker functions of an irrep are given by a character of the "L-group" applied to the conjugacy class parameterizing it.

History: The L-group was introduced by Langlands in 1967.

He conjectured about the role of these L-groups in the subsequent years— including this formula.

- Corresponding formula for \( GL_n \) proven by Shintani in 1976.


- Not only significant for this connection— but also aids in calculations involving L-functions, Pariin-Selberg method, Langlands-Shahidi method.
Sources:
- Automorphic Forms and Representations" Bump
- "The L-group" Casselman
- Katy’s notes for nice background on reptheory of $GL_2(\mathbb{Q}_p)$.